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THE MEAN VALUE PROPERTY FOR NONSTRICTLY HYPERBOLIC SECOND ORDER QUASILINEAR EQUATIONS AND THE NONLOCAL PROBLEMS*

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Abstract. In the present paper we consider a class of quasilinear nonstrictly hyperbolic equations with degeneration of order and type. For these problems the nonlinear analogue of Asgeirsson's mean value property, the so-called property of proportionality of an argument, is established. General solutions are constructed in the form of superposition of arbitrary functions. On the basis of these representations and mean value properties, three nonlinear generalizations of the Goursat problem, including the problem with a free characteristic, are developed and investigated.

რეზიუმე. განხილულია ნამდვილმახასიათებლებიან კვაზიწრფივ განტოლებათა კლასი რიგის და ტიპის გადაგვარებით. მათთვის დადგენილია ასგეირსონისეული საშუალო მნიშვნელობის თვისების არაწრფივი ანალოგი-ე.წ. არგუმენტის პროპორციულობის თვისება. აგებულია აგრეთვე მათი ზოგადი ამონახსნები ნებისმიერ ფუნქციათა სუპერპოზიციის სახით. მათ საფუძველზე შემუშავებულია და გამოკვლეული გურსას ამოცანის სამი არაწრფივი ვარიანტი. მათ შორის გურსას არაწრფივი ამოცანა თავისუფალი მახასიათებლით.

§ 1. Introduction

In the present paper we consider some problems which are connected with a general representation of solutions of quasilinear second order equations with two independent variables x and t. It is assumed that these equations have two families of real characteristics which on some sets of points (depending on values of a solution and on its first derivatives) may merge. This class of parabolically degenerating hyperbolic equations is wide enough and in some cases they admit an explicit representation of general solutions of verious structure. A number of arbitrary elements, arbitrary

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functions in these representations does not always correspond to the equation order. Their number may even be unrestricted (see [1], [2]). Frequently, general solutions contain arbitrary functions together with their derivatives and primitives. We are also familiar with classes of equations in general solutions of which one arbitrary function is a part of an argument of another arbitrary function [3]. For example, a general solution of the equation

$$(1 + u_t)u_{xx} + (u_t - u_x + 1)u_{xt} - u_x u_{tt} = 0$$

is represented by two arbitrary functions $f,\, g \in C^2(R^1)$ as follows:

$$u(x,t) = -t + f\big[x + g(x-t)\big].$$

Solutions $u(x,l)$, for which $u_x + u_t + 1 \neq 0$, are hyperbolic. If on some set of points the above-mentioned inequality violates, we deal with parabolically degenerating hyperbolic solutions [3].

There naturally arises the question whether anyone combination of two arbitrary functions of similar structure will be a general solution of some equation of the second order. For illustration, we consider the following simplest structure:

$$u(x,t) = f\big\{x + \alpha(x,t)g(x-t)\big\},$$

which in the case of invertible arbitrary functions f is equivalent to the relation

$$f(u) - \alpha(x,t)g(x-t) = x. \tag{1}$$

We have to define a concrete equation or equations of second order for which relation (1) is the general integral. It turns out that such a correspondence between the general integral of type (1) and some equation of second order is not always possible.

The following proposition is valid: relation (1) with arbitrary functions $f,\, g \in C^2(R^1)$ may be a general integral of the equation of second order if and only if a coefficient α is a linear function of the argument x, and this equation is of the form

$$L(u) \equiv u_t u_{xx} - (u_x - u_t)u_{xt} - u_x u_{tt} = -\frac{c_0}{c_0 x + c_1} u_t(u_x + u_t) \tag{2}$$

for $\alpha = c_0 x + c_1$. Otherwise, relation (1) may be a general integral of the equation of order higher than the second one.

The proof is performed in a standard manner, by differentiating relation (1) with respect to the arguments x, t and excluding subsequently arbitrary functions f and g together with their derivatives up to the second order, inclusive.

If we exclude derivatives f'' and g'' from the following three relations

$$\begin{cases} u_x^2 f'' - \alpha g'' = -u_{xx} f' + \alpha_{xx} g + 2\alpha_x g', \\ u_t^2 f'' - \alpha g'' = -u_{tt} f' + \alpha_{tt} g - 2\alpha_t g', \\ u_x u_t f'' + \alpha g'' = -u_{xt} f' + \alpha_{xt} g + (\alpha_t - \alpha_x) g', \end{cases}$$

which are obtained by the differentiation of relation (1) twice with respect to the arguments x and t, we will finally get the equality

$$\frac{1}{u_x - u_t}\big\{ - L(u) f' + M(\alpha) g + (\alpha_x + \alpha_t)(u_x + u_t) g' \big\} = 0. \tag{3}$$

Here we introduce the notation

$$M(\alpha) \equiv u_t \alpha_{xx} - (u_x - u_t)\alpha_{xt} - u_x \alpha_{tt},$$

and define the operator L by formula (2). The latter equality involves the first order derivatives f' and g'. To exclude them, we have first to supplement the above equality with two another relations

$$\begin{cases} u_x f' - \alpha g' = 1 + \alpha_x g, \\ u_t f' + \alpha g' = \alpha_t g. \end{cases}$$

which are resulted from a single differentiation of (1).

Defining the derivatives f' and g' through g and substituting them in (3), we obtain

$$L(u) - \frac{1}{\alpha}(\alpha_x + \alpha_t) u_t (u_x + u_t) =$$

$$= \big[- (\alpha_x + \alpha_t) L(u) + M(\alpha)(u_x + u_t) - \frac{\alpha_x + \alpha_t}{\alpha}(\alpha_t u_x - \alpha_x u_t)(u_x + u_t) \big] g.$$

In order for this equality to have no arbitrary function g, it is necessary to require that its coefficient vanish. In such a case a desired equation is obtained if we equate the left-hand side of the equality to zero. But the combination $L(u)$ is a part of the coefficient of the function α in the right-hand side of the equation. To select the function $L(u)$, we define the expression g from the condition of vanishing of the coefficient for u and substitute it in the left-hand side which is equated to zero. As a result, we have

$$M(\alpha) + \frac{1}{\alpha}(\alpha_x + \alpha_t)\alpha_t (u_x + u_t) = 0$$

for all values u_x, u_t. Thus we can conclude that the function α should be selected in such a way that the expression would not in the whole depend on the derivatives u_x, u_t. And this is quite possible if either $\alpha = \alpha(x - t)$, or $\alpha = c_0 x + c_1$.

An analogous result is obtained if we exclude another pair f', g of free parameters. The result will be somewhat different if we exclude an arbitrary function g together with its derivative, express them in terms of the derivative f' and substitute the obtained values in (3). Thus we have the relation

$$L(u) = (u_x + u_t)\Big\{\frac{M(\alpha)}{\alpha_x + \alpha_t} + \frac{\alpha_t u_x - \alpha_x u_t}{\alpha}\Big\} + \\ + \frac{1}{f'}\Big\{\frac{M(\alpha)}{\alpha_x + \alpha_t} - \alpha_t \frac{u_x + u_t}{\alpha}\Big\}. \tag{4}$$

In order for this relation to have no arbitrary element, it is necessary to select a function $\alpha(x,t)$ such that the coefficient for $1/f'$ to be identically equal to zero. This is equivalent to the requirement

$$u_t\Big[\alpha_{xx} + \alpha_{xt} - \frac{\alpha_t}{\alpha}(\alpha_x + \alpha_t)\Big] - u_x\Big[\alpha_{xt} + \alpha_{tt} + \frac{\alpha_t}{\alpha}(\alpha_x + \alpha_t)\Big] = 0.$$

To fulfil the above requirement, we can consider the obtained differential relation as the second order equation with respect to $\alpha(x,t)$, containing derivatives u_x and u_t as parameters. In such a case the function α would depend on these parameters and put us beyond the frames of representation (1). And the main thing is that representation (1) would have not been general, but an intermediate integral of anyone equation. Therefore it is necessary that the obtained relation contain no derivatives u_x, u_t. But this is possible only if these derivatives have zero coefficients. Consequently, it is necessary that

$$(\alpha_x + \alpha_t)_x = \frac{\alpha_t}{\alpha}(\alpha_x + \alpha_t), \quad (\alpha_x + \alpha_t)_t = -\frac{\alpha_t}{\alpha}(\alpha_x + \alpha_t).$$

The second equality immediately yields

$$\alpha_x + \alpha_t = \frac{c(x)}{\alpha},$$

where $c(x)$ is an arbitrary function. Substituting the above expression into the sum of both equalities, we obtain the relation

$$c'(x)\alpha^2(x,t) - c^2(x) = 0.$$

which implies that the function α may depend on the argument x only. Taking into account that $\alpha_t = 0$, the first equality allows us to conclude that $\alpha_{xx} = 0$. Therefore our choice is very restrictive, i.e., α may be a linear function only of the argument x,

$$\alpha = c_0 x + c_1$$

with arbitrary constants c_0, c_1.

Using the linear function α, by means of (4), we come to a class of second order equations:

$$(c_0 x + c_1)L(u) + c_0 u_t(u_x + u_t) = 0 \tag{2}$$

with the corresponding general integral

$$f(u) - (c_0 x + c_1)g(x - t) = x. \tag{5}$$

If we now give up this restriction with respect to the function α, then equality (3) will fail to provide us with the relation containing no one arbitrary function. Since relation (4) allows one to make more wide choice, from the three possible versions we will dwell on it. Rewriting the expression $(-f')^{-1}$ in terms of $F(u)$, we define this arbitrary function from (4) as follows:

$$F(u) = v(x,t),$$

where we have introduced the notation

$$v(x,t) = \Big\{L(u) - (u_x + u_t)\Big[\frac{M(\alpha)}{\alpha_x + \alpha_t} + \frac{\alpha_t u_x - \alpha_x u_t}{\alpha}\Big]\Big\}\times$$
$$\times\Big\{\frac{M(\alpha)}{\alpha_x + \alpha_t} - \alpha_t \frac{u_x + u_t}{\alpha}\Big\}^{-1}.$$

Differentiating the obtained equality with respect to independent variables x and t and excluding subsequently the derivative F' from the obtained two relations $u_x F' = v_x$ and $u_t F' = v_t$, we have

$$u_t v_x - u_x v_t = 0.$$

Substituting now the expression v in the latter equality, we can see that combination (1) with arbitrary functions f and g is the general integral of the equation which is now not of the second, but of the third order. Thus our proposition is proved completely.

The above reasoning shows that in most cases the combination with two arbitrary functions of type (1) will appear to be a general integral of an equation of order more higher than the second one. Only in rare cases it satisfies the equation of second order, i.e., if α is a linear function of argument x. Thus we have faced the phenomenon when a number of arbitrary functions appearing in the general integral does not correspond to the equation order.

§ 2. Some Properties of Equation (2)

We will now proceed to investigate the properties of equation (2). First of all, we note its invariance with respect to the linear transformation $\xi = c_0 x + c_1$, $\eta = c_0 t + c_2$. Relying on this fact, we can, without restriction of generality, instead of equation (2) consider another equation

$$xL(u) + u_t(u_x + u_t) = 0, \tag{6}$$

whose general solution has the form

$$u(x,t) = f\big[xg(x-t)\big], \tag{7}$$

where f, $g \in C^2(R^1)$ are arbitrary functions.

The characteristic roots

$$\lambda_1 = 1, \quad \lambda_2 = -u_x(u_t)^{-1}$$

define two families of characteristic curves. One family, corresponding to the root λ_1, is defined completely by means of the dependencee $x-t = \text{const}$ between the arguments x and t, as it often takes place in the case of linear equations. The nonlinear effect arises on the other family of characteristics, since this family is defined by the relation

$$u(x,t) = \text{const},$$

which is the family of lines of a level of unknown solutions of equation (2). Consequently, it can be defined simultaneously with a solution.

Moreover, the roots λ_1, λ_2 show that characteristic directions may coincide on a set of points defined by the relation

$$u_x + u_t = 0 \tag{8}$$

between derivatives of a solution. Solutions for which equality (8) is everywhere fulfilled, are parabolic ones. For hyperbolic solutions, condition (8) must be violated everywhere.

There naturally exist the solutions for which the above-mentioned equality takes place only on some sets of points. Therefore equation (6) should be referred to a class of mixed type quasilinear, hyperbolic-parabolic equations (see [4–8]).

Besides an admissible parabolic degeneration, equation (6) is, in addition, characterized by an order degeneration which takes place on the ordinate axis and does not depend on behaviour of a solution. Consequently, it can be attached to the well-known Euler-Darboux equations [9] as one of their nonlinear versions. At the same time, equation (6) differs significantly from the Euler-Darboux equation: judging by (7), any its solution on the line $x = 0$ of order degeneracy is constant. Consequently, no matter how the solution is, the line of order degeneracy of equation (6), at the same time, is characteristic of the family corresponding to the root λ_2. This is the only one singular characteristic of the given family which is defined stringently irrespective of values of the solutions.

In addition, the fact of partitioning of the differential operator of second order presented by the left-hand side of equation (6) is it should be pointed out. To this end, we assign the corresponding differential characteristic relations of both families and then, using the Poisson brackets, extend them to the complete Jacobi joint systems. It turns out that each of these systems consist of three equations. This allows us to conclude that characteristic families have two invariants [1].

Relying on the structure of these invariants, we can conclude that equation (6) is equivalent to the following conservation law [10]:

$$\begin{cases} (x+v)_x + (x-v)_t = 0, \\ xu_x - vu_t = 0, \end{cases} \tag{9}$$

whose integration leads naturally to the solution (7).

For hyperbolic equations the Asgeirsson's principle called as the mean value property, is well-known [11]. For example, for the string equation it is formulated simply as follows: the sums of values of a solution at opposite vertices of an arbitrary characteristic quadrangle are equal. Equation (6) has its own nonlinear analogue of that property. It easily follows from the representation of the general solution (7) by taking the values of an arbitrary solution at the vertices of an arbitrarily taken characteristic quadrangle. We denote these vertices by (x_i, t_i), $i = 1, \dots, 4$, where $x_1 - t_1 = x_2 - t_2$ and $x_3 - t_3 = x_4 - t_4$. Since the vertices (x_1, t_1) (x_4, t_4) on the one hand, and the points (x_2, t_2), (x_3, t_3) on the other hand, pairwise lie on one and the same of the characteristic of the family of the root λ_2 the values of the solution of equation (6) in them must be equal:

$$f\big[x_1 g(x_1 - t_1)\big] = f\big[x_4 g(x_4 - t_4)\big],$$
$$f\big[x_2 g(x_2 - t_2)\big] = f\big[x_3 g(x_3 - t_3)\big].$$

Taking into account that the characteristic quadrangle is taken arbitrarily, from the obtained relations we conclude that the corresponding values of the argument of the function f are equal. Thus we have the following relations:

$$x_1 g(x_1 - t_1) = x_4 g(x_4 - t_4),$$
$$x_2 g(x_2 - t_2) = x_3 g(x_3 - t_3).$$

But, on the other hand, the points (x_1, t_1), (x_2, t_2) like the second pair of points $(x_3, t_3)(x_4, t_4)$, lie on one characteristic of the family corresponding to the root λ_1. Therefore $g(x_1 - t_1) = g(x_2 - t_2)$ and $g(x_3 - t_3) = g(x_4 - t_4)$. In view of these equalities, we come to the following concluding proposition:

The products of abscissas of opposite vertices of an arbitrary characteristic quadrangle are equal,

$$x_1 x_3 = x_2 x_4. \tag{10}$$

Just this is the nonlinear analogue of the Asgeirsson's mean value property in the case of equation (6) which in the sequel will be called the property of proportionality of the argument. This simple property combined with another methods simplifies considerably the investigation of problems formulated for equation (6). The structure of the general solution itself, as well as the property (10) can be widely applied for correct statements of a number of nonlinear analogues of the well-known linear characteristic problems. One of the evident examples of such an application is the nonlinear Goursat problem.

§ 3. The Nonlinear Goursat Problem

For linear equations this problem consists in finding a solution by means of its values assigned on the characteristic arcs of both families, coming out of an arbitrarily given point in common.

In a linear case, characteristics of the equation are defined completely by the relations between independent variables. So, it is not difficult to formulate the problem correctly. Complications do not arise for equation (6) either when assigning values of a solution on the characteristic of the family of the root λ_1. This family is given by the relation $x-t=\text{const}$, and from these lines we can choose one, passing through the given point. For the sake of simplicity, we take the line $t=x$ and on its interval $0\le x\le a$ assign values of an disired solution

$$u\big|_{t=x}=\varphi(x),\quad 0\le x\le a, \tag{11}$$

just in the same way as in the linear case. For a solution to be regular, the function is required to be sufficiently smooth, say $\varphi\in C^2[0,a]$.

Suppose that another characteristic arc of the family of the root λ_2 comes out of the end point (a,a) of the above-mentioned interval. There now arises discrepancy with the linear theory. First, this characteristic is not assigned, and it is not clear where values of a solution can be preassigned. Second, if this characteristic is defined, then it is impossible to assign values of a solution arbitrarily because all characteristics of this family are, in fact, the level lines of solutions of equation (6). Consequently, on the characteristic of the family root λ_2, still unknown, the solution $u(x,t)$ is constant and equal to $\varphi(a)$. This deficiency can be compensated in different ways, in particular, by assigning the arc itself of the given characteristic family.

Suppose that the characteristic of the family corresponding to the root λ_2, coming out of the point (a,a), is given explicitly by the equation

$$t=\mu(x),\quad a\le x\le b,\quad \mu(a)=a, \tag{12}$$

has continuous curvature, intersects with straight lines of the family $t-x=\text{const}$ not more than at one point and uniquely projects to the coordinate axes. Thus we arrive at the following statement of the problem: Find a domain of definition of a regular solution of equation (6) and simultaneously a solution $u(x,t)$ itself, if it satisfies condition (11) and along that solution the arc of the curve (12) is characteristic one.

In this manner is formulated one of nonlinear versions of the Goursat characteristic problem which is quite suitable for practical purposes. It is not difficult to solve this problem by using the general solution (7). However, the values of the solution can be defined without its application.

Indeed, let us take the point (x_0,t_0) at which we are to define the value $u(x_0,t_0)$ of the solution $u(x,t)$. To this end, we draw through it the characteristic $t=x-x_0+t_0$ and find a point of its intersection with the characteristic (12), if such point exists. Denote its abscissa by x_1. It can be defined by the relation $\mu(x) = x - x_0 + t_0$, as the equation with respect to the value x. Thus it is necessary to assume the existence of the real inverse to the function $x - \mu(x)$, which we denote by M. The necessary for that aim condition

$$\mu'(x) \neq 1 \tag{13}$$

excludes the parabolic degeneracy of equation (6) on the characteristic arc (12).

Thus the abscissa x_1 is defined by the values x_0, t_0 by means of the formula

$$x_1 = M(x_0 - t_0). \tag{14}$$

Through the point (x_0, t_0) must likewise pass another characteristic of the family of the root λ_2. If it intersects with data carrier (11), then the abscissa of the point of intersection can be defined with the help of the property of proportionality of the argument. It is equal to

$$x^* = \frac{ax_0}{M(x_0 - t_0)}.$$

Since the points (x^*, x^*) and (x_0, t_0) lie on a common characteristic of the family of the root λ_2, the values of the solution in them must be equal. But according to the condition (11), $u(x^*, x^*) = \varphi(x^*)$. Consequently,

$$u(x_0, t_0) = \varphi\Big[\frac{ax_0}{M(x_0 - t_0)}\Big] \tag{15}$$

at all points (x_0, t_0) which lie in the domain bounded by the characteristics $t = x$, $t = x - b + \mu(b)$, $t = \mu(x)$, $x = 0$. The domain of definition of a solution does not contain another points because the characteristics coming out of them do not reach the data carriers of the problem under consideration. This proves the following theorem.

Theorem. *If the curve presented by equation* (12) *has nowhere characteristic direction of the family of the root λ_1, and the function $\mu(x) - x$ has the only one real inverse M, then there exists a unique solution of problem* (6), (11), (12) *which is represented by the formula*

$$u(x, t) = \varphi\Big[\frac{ax}{M(x - t)}\Big] \tag{15}$$

and defined in the domain

$$D = \Big\{(x, t) : x - t \in \big[0, b - \mu(b)\big],\ \frac{x}{M(x - t)} \in [0, 1]\Big\}. \tag{16}$$

We can obtain this result if we subject the general solution (7) of equation (6) to the conditions (11), (12). In our reasoning there is no need to require for the function $\mu(x)$ to be increasing or decreasing. Therefore one can neglect the condition $b > a$. In such a case all characteristic arcs of the root λ_2 can be defined in terms of a one-parametric family. In the capacity of the parameter c it is more convenient to take the abscissa of the point (c,c) of the data carrier (11). Such characteristic is given by the relation $ax = cM(x-t)$ from which it immediately follows that

$$t = x - \frac{ax}{c} + \mu\Big(\frac{ax}{c}\Big),\ x \in \Big[c, \frac{bc}{a}\Big] \tag{17}$$

for every $c \in (0,a]$, while for $c = 0$ we have a segment $t \in [0], \mu(b) - b]$ of the coordinate axis.

The case for $b = 0$, which corresponds to tending of the characteristic (12) to the line of order degeneracy of equation (6), needs special consideration. Under such an assumption, by the conditions of the problem, two characteristics $x = 0$ and $t = \mu(x)$ have a point $(0, \mu(0))$ in common. Moreover, the representation (17) of characteristic arcs of the given family shows that all of them converge to the line $x = 0$ at the same point. Thus we conclude that the point $(0, \mu(0))$ is the node of characteristic arcs of the family corresponding to the root λ_2, and the limiting values of the solution

$$\lim_{\substack{x\to 0\\ t\to\mu(0)}} u(x,t)$$

depend on the path in which the moving point (x,t) tends to the point $(0,\mu(0))$. This very fact describes one of the effects of order degeneracy of equation (6).

It should also be noted that the property of proportionality of the argument allows one to investigate the Goursat problem in the case of parabolic degeneracy of equation (6) on the characteristic curve (12). Here the following theorem holds.

Theorem. *If at some point* $(x_0, \mu(x_0))$ *the curve has direction of the characteristic family of the root* λ_1*, and the conditions of the previous theorem are fulfilled, then equation* (6) *along the characteristic arc*

$$t = x - x_0 + \mu(x_0), \tag{18}$$

lying in the domain D *of definition of the solution* (15)*, degenerates parabolically, and the latter is noncharacteristic, of Tricomi type.*

Indeed, the characteristic of the family the root λ_2, coming out of the point (c,c) and defined by formula (17), has a slope

$$\frac{dt}{dx} = 1 - \frac{a}{c}\Big[1 - \mu'\Big(\frac{ax}{c}\Big)\Big], \quad x \in \Big[c, \frac{bc}{a}\Big].$$

which for $x = x^* = \frac{cx_0}{a}$ is equal to the unity and coincides with the slope of characteristics of the family of the root λ_1. Consequently, the parabolic degeneracy of the curve (12) at the point $(x_0, \mu(x_0))$ generates an analogous degeneration of equation (6) on the characteristic arc (17) at the point (x^*, t^*), where

$$\begin{cases} x^* = \dfrac{cx_0}{a}, \\ t^* = \Big(\dfrac{c}{a} - 1\Big)x_0 + \mu(x_0). \end{cases} \qquad (19)$$

Since the point (c, c) is taken arbitrarily, equation (6) degenerates parabolically on all characteristics coming out of the points of the data carrier (11). Therefore if we take the value c as a parameter with values from the interval $[0, a]$, then we will get an equation of the arc of some curve on which equation (6) degenerates parabolically. Omitting the parameter c from the relations (19), we obtain the straight line (18), which was to be demonstrated.

§ 4. The Nonlocal Characteristic Problem

This problem is likewise a certain nonlinear generalization of the Goursat problem.

Let there be explicitly given some arc γ of a curve defined by the equation

$$t = \mu(x), \quad 0 \le x \le b,$$

where the function $\mu \in C^2[0, b]$ satisfies the conditions

$$\mu(0) = 0, \quad \mu'(x) \ne 0, \quad \mu(x) \ne x; \qquad (20)$$

it lies in the strip

$$x - t \in \big[0, b - \mu(b)\big] \qquad (21)$$

and intersects with the characteristics $x - t = \text{const}$ of that strip not more than once.

Another function $\nu \in C^2[0, b]$ given on the same interval $[0, b]$ maps it onto the interval $0 \le x \le \nu(b)$. This function is assumed to be monotonically increasing, and $\nu(0) = 0$. By means of the function $\nu(x)$ we find that there exists one-to-one correspondence between the points of the arc γ and the characteristic segment $t = x + h$, $0 \le x \le \nu(b)$ which lies outside the strip (21).

The nonlocal characteristic problem: *Find a regular solution $u(x, t)$ of equation* (6) *simultaneously with the domain of its definition, if it satisfies the conditions*

$$u\Big|_{x=t} = \varphi(x), \quad 0 \le x \le a, \qquad (11)$$

$$u\big[\nu(x), \nu(x) + h\big] = u\big[x, \mu(x)\big], \quad 0 \le x \le b. \qquad (22)$$

As is becomes clear, the solvability of that problem depends essentially on the order of vanishing of the function $\nu(x)$ at the point $x = 0$.

First of all, we note that the non-local condition (22) defines completely all characteristics of the family of the root λ_2 in the strip (21). Indeed, let us take arbitrarily the point $(x_0, \mu(x_0))$ on the arc γ and the corresponding point $(\nu(x_0), \nu(x_0) + h)$ on the characteristic $t = x + h$. According to the condition (22), they must lie on the same characteristic of the same family of the root λ_2.

Analogously, we take some value $c \in [0, b]$ and the corresponding point $(\nu(c), \nu(c) + h)$. By our assumption, it lies on the same characteristic of the family of the root λ_2 as the point $(c, \mu(c))$ of the curve γ. This characteristic must intersect the characteristic line $t = x - x_0 + \mu(x_0)$ at some point whose abscissa we denote by x. Thus we have constructed the characteristic quadrangle for which the pairs of values x_0, $\nu(c)$ on the one hand, and x, $\nu(x_0)$ on the other hand, are the abscissas of opposite vertices. Three of these vertices x_0, $\nu(x_0)$ and $\nu(c)$ are given, while the fourth one is defined by means of the above vertices on the basis of the property of proportionality of the argument:

$$x = \frac{x_0}{\nu(x_0)}\,\nu(c).$$

Assume that the function $\nu(c)$ is representable in the form

$$\nu(x) = x^\alpha \nu_0(x), \quad \alpha > 1, \quad \nu_0(0) \neq 0. \tag{23}$$

Taking into account that the point with the abscissa x lies on the characteristic $t - x = \mu(x_0) - x_0$ of the family of the root λ_1, we define both of its coordinates:

$$\begin{cases} x = \dfrac{x_0^{1-\alpha}}{\nu_0(x_0)}\nu(c), \\ t = \dfrac{x_0^{1-\alpha}}{\nu_0(x_0)}\nu(c) - x_0 + \mu(x_0), \quad 0 \leq x_0 \leq b \end{cases}. \tag{24}$$

If now we take the value x_0 in the capacity of the parameter which passes through all values of the interval $[0, b]$, we will get all points of the arc of the characteristic of the family of the root λ_2 lying in the strip (21) and passing through the point $(c, \mu(c))$ of the arc γ. Indeed, if in (24) we take $x = c$, then the first equality by virtue of (23) will imply $\frac{\nu(x_0)}{x_0} = \frac{\nu(c)}{c}$. Therefore under the assumption that the function $\frac{\nu(x)}{x}$ is monotone, we find that $x_0 = c$, and then the second relation in (24) yields $t = \mu(c)$. This implies that the curve represented parametrically by formulas (24) intersects with the curve γ at the point $(c, \mu(c))$. It should merely be noted that the characteristic arc (24) is defined only within the strip (21). Although, as is known, it passes through the point $(\nu(c), \nu(c) + h)$ of the characteristic $t = x + h$ which lies outside the strip (21); note that h can be taken arbitrarily.

If we take c in terms of the parameter with the values from the interval $[0, b]$, we will obtain representations of all arcs of characteristics of the family of the root λ_2, passing through the points of the curve γ. To obtain an explicit equation from the parametric representation (24) of the characteristic arc, it is necessary to omit the parameter x_0. According to the above assumption, the relation

$$\frac{\nu(x)}{x} = x^{\alpha-1}\nu_0(x) = \zeta, \tag{25}$$

being the equation with respect to the value x, has the unique real solution which we denote by

$$x = N(\zeta), \quad \zeta \in \big[\lim_{x\to 0} x^{\alpha-1}\nu_0(x), b^{\alpha-1}\nu_0(b)\big]. \tag{26}$$

Then the one-parametric family of characteristics of the root λ_2 will be represented as follows:

$$t = x - N\Big[\frac{\nu(c)}{x}\Big) + \mu\Big[N\Big(\frac{\nu(c)}{x}\Big)\Big], \tag{27}$$

with the parameter c having the values from the interval $[0, b]$.

As for the interval of changing the variable x, it can be defined in different ways, depending on the exponent α appearing in (23). We have

$$x \in \Big[0, b^{1-\alpha}\frac{\nu(c)}{\nu_0(a)}\Big] \ \text{ for } \ 0 < \alpha < 1,$$
$$x \in \Big[\frac{\nu(c)}{\nu_0(0)}, \frac{\nu(c)}{\nu_0(b)}\Big] \ \text{ for } \ \alpha = 1,$$
$$x \in \Big[\frac{\nu(c)}{b^{\alpha-1}\nu_0(b)}, \infty\Big] \ \text{ for } \ \alpha < 1.$$

In the first case, for $\alpha > 1$, by the relations (24), irrespective of the value of the parameter $c \in [0, b]$, we have $x = t = 0$ as $x_0 \to 0$. Consequently, all characteristic arcs of the family of the root λ_2 converge at the origin. Moreover, the slope of these characteristics coincides at the origin with that of the characteristic $t = x$ of another family. This implies that at the given point there takes place simultaneous degeneration of an order of equation (6) as well as of its type, i.e., hyperbolic equation (6) at the given point $(0, 0)$ degenerates parabolically, and this degeneracy is characteristic one. Indeed, differentiating the values x, t with respect to the argument x_0 and then constructing the derivative

$$\frac{dt}{dx} = 1 - x_0^{\alpha}\frac{1 - \mu'(x_0)}{(1-\alpha)\nu_0(x_0) - x_0\nu'(x_0)} \cdot \frac{\nu_0^2(x_0)}{\nu(c)}, \tag{28}$$

we can easily see that

$$\lim_{x_0\to 0}\frac{dt}{dx} = 1, \quad \forall c \in [0, b].$$

This allows us to conclude that the origin of coordinates for one-parametric family (27) is a node. Therefore the limiting value of the solution $u(x, t)$ at that point depends on a path in which the point (x, t) moves to the point $(0, 0)$. But here another circumstance is much more sufficient: namely, none of characteristics (27), except the origin, intersects the data carrier (11). Therefore these data do not extend outside the characteristic $t = x$. Consequently, in the given case the problem is unsolvable.

Under the assumption $\alpha > 1$ we have an analogous conclusion. As is seen from the parametric representation (24) of the characteristic arcs (27), the both coordinates x, t grow infinitely as $x_0 \to 0$. The straight line $t = x$ is an asymptote of each of these characteristics, with the exclusion of the curve which corresponds to the value of the parameter $c = 0$. For every specific value of the parameter $c \in (0, b]$ the curve (27) takes its origin at the point $(\frac{b^{1-\alpha}}{\nu_0(b)}\nu(c), \frac{b^{1-\alpha}}{\nu_0(b)}\nu(c) - b + \mu(b))$, passes through the point $(c, \mu(c))$ of the curve γ and then tends to infinity. None of the curves (27) has general points with the data carrier (11), and hence the problem (6), (11), (22) has no solution.

It remains to consider the case $\alpha = 1$ when the value x defined by the parameter x_0 by formula (24) passes over a finite interval. In our assumptions made with respect to the functions $\mu(x)$ and $\nu(x)$, the curves of the family (24) are mutually disjoint, have general point with the data carrier (11)

$$x = t = \frac{\nu(c)}{\nu_0(0)},$$

pass through the point $(c, \mu(c))$ of the curve γ and finish at the point

$$\Big(\frac{\nu(c)}{\nu_0(b)}, \frac{\nu(c)}{\nu_0(b)} - b + \mu(b)\Big)$$

of the characteristic $t = x - b + \mu(b)$ of the family of the root λ_1 coming out of the endpoint $(b, \mu(b))$ of the curve γ.

Moreover, as the relation (28) shows, none of the characteristics (24) has direction of characteristics of the family of the root λ_1. This implies that in the strip (21), in which are located all characteristic arcs (27), equation (6) has no parabolic degeneration.

Summarizing all the above said, we can state that *if the function* μ, $\nu \in C^2(0, b]$ *satisfies the conditions*

$$\begin{gathered}\mu'(x) \neq 1, \ \ \mu'(x) \neq 0, \ \ \nu(x) = x^\alpha \nu_0(x), \ \ \alpha > 0, \ \ \nu_0(x) \neq 0, \\ \big[x^{\alpha-1}\nu_0(x)\big]' \neq 0, \end{gathered} \tag{29}$$

then for $\alpha \neq 1$ *the problem* (6), (11), (12) *has no solution, while for* $\alpha = 1$ *it can be solvable.*

The second part of our statement is based on the reduction of the problem under consideration to the previous non-linear Goursat problem. Indeed, if from the family of characteristics (27) we first select one characteristic with the value $c = b$, passing through the point $(a, \mu(a))$ of the curve γ, and then consider it together with the condition (11) for $a = \nu(b)\nu_0^{-1}(0)$, we will get the statement of the problem (6), (11), (12). Combining now the sufficient conditions for its solvability and the conditions of the above statement, we conclude that the following theorem is valid.

Theorem. *If in a half-strip* $0 \le x-t \le h$, $x > 0$ *there exists a solution of equation* (6)*; the conditions* (29) *are fulfilled;* $\alpha = 1$*; the function* $x - \mu(x)$ *has the only one real inverse; the equation* $\nu_0(x) = \xi$ *has a unique real solution which is normalized by the conditions* $\nu(0) = 0$, $\nu \in [0, \nu(b)]$ *then for* $a = \nu(b)\nu_0^{-1}(0)$ *the problem* (6), (11), (12) *has a solution even in a domain lying in the strip* (21), *bounded by the characteristics* $x = 0$, $t = x - N(\frac{\nu(b)}{x}) + \mu[N(\frac{\nu(b)}{x})]$ *and this solution is unique.*

As for the case $\alpha \neq 1$, the problem will become completely solvable if the condition (11) is assigned not on the segment of the straight line $t = x$, but on any other characteristic $t = x + \delta$ lying in the strip (21).

In all non-local problems under consideration the most important fact is that the domain of definition of their solution does not involve the characteristic $t = x + h$, although the condition (22) defines completely characteristic arcs of the family of the root λ_2. A noteworthy is also the fact that remoteness of the interval of that characteristic from the strip (21) does not affect the solvability of the problem.

§ 5. The Nonlinear Goursat Problem With a Free Characteristic

The problems with free boundaries belong to the boundary value problems in which one part of the boundary is known and the other one is free. We assign only a general type of free boundaries; it is required to define their exact form. This is one of the component parts of the problem. This type of problems have been considered as far as in [12], when developing the hodograph method for subsonic gas flows. For the first time, the connection both of the theory of transonic flows and of the problems with free boundaries with the Tricomi problem formulated for mixed type equation [4] has been shown in [13].

For equation (6), the nonlinear Goursat problem with a free characteristic is formulated as follows: simultaneously with its domain of definition, find a regular solution of equation (6) if it satisfies the condition (11) and

$$\big(\alpha u_x + \beta u_t\big)\Big|_{\gamma} = \nu(x), \quad a \le x \le b, \tag{30}$$

where $\nu(x)$ is the twice continuously differentiable function given on the interval $[a, b]$, and γ is the arc of a free characteristic of the family of the root λ_2, coming out of the endpoint (a, a) of the data carrier (11).

α and β are arbitrary constants for which the only one condition

$$\alpha \neq \beta, \tag{31}$$

is required.

The purpose of such a restriction lies in the following: on all characteristics of the family of the root λ_2 not only solutions of equation (6), but also the expression $x(u_x + u_t)$ containing the first derivatives u_x, u_t of the solution $u(x,t)$ are constant. This combination for equation (6) is the characteristic invariant of the family under consideration. Therefore in the right-hand side of the condition (30) it is necessary to take in the capacity of the function $\nu(x)$ for $\alpha = \beta$ the expression inversely proportional to the argument x.

We begin our investigation of the problem (6), (11), (30) with subjecting the general solution (7) to the condition (11). As a result we define an arbitrary function f and for a solution we obtain

$$u(x,t) = \varphi\Big\{\frac{x}{g(0)}\, g(x-t)\Big\}, \tag{32}$$

where the argument of the function φ must run through the values from the interval $[0,a]$. On an unknown characteristic γ with the condition (30) the solution $u(x,t)$ is constant and equal to $\varphi(a)$. Under the requirement for the function φ to be strictly monotone this value will no longer be repeated.

Assume that the free characteristic is represented explicitly in the form $t = \mu(x)$, where $\mu(x)$ is to be defined.

Using now formula (32), we construct a combination of first order derivatives appearing in the left-hand side of the condition (30) and then take its value on the arc γ of an unknown characteristic. We obtain

$$\frac{\varphi'(a)}{g(0)}\Big\{\alpha g\big(x-\mu(x)\big) + (\alpha-\beta)xg'\big(x-\mu(x)\big)\Big\} = \nu(x). \tag{33}$$

But taking into account that

$$u\big|\gamma = \varphi\Big\{\frac{x}{g(0)} g\big(x-\mu(x)\big)\Big\} = \varphi(a),$$

we get

$$g\big[x-\mu(x)\big] = \frac{ag(0)}{x}.$$

Substituting the obtained expression into equation (33), we define the derivative

$$g'\big[x-\mu(x)\big] = \frac{g(0)}{\alpha-\beta}\Big\{\frac{\nu(x)}{\varphi'(a)x} - \frac{\alpha a}{x^2}\Big\}.$$

Integration of the latter relation yields

$$g\big[x-\mu(x)\big] = g(0) + \frac{g(0)}{\alpha-\beta}\int\limits_a^x \Big\{\frac{\nu(z)}{\varphi'(a)z} - \frac{\alpha a}{z^2}\Big\}\big[1-\mu'(z)\big]\,dz.$$

As is seen, one and the same function $g[x-\mu(x)]$ is defined by means of two different ways. If we first equalize them and then differentiate we will obtain the following relation:

$$\big(1-\mu'(x)\big)\Big[-\frac{\alpha a}{x^2}+\frac{\nu(x)}{\varphi'(a)x}\Big]=-\frac{\alpha-\beta}{x^2}a,$$

from which by integration we can easily define the unknown function $\mu(x)$,

$$\mu(x)=\mu(a)+\int\limits_a^x\Big\{1+\frac{(\alpha-\beta)a\varphi'(a)}{z\nu(z)-\alpha a\varphi'(a)}\Big\}\,dz.$$

.

Taking into account that $\mu(a)=a$, we finally obtain

$$\mu(x)=x+(\alpha-\beta)\int\limits_a^x\Big\{\frac{z\nu(z)}{a\varphi'(a)}-\alpha\Big\}^{-1}dz. \tag{34}$$

Thus the condition (30) allows us to define the unknown arc γ of the characteristic of the family of the root λ_2, coming out of the endpoint (a,a) of the data carrier (11). Consequently, the problem under consideration is reduced to the Goursat problem (6), (11), (12). It now remains to rephrase the conditions for the solvability in terms of the function $\nu(x)$ appearing in the condition (30).

References

1. E. Goursat, Leçons sur l'intégration des équations aux dérivées partielles du second ordre, Tome II. *Hermann, Paris,* **2** 1898.
2. J. H'Adamard, Leçons sur la propagation des ordes et les équations de l'hydrodinamique. *Hermann, Paris,* 1903.
3. J. Gvazava, Nonlocal and initial problems for quasilinear nonstrictly hyperbolic equations with general solutions represented by superposition of arbitrary functions. *Georgian Math. J.* **10**(2003), No. 4, 687-707.
4. F. Tricomi, Sulle equationi alle derivate parziali di 2^0 ordine di tipo misto. *Memorie della Academia Nazionale dei Lincei, serie V,* **14**(1923), No. 7. *Russian transl.: OGIZ, Moscow–Leningrad,* 1947.
5. S. Gellerstedt, Sur un probleme aux limites pour une equation lineaire aux derivees partielles du second ordre de type mixte. *Almqvist-Wiksels doktr. Uppsala,* 1935.
6. A. Bitsadze, Some classes of partial differential equations. (Russian) *Nauka, Moscow,* 1981.
7. M. Cibrario, Primi studii intorno alle equazioni lineari alle derivate parziali del secondo ordine di tipo misto iperbolo-paraboliche. *Rendiconti del Circolo Matematico di Palermo* **56**(1932), 385–418.
8. E. Holmgren, Sur un probleme aux limites pour l'equation $y^m u_{xx}+u_{yy}=0$. *Arkiv för Matematik, Astronomi och Fysik* **14**(1927), 1–3.
9. G. Darboux, Lecons sur la theorie des surfaces generale et les applications geometriques du calcus infinitesimal. *Gauthier-Villars, Paris,* 1894.

10. D. Serre, Le complicate par compensation pour les systemes hyperboliques non lineares de deux equations á une dimension d'espace. *J. Math. Pures Appl.* **65**(1986), 423–468.
11. L. Asgeirsson, Über eine Mittelwerteigenschaft vor Lösungen homogener linearer partieller Differentialgleichungen zweizer Ordinung. *Math. Ann.* **113**(1937), 321–346.
12. S. Chaplygin, On gas streams. (Russian) *Complete collection of works, Izd. AN SSSR, vol.* 2*, Leningrad*, 1933.
13. F. Frankl, On the Cauchy problem for mixed type elliptic-hyperbolic equations with initial data on transient curve. (Russian) *Izv. Akad Nauk SSSR, ser. Mathematics* **8**(1944), 195–224.

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