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ON ONE NONLINEAR VERSION OF CHARACTERISTIC PROBLEM WITH A FREE SUPPORT OF DATA

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Abstract. For quasi-linear, second-order, non-strictly hyperbolic equations, the problem with an oblique derivative prescribed on unknown characteristics is considered. Using complete systems of characteristic invariants, the integral of the problem and its domain of definition are constructed explicitly.

რეზიუმე. მეორე რიგის არამკაცრად ჰიპერბოლური კვაზიწრფივი განტოლებებისათვის განხილულია დახრილწარმოებულებიანი ამოცანა უცნობ მახასიათებლებზე მოცემული პირობებით. მახასიათებელი ინვარიანტების სრული სისტემების საფუძველზე ცხადი სახით არის წარმოდგენილი ამოცანის ინტეგრალი და მისი განსაზღვრის არე.

Statement of the Problem

On a plane of variables x, t we consider second order the equation of with real characteristics

$$\mathcal{L}(u) = Au_{xx} + Bu_{xt} + Cu_{tt} = F, \tag{0.1}$$

where the coefficients and the right-hand side are the given, sufficiently smooth functions which may depend on the arguments x, t, on a solution $u(x,t)$ and its first order derivatives $u_x(x,t)$ and $u_t(x,t)$. According to our assumption, through every point of the domain of representation of the equation passes only by one characteristic from two different families. In one case, these characteristics can be defined completely, for example, if the principal part of the given equation is linear. In the other case, these curves may depend on the values of an unknown solution u, and hence be undefined.

Let (x_0,t_0) be an arbitrary point of the domain of representation of the equation. Characteristics of different families passing through this point we denote by Γ and Δ. Note that the directions of the curves at the point (x_0,t_0) may coincide.

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Consider the following problem: simultaneously with the domain of its definition, find a regular solution of equation (0.1) which satisfies the conditions:

$$u(x_0, t_0) = u_0 \tag{0.2}$$

$$(\alpha_1 u_x + \beta_1 u_t)|_\Gamma = \gamma, \tag{0.3}$$

$$(\alpha_2 u_x + \beta_2 u_t)|_\Delta = \delta, \tag{0.4}$$

where u_0 is the given number. Coefficients and the right-hand sides in the conditions (0.3), (0.4) depend on the argument in which terms it is more convenient to represent characteristic curves Γ and Δ. Below, this aspect will be considered in detail.

Consider the problem (0.1)-(0.4) for three different cases, when:

a) both characteristics Γ and Δ are defined;

b) one of the characteristics is defined and the other depends on the values of an unknown solution;

c) both characteristics are unknown.

Thus step by step we will try, at least partially, to clear up to what extent and how the nonlinearity of equation (0.1) affects the statement and solvability of the problem under consideration.

1. The Linear Case

For the time being, we will restrict ourselves to the consideration of the simplest linear equation

$$\Box\, u \equiv u_{xx} - u_{tt} = 0, \tag{1.1}$$

for which the characteristics $\Gamma = \{(x,t) : t = x - x_0 + t_0, x \in [x_0, x_1]\}$ and $\Delta = \{(x,t) : t = -x + x_0 + t_0, x \in [x_0, x_2]\}$ are completely defined. Therefore we have no undefined components in the statement of the problem. Thus using three particular conditions

$$u(x_0, t_0) = u_0 \tag{1.2}$$

$$(\alpha_1(x) u_x + \beta_1(x) u_t)|_\Gamma = \gamma(x), \quad x \in [x_0, x_1] \tag{1.3}$$

$$(\alpha_2(x) u_x + \beta_2(x) u_t)|_\Delta = \delta(x), \quad x \in [x_0, x_1] \tag{1.4}$$

we have to construct a solution of equation (1.1) and find a domain of its definition. Parameters of the conditions (1.3), (1.4) are required to be continuously differentiable.

It is not difficult to see that the problem (1.1)-(1.4) is well-posed. Without restriction of generality, we assume that $x_0 = t_0 = u_0 = 0$. As it becomes clear, the conditions (1.2), (1.3), (1.4) specify the values $u(x,t)$, $u_x(x,t)$ and $u_t(x,t)$ first at the point (x_0, t_0) and then on the characteristics

Γ and Δ. Towards this end, the functions $\alpha_i(x)$ and $\beta_i(x)$ are required to have supplementary conditions. For example, the condition

$$\alpha_1(0)\beta_2(0) \neq \alpha_2(0)\beta_1(0) \tag{1.5}$$

allows us to obtain values of the first order derivatives u_x and u_t at the point $(x_0 = 0, t_0 = 0)$:

$$u_x(0,0) = \frac{\gamma\beta_2 - \delta\beta_1}{\alpha_1\beta_2 - \alpha_2\beta_1}\Big|_{x=0} \equiv p_0, \quad u_t(0,0) = \frac{\delta\alpha_1 - \gamma\alpha_2}{\alpha_1\beta_2 - \alpha_2\beta_1}\Big|_{x=0} \equiv q_0.$$

To determine values of these derivatives along the characteristics Γ and Δ, we will, naturally, need the conditions (1.3), (1.4). But these conditions are not sufficient, hence they should be supplemented with relations and conditions inherent in all solutions of equation (1.1).

In particular, along every characteristic of the family $t = x + \text{const}$ to which the curve Γ belongs, the difference $u_x - u_t$ of the the first order derivatives for any solution is constant. This difference is one of the characteristic invariants on the basis of which we construct a general solution of equation (1.1). Thus we find that along the curve Γ, the difference $u_x - u_t$ of the derivatives retains the same value as it has at the point $(0,0)$.

Consequently, we have

$$u_x(x,x) - u_t(x,x) = \theta_1,$$

where the constant

$$\theta_1 = p_0 - q_0 = \frac{\gamma(\beta_2 + \alpha_2) - \delta(\beta_1 - \alpha_1)}{\alpha_1\beta_2 - \alpha_2\beta_1}\Big|_{x=0}.$$

If we assume that

$$\alpha_1(x) + \beta_1(x) \neq 0, \quad x \in [x_0, x_1], \tag{1.6}$$

then on the characteristic Γ the condition (1.3) and the above-obtained equality guarantee values of the derivatives

$$u_x(x,x) = \frac{\gamma(x) + \theta_1\beta_1(x)}{\alpha_1(x) + \beta_1(x)}, \quad u_t(x,x) = \frac{\gamma(x) - \theta_1\alpha_1(x)}{\alpha_1(x) + \beta_1(x)}.$$

The inequality

$$\alpha_2(x) - \beta_2(x) \neq 0, \quad x \in [x_0, x_2] \tag{1.7}$$

similar to the condition (1.6), guarantee values of the derivatives u_x, u_t on the characteristic Δ:

$$u_x(x,-x) = \frac{\delta(x) - \theta_2\beta_2(x)}{\alpha_2(x) - \beta_2(x)}, \quad u_t(x,-x) = \frac{\theta_2\alpha_2(x) - \delta(x)}{\alpha_2(x) - \beta_2(x)},$$

where

$$\theta_2 = \frac{\gamma(\beta_2 - \alpha_2) - \delta(\beta_1 - \alpha_1)}{\alpha_1\beta_2 - \alpha_2\beta_1}\Big|_{x=0} = p_0 + q_0.$$

This follows from the characteristic invariant $u_x + u_t$. Using the values of the derivatives u_x, u_t and the condition (1.2), we can define uniquely the values of the unknown solution on the characteristics Γ and Δ:

$$u|_\Gamma = u(x,x) = \int_0^x \frac{2\gamma(z) + \theta_1[\beta_1(z) - \alpha_1(z)]}{\alpha_1(z) + \beta_1(z)} dz \equiv \varphi(x), \tag{1.8}$$

for $x \in [x_0, x_1]$, and

$$u|_\Delta = u(x,-x) = \int_0^x \frac{2\delta(z) - \theta_2[\beta_2(z) + \alpha_2(z)]}{\alpha_2(z) - \beta_2(z)} dz \equiv \psi(x), \tag{1.9}$$

for $x \in [x_0, x_2]$.

Relations (1.8) and (1.9) allow one to determine uniquely the solution of the Goursat problem in the rectangle

$$R = \{\, (x,t) : x + t \in [0, 2x_1], \quad x - t \in [0, 2x_2 \,\}.$$

The solution itself has the form

$$u(x,t) = \int_0^{\frac{x+t}{2}} \frac{2\gamma(z) + \theta_1[\beta_1(z) - \alpha_1(z)]}{\alpha_1(z) + \beta_1(z)} dz + \\ + \int_0^{\frac{x-t}{2}} \frac{2\delta(z) - \theta_2[\beta_2(z) + \alpha_2(z)]}{\alpha_2(z) - \beta_2(z)} dz. \tag{1.10}$$

As is seen, (1.5), (1.6) and (1.7) are the conditions for the unique solvability of the above-formulated problem.

We will follow this scheme in considering the problem (0.2), (0.3), (0.4) for the equation whose one family is given just as in the case of equation (1.1). Another family of characteristics is unknown.

2. The Case of Equation with One Unknown Characteristic

Such kind of equations exceeds, naturally, the limits of linear equations. One of the simplest equations of that class has the form

$$(1 + u_t)u_{xx} + (1 + u_t - u_x)u_{xt} - u_x u_{tt} = 0. \tag{2.1}$$

The roots

$$\lambda = 1, \quad \mu = -u_x(1 + u_t)^{-1}$$

of the characteristic equation corresponding to (2.1) specify two families of characteristics. The family, corresponding to the characteristic root λ is defined by two combinations of independent variables and first order derivatives of an unknown solution

$$\xi = t - x, \quad \xi_1 = \frac{u_x}{1 + u_t}. \tag{2.2}$$

Combinations ξ, ξ_1 called as characteristic invariants, are constant along every curve of characteristic family under consideration. As is seen from the structure of the invariant ξ, this family of characteristics is represented by the straight lines $t - x = \text{const}$.

In this respect, equations (2.1) and (1.1) are characterized equally. The only difference is that equation (2.1) has instead of $u_t - u_x$ the invariant ξ_1.

An essential difference between equations (2.1) and (1.1) becomes apparent from the invariants of the characteristic family defined by the root μ:

$$\eta = u + t, \quad \eta_1 = u_x + u_t. \tag{2.3}$$

As is seen, both invariants η, η_1 depend either on the values of an unknown solution, or on its first order derivatives. Consequently, in this case, direct interconnection between independent variables x, t which define the given characteristic family does not take place. Therefore the characteristic family of the root μ is not defined yet. This is one of the nonlinear effects of equation (2.1) which in fact determines a general solution of the equation

$$u(x,t) = -t + f[x + g(x - t)], \tag{2.4}$$

where f and g are arbitrary, twice continuously differentiable functions ([1]).

Moreover, it should be noted that the type of equation (2.1) depends also on the solution. The coincidence of the values of characteristic roots λ, μ expressed by the relation $u_t + 1 = u_x$ determines parabolic degeneration of equation (2.1). Consequently, the class of hyperbolic solutions is defined by the inequality

$$u_x + u_t + 1 \neq 0. \tag{2.5}$$

Thus equation (2.1), unlike equation (1.1), is not strictly hyperbolic and it can be attributed to the class of parabolically degenerating hyperbolic equations (see [2], [3], [4]).

Information we have presented here is sufficient enough to proceed to considering the problem (0.2-4) for equation (2.1).

First of all, it should be noted that equation (2.1) is invariant with respect to the parallel displacement and does not explicitly contain a solution $u(x,t)$. Therefore again, without restriction of generality, we assume that $x_0 = t_0 = u_0 = 0$.

The support Γ of the condition (0.3) will be assumed to be the segment of the characteristic family, corresponding to the root $\lambda = 1$:

$$\Gamma = \{(x,t) : t = x, \quad x \in [0, x_1]\},$$

where x_1 is an arbitrarily given number. Since the segment Γ is defined by the relation in which the value t is written in terms of the argument x, the coefficients and the right-hand sides of the conditions (0.3), (0.4) are assumed to be the functions of the same argument. If the condition (1.5) is fulfilled at the origin of coordinates, which at the same time is the common point of characteristic support of data, then we will have at hand the values $p_0 = u_x(0,0)$, $q_0 = u_t(0,0)$. This can be realized in a complete analogy with the previous case.

When determining the derivatives u_x and u_t along Γ there may take place insignificant discrepancy. The values of the invariant ξ_1 on Γ and at the point $(0,0)$ are the same. Therefore

$$[u_x(1+u_t)^{-1}]\big|_\Gamma = \xi_0,$$

where the constant $\xi_0 = p_0(1+q_0)^{-1}$. Thus we have the relation

$$u_x(x,x) - \xi_0 u_t(x,x) = \xi_0, \quad x \in [0,x_1],$$

which we consider together with the condition (0.3) in the capacity of a system for finding derivatives u_x and u_t on the segment Γ. The condition

$$\beta_1(x) + \xi_0\alpha_1(x) \neq 0, \quad x \in [0,x_1] \tag{2.6}$$

ensures the solvability of that system, and we calculate the values u_x and u_t:

$$u_x(x,x) = \xi_0 \frac{\beta_1(x)+\gamma(x)}{\beta_1(x)+\xi_0\alpha_1(x)},$$
$$u_t(x,x) = \frac{\gamma(x)-\xi_0\alpha_1(x)}{\beta_1(x)+\xi_0\alpha_1(x)}.$$

By virtue of the values obtained above, we find that

$$u(x,x) = \int\limits_0^x \frac{(1+\xi_0)\gamma(z)+\xi_0(\beta_1(z)-\alpha_1(z))}{\beta_1(z)+\xi_0\alpha_1(z)}\,dz \equiv$$
$$\equiv \varphi(x), \quad x \in [0,x_1] \tag{2.7}$$

Thus we have found all possible values connected with the characteristic segment Γ, and we achieved this by two simple conditions, one of which (1.5) is point wise, and the other one (2.6) is given on the segment $[0,x_1]$.

Let us now find out whether it is possible to calculate all these values on another characteristic, when the domain of representation of the condition (0.4) is unknown. Here we deal with the version of the problem with a free boundary. Such kind of problems take place in the theory of transonic flows in aerohydrodynamics (see [5], [6], [7]). In these problems, a free portion of the boundary should be defined simultaneously with a solution. Instead of a missing portion of the boundary it is necessary to assign supplementary boundary conditions. However, our case does not involve such conditions. Thus we have to find out whether the condition (0.4) and invariants (2.3) are sufficient for determination of an exact form of the characteristic arc Δ.

Although the arc Δ is unknown, the invariant $\xi = u + t$ is constant along the whole characteristic. The origin of the coordinates lies on Δ, and therefore $(u+t)|_{\Delta}=0$. This fact allows us to conclude that

$$u|_{\Delta}=-t|_{\Delta}, \quad x\in[0,x_2]. \tag{2.8}$$

Thus it is obvious that if we find an explicit representation of the curve Δ, then we will be able to find all values of the unknown solution, and vice versa. But neither the first, nor the second is known yet. As for the first order derivatives u_x and u_t on that arc, from the representation of the invariant η_1 we can conclude that

$$(u_x+u_t)|_\Delta=p_0+q_0\equiv\eta_0.$$

It is this relation that should be considered together with the condition (0.4).

Assuming that for the above system the condition

$$\alpha_2(x)\neq\beta_2(x), \quad x\in[0,x_2], \tag{2.9}$$

is fulfilled, we define uniquely

$$u_x|_\Delta=\frac{\eta_0\beta_0(x)-\delta(x)}{\beta_2(x)-\alpha_2(x)}, \quad u_t|_\Delta=\frac{\delta(x)-\eta_0\alpha_2(x)}{\beta_2(x)-\alpha_2(x)}.$$

Substituting the obtained on the arc Δ values of derivatives into the expression of characteristic root μ, we can find direction of that arc at every point. Hence for the function representing explicitly the arc Δ, we have the first order ordinary differential equation

$$\frac{dt}{dx}=\frac{\eta_0\beta_0(x)-\delta(x)}{(\eta_0+1)\alpha_2(x)-\delta(x)-\beta_2(x)}, \quad x\in[0,x_2]. \tag{2.10}$$

which should be supplemented with the initial condition $t|_{x=0}=0$. The solution of the Cauchy problem

$$t=\int\limits_0^x\frac{\eta_0\beta_2(z)-\delta(z)}{(\eta_0+1)\alpha_2(z)-\delta(z)-\beta_2(z)}\,dz=\psi(x), \quad x\in[0,x_2] \tag{2.11}$$

provides us with the explicit representation of the characteristic arc Δ. It is clear that for the solution to be regular and to exist, it is necessary to have some conditions with respect to the right-hand side of the relation (2.10). Towards this end, we will require a sufficiently strict condition

$$\delta(x)+\beta_2(x)\neq(\eta_0+1)\alpha_2(x) \tag{2.12}$$

with respect to the parameters of the condition (0.4). Thus the validity of the following proposition is stated.

If the conditions (1.5), (2.9) and (2.12) are fulfilled, then there exists the characteristic arc Δ of regular curvature representable explicitly by the formula (2.11).

As it can be seen from the above reasoning that the above-mentioned conditions are not only enough for determining the characteristic arc Δ, but also for finding derivatives u_x and u_t of the unknown solution and hence the values of the solution itself along Δ according to formula (2.8).

Theorem. *If the conditions* (1.5), (2.6), (2.9) *and* (2.12) *are satisfied, the problem* (2.1), (0.2), (0.3), (0.4) *is equivalent to the following Goursat problem: Simultaneously with its domain of definition, find a regular solution* $u(x,t)$ *of equation* (2.1) *satisfying the characteristic conditions*

$$u|_{t=x} = \varphi(x), \quad x \in [0, x_1] \tag{2.13}$$

$$u|_{t=\psi(x)} = -\psi(x), \quad x \in [0, x_2] \tag{2.14}$$

where the functions φ *and* ψ *are defined by formulas* (2.7) *and* (2.11), *respectively.*

Let us now proceed to constructing a solution of the problem (2.1), (2.13), (2.14). To this end, we use the representation (2.4) of the general solution of equation (2.1) and subject it first to the condition (2.13),

$$u|_{t=x} = -x + f[x + g(0)] = \varphi(x),$$

whence

$$f(\zeta) = \varphi(\zeta - g(0)) + \zeta - g(0).$$

Taking into account the type of the function φ defined by formula (2.7), the last relation can be rewritten as follows:

$$f(\zeta) = (1 + \xi_0) \int\limits_0^{\zeta - g(0)} \frac{\beta_1(z) + \gamma(z)}{\beta_1(z) + \xi_0 \alpha_1(z)}\, dz.$$

As is seen from the above formula, the function f can be defined for $\xi_0 \neq -1$. In the opposite case this means that the value of the characteristic root μ at the origin of the coordinates is equal to unity and coincides with that of the root λ. Thus for $\xi_0 = -1$, equation (2.1) parabolically degenerates at the common point of characteristic arcs Γ and Δ, but for the present such a degeneration of equation (2.1) will be excluded.

Substituting the function f in the general solution (2.4),

$$u(x,t) = -t + \int\limits_0^{x + g(x-t) - g(0)} (1 + \xi_0) \frac{\beta_1(z) + \gamma(z)}{\beta_1(z) + \xi_0 \alpha_1(z)}\, dz,$$

the condition (2.14) is satisfied. Since (2.8) holds on the characteristic arc Δ, we have

$$\int\limits_0^{x + g[x - \psi(x)] - g(0)} (1 + \xi_0) \frac{\beta_1(z) + \gamma(z)}{\beta_1(z) + \xi_0 \alpha_1(z)}\, dz = 0,$$

for all $x \in [0, x_2]$. But this is possible in two cases: either the integrand or the upper limit of integration are identically equal to zero. The first assumption transforms the condition (0.3) into the relation $\xi_1|_\Gamma = \xi_0$, and hence it can be neglected. The second assumption

$$x + g[x - \psi(x)] - g(0) \equiv 0,$$

should be identically fulfilled on the whole segment $[0, x_2]$. With regard for the formula (2.11), we rewrite it in the form

$$g\Big\{(1+\eta_0)\int\limits_0^x \frac{\alpha_2(z)-\beta_2(z)}{(1+\eta_0)\alpha_2(z)-\delta(z)-\beta_2(z)}\Big\} = g(0) - x.$$

The problem of finding an arbitrary function g is closely connected with the solvability of the functional equation

$$\int\limits_0^x \frac{\alpha_2(z)-\beta_2(z)}{(1+\eta_0)\alpha_2(z)-\delta(z)-\beta_2(z)}\,dz = \zeta \tag{2.15}$$

with respect to the value x as the function of the argument ζ. The derivative of that function with respect to x is bounded and different from zero, according to the conditions (2.12) and (2.9). Consequently, the integrand function does not change its sign everywhere on $[0, x_2]$. Therefore from $\zeta = 0$ it immediately follows that $x = 0$. Formally, all the conditions of the theorem on the implicit function are satisfied. But in such a way we will arrive only at a local result. To achieve the result in the whole, we assume that there exists the unique branch, inverse to the left-hand side of the relation (2.15),

$$x = G(\zeta), \quad G(0) = 0.$$

Then the function g will be defined,

$$g(\zeta) = g(0) - G\Big(\frac{\zeta}{1+\eta_0}\Big), \quad \zeta \in [0, \zeta_1],$$

where the number

$$\zeta_1 = \int\limits_0^{x_2} \frac{\alpha_2(z)-\beta_2(z)}{(1+\eta_0)\alpha_2(z)-\delta(z)-\beta_2(z)}\,dz.$$

Substituting the above obtained values of the function $g(\zeta)$ into the general solution, we get

$$u(x,t) = -t + \int\limits_0^{x-G(\frac{x-t}{1+\eta_0})} (1+\xi_0)\frac{\beta_1(z)+\gamma(z)}{\beta_1(z)}dz + \xi_0\alpha_1(z). \tag{2.16}$$

which in fact determines the solution of the problem (2.1), (2.13), (2.14) and hence of the initial problem with a free support of data Δ.

In the statement of the problem it is required to find the domain of definition of its solution. As is known ([8]), such a domain is bounded by characteristics, i.e. by data supports and by the characteristics coming out of finite points of the support. This implies that the arcs Γ and Δ represent a part of the boundary of the unknown domain. But they are known. It is required to find an exact form of characteristics coming out of the end points of the support (x_1, x_1) and $(x_2, \psi(x_2))$. Equation for one of the characteristics has a simple form,

$$t = x + \psi(x_2) - x_2. \tag{2.17}$$

Equation for the other characteristic can be obtained from the solution of the problem (2.16). Note that along the unknown characteristic the value $u + t = \eta$ must be constant and is defined by the value $u(x_1, x_1) + x_1$. Consequently,

$$\int\limits_0^{x - G(\frac{x-t}{1+\eta_0})} \frac{\beta_1(z) + \gamma(z)}{\beta_1(z) + \xi_0 \alpha_1(z)}\, dz = \int\limits_0^{x_1} \frac{\beta_1(z) + \gamma(z)}{\beta_1(z) + \xi_0 \alpha_1(z)}\, dz.$$

This relation is, in fact, the equation of the unknown characteristic, and we rewrite it as follows:

$$\int\limits_{x_1}^{x - G(\frac{x-t}{1+\eta_0})} \frac{\beta_1(z) + \gamma(z)}{\beta_1(z) + \xi_0 \alpha_1(z)}\, dz = 0. \tag{2.18}$$

Thus we have proved that the following theorem is valid.

Theorem. *If the conditions* (1.5), (2.6), (2.9) *and* (2.12) *are fulfilled, and the functional equation* (2.15) *is uniquely solvable, then the problem* (2.1), (0.2), (0.3), (0.4) *has the unique solution which is defined in the domain bounded by the curves* Γ, (2.14), (2.17) *and* (2.18).

It should be noted that when the conditions (2.9) and (2.12) violate simultaneously at some point $(a, \psi(a)$ of the characteristic Δ, equation (2.1) degenerates parabolically. The point itself is singular for the family of characteristics of the root μ. We put these questions aside and proceed to considering the case when both families of characteristics of the given equation depend the values of an unknown solution.

3. The Problem with Two Free Characteristics

In this section we also consider the simplest equation

$$(u_t^2 - 1)u_{xx} - 2u_x u_t u_{xt} + u_x^2 u_{tt} = 0 \tag{3.1}$$

with roots of the characteristic equation

$$\lambda = -\frac{u_x}{u_t + 1}, \quad \mu = -\frac{u_x}{u_t - 1}.$$

Characteristic invariants of the family, corresponding to the root λ, have the form

$$\xi = u(x,t) + t, \quad \xi_1 = \frac{u_x}{u_t - 1}. \tag{3.2}$$

The family of characteristic curves defined by the root μ has the following invariants:

$$\eta = u(x,t) - t, \quad \eta_1 = \frac{u_x}{u_t + 1}. \tag{3.3}$$

As is seen, unlike equation (2.1), none of the characteristic invariants provides us with the definition of characteristic curves. Therefore the curves Γ and Δ depend on the values of an unknown solution and should be defined together.

The general integral of equation (3.1) can be represented by two arbitrary functions f, $g \in C^2(R^1)$ as follows ([9]):

$$f(u,t) + g(u-t) = x. \tag{3.4}$$

A class of hyperbolic solutions is defined by a simple condition

$$u_x \neq 0, \tag{3.5}$$

and equation (3.1), just as (2.1), belongs to the class of parabolically degenerating hyperbolic equations.

Reasoning analogously as in the previous cases, without loss of generality, we assume that $x_0 = t_0 = u_0 = 0$ and proceed to investigating the problem (0.2-4) for equation (3.1).

In this case, the whole data support is unknown with the exclusion at the origin from which the characteristics Γ and Δ come out. Unlike the foregoing cases, none of the characteristic invariants depends on the argument x. Therefore it is more convenient to assume that the coefficients and the right-hand sides in the conditions (0.3), (0.4) are the given functions of the argument t. Let α_1, β_1, $\gamma \in C^2[0,t_1]$ and α_2, β, $\gamma \in C^2[0,t_2]$ where t_1 and t_2 are the given numbers.

To study the problem, we will again take advantage of the method suggested above. First of all, let the first order derivatives $u_x(0,0) \equiv p_0$ and $u_t(0,0) \equiv q_0$ of the unknown solution be endowed at the origin with the assumption (1.5). Using the already known values p_0, q_0 and $u_0 = u(0,0)$, we can find constant values of the invariants ξ, ξ_1 and η, η_1 along Γ and Δ, respectively:

$$\xi|_\Gamma = (u+t)|_\Gamma = u_0 + t_0 = 0, \quad \xi_1|_\Gamma = \frac{u_x}{u_t - 1}\Big|_\Gamma = \frac{p_0}{q_0 - 1} = \xi_0,$$
$$\eta|_\Delta = (u-t)|_\Delta = u_0 - t_0 = 0, \quad \eta_1|_\Delta = \frac{u_x}{u_t + 1}\Big|_\Delta = \frac{p_0}{q_0 + 1} = \eta_0.$$

From the above invariants it follows that along Γ

$$\begin{aligned} u|_{\Gamma} &= -t \\ (u_x - \xi_0 u_t)|_{\Gamma} &= -\xi_0, \end{aligned} \tag{3.6}$$

and along Δ we have

$$\begin{aligned} u|_{\Delta} &= t \\ (u_x - \eta_0 u_t)|_{\Delta} &= -\eta_0. \end{aligned} \tag{3.7}$$

This does not, surely, imply that the values of the solution $u(x,t)$ are defined on the arcs of characteristics Γ and Δ. Such a conclusion can be made only after the functions representing these curves are found explicitly.

To determine these functions, it is necessary to consider pairs of relations (0.3), (3.6) and (0.4), (3.7) along the curves Γ and Δ, respectively.

The first pair, as a system of two linear algebraic equations with respect to the derivatives u_x and u_t along Γ, is solvable under the condition

$$\beta_1(t) + \xi_0\alpha_1(t) \neq 0, \quad t \in [0, t_1]. \tag{3.8}$$

The values of these derivatives

$$u_x|_{\Gamma} = \xi_0 \frac{\gamma(t) - \beta_1(t)}{\beta_1(t) + \xi_0\alpha_1(t)}, \quad u_t|_{\Gamma} = \xi \frac{\gamma(t) + \xi_0\alpha_1(t)}{\beta_1(t) + \xi_0\alpha_1(t)}$$

are defined everywhere in the interval $[0, t_1]$.

Reasoning analogously, we obtain the condition

$$\beta_2(t) + \eta_0\alpha_2(t) \neq 0. \tag{3.9}$$

which provides us with the values of the above derivatives:

$$u_x|_{\Delta} = \eta_0 \frac{\beta_2(t) + \delta(t)}{\beta_2(t) + \eta_0\alpha_2(t)}, \quad u_t|_{\Delta} = \frac{\delta(t) - \eta_0\alpha_2(t)}{\beta_2(t) + \eta_0\alpha_2(t)}, \quad t \in [0, t_2]$$

along the curve Δ.

Having obtained the values of the derivatives u_x, u_t at all points of the curve Γ, we can define inclination of its tangents everywhere. This can be achieved by substituting the values u_x and u_t in the expression of the characteristic root λ. Thus we have

$$\frac{dt}{dx} = \xi_0 \frac{\beta_1(t) - \gamma(t)}{2\xi_0\alpha_1(t) + \beta_1(t) + \gamma(t)}, \quad t \in [0, t_1].$$

Taking into account the initial condition $x(0) = 0$, after integration the obtained ordinary differential equation gives the explicit representation of the characteristic arc Γ,

$$x = \frac{1}{\xi_0} \int\limits_0^t \frac{2\xi_0\alpha_1(z) + \beta_1(z) + \gamma(z)}{\beta_1(z) - \gamma(z)} dz \equiv \varphi(t), \quad t \in [0, t_1]. \tag{3.10}$$

Repeating the same operations and substituting the values of the derivatives u_x, u_t on the arc Δ into the characteristic root μ, we obtain the differential relation of the first order. The solution of the Cauchy problem with the condition $x(0) = 0$ allows us to get an explicit representation of the curve Δ. It has the form

$$x = \frac{1}{\eta_0} \int_0^t \frac{\delta(z) - \beta_2(z) - 2\eta_0\alpha_2(z)}{\beta_2(z) + \delta(z)}\, dz \equiv \psi(t), \quad t \in [0, t_2]. \tag{3.11}$$

It should be noted that the curves Γ and Δ are defined by means of equations (3.10) and (3.11) on the corresponding intervals.

As is seen, the characteristic conditions (0.4) and (0.4) together with the condition (0.2) guite enough for determination of unknown data supports Γ and Δ explicitly and globally. For this we only need the conditions (1.5), (3.8) and (3.9) under the assumption that the denominators of integrands in (3.10) and (3.11) are different from zero.

Having constructed (3.10) and (3.11), we can proceed to the final step of our investigating of the problem formulated above. This can be realized on the basis of the general integral (3.4) of equation (3.1).

First, let us consider the relation which follows from the representation of the general integral (3.4) taken on the arc Γ,

$$[f(u+t) + g(u-t)]_\Gamma = x|_\Gamma. \tag{3.12}$$

Along Γ, the sum $u+t$ is constant and equal to zero. The difference $u - t = u + y - 2t = -2t$. Therefore

$$f(0) + g(-2t) = \varphi(t), \tag{3.13}$$

where the function $\varphi(t)$ is defined by formula (3.10). This expression allows one to determine an arbitrary function g to within the constant summand

$$g(\zeta) = \varphi\Big(-\frac{\zeta}{2}\Big) - f(0), \quad \zeta \in [0, -2t_1].$$

On the other characteristic arc Δ, reasoning analogously, we obtain the equality $f(2t)+g(0) = \psi(t)$ from which one can define an arbitrary function f, appearing in the representation of the general integral

$$f(\zeta) = \psi\Big(\frac{\zeta}{2}\Big) - g(0), \quad \zeta \in [0, 2t_2]$$

also to within the constant summand.

Substituting the obtained in such a way functions f and g in (3.4), we get the integral of the problem (3.1), (0.2), (0.3), (0.4). The integral has the form

$$\psi\Big(\frac{u+t}{2}\Big) + \varphi\Big(\frac{t-u}{2}\Big) = x, \tag{3.14}$$

since $f(0)+g(0) = 0$. As is seen from the representation (3.14), the integral of the problem under consideration does not contain arbitrary parameters. This integral is defined on the whole by means of the conditions (0.2-4). As for the solvability on the whole, there may arise complications if the point is the construct not of an integral, but of a solution of the problem. In such a case, the relation (3.14) is considered as a functional equation with respect to the value $u(x,t)$, and from all possible solutions we choose those which satisfy the condition (0.2) for normalization.

Clear now up the structure of the domain in which we have defined the integral (3.14). To this end, we will need all characteristics of both families, coming out of the points of already known supports of the data Γ and Δ.

First, we consider characteristics of the μ-family which come out of the points of the characteristic arc Γ. An arbitrarily taken value $t = t_0$ from the interval $[0, t_1]$ allows us to find the point of the arc Γ. $(\varphi(t_0), t_0)$ are the coordinates of the point at which the solution of the problem is known. We define it by the first of formulas (3.6) and find that it is equal to $(-t_0)$. Besides the arc Γ, through that point passes the characteristic of another family which corresponds to the root μ. The invariants η, η_1 along that characteristic should be constant. The constant value of the invariant η on the entire characteristic is the same as at the point $(\varphi(t_0), t_0)$.

Thus along the characteristic we have $\eta = u - t = -2t_0$. As for the values of another invariant ξ, on the characteristic we have

$$\xi = 2(t - t_0), \quad t_0 \in [0, t_1]. \tag{3.15}$$

If we substitute the values of the invariants $\eta = -2t$ and (3.15) into the representation of the integral (3.14), we will obtain the relation connecting the values x and t along the whole characteristic. The form of the relation looks as

$$x = \psi(t - t_0) + \varphi(t_0), \quad t_0 \in [0, t_1], \quad t \in [t_0, t_2 + t_0]. \tag{3.16}$$

It represents an explicit equation of arc of the characteristic curve of the family of the root μ coming out of the point $(\varphi(t_0), t_0)$. Taking into account the fact that $t_0 \in [0, t_1]$ is chosen arbitrarily, we can conclude: formula (3.16) makes it possible to determine the family of all characteristics of the root μ, coming out of all points of the arc Γ. This family is one-parametric, and the value t_0 of the ordinate of the point Γ plays the role of a parameter. For the parameter $t_0 = 0$, we have equation (3.11) of the arc Δ.

Omitting quite similar reasoning, we can write out the final form of the family of characteristics of the root λ, coming out of the points $(\psi(\tau_0), \tau_0)$ of the arc Δ.

$$x = \varphi(t - \tau_0) + \psi(\tau_0), \quad \tau_0 \in [0, t_2], \quad t \in [\tau_0, t_1 + \tau_0]. \tag{3.17}$$

This family is likewise one-parametric with the ordinate τ_0 of an arbitrary point of the arc Δ taken as a parameter.

It should be noted that in all the above reasoning we did not restrict ourselves to the class of hyperbolic solutions of equation (3.1). As it was said, hyperbolicity of the solution of equation (3.1) is defined by a simple condition $u_x(x,t) \neq 0$. Therefore to avoid parabolic degeneration of the solution on the supports of data Γ and Δ, we should, according to our calculations, require that

$$\beta_1(t) \neq \gamma(t), \quad \beta_2(t) \neq -\delta(t) \tag{3.18}$$

respectively. Our reasoning admits violation of these requirements within the limits of the conditions for the existence of integrals in formulas (3.10) and (3.11). Consequently, at some isolated points of the arcs Γ and Δ the parabolic degeneration of equation (3.1) is admissible.

However, under the assumption of parabolic degeneration at isolated points of supports of the data Γ and Δ, the numerators of integrands in (3.10) and (3.11) will be different from zero, according to the conditions (3.8) and (3.9). Therefore we should not expect the existence of any discriminant points or singular curves in the families (3.16) and (3.17).

Thus the following theorem is valid.

Theorem. *If the conditions* (1.5), (3.8), (3.9) *and* (3.18) *are fulfilled, then there exists the integral of the problem* (3.1), $(0.2-4)$ *which is represented by formula* (3.14) *and defined in the domain bounded by the characteristics* (3.10), (3.11), (3.16) *for* $t_0 = t_1$ *and* (3.17) *for* $\tau_0 = t_2$.

In conclusion, it should be noted that the above-formulated problem with the data (0.2-4) for equation (1.1) and, in general, for linear equations and systems is, in fact, the ordinary characteristic problem which has been investigated by using various methods in various functional spaces (see, for e.g., [10-13]). Consequently, for equation (1.1) we have not present here any noval results of important scientific value, but simply push into the foreground the details which subsequently were used in the case of nonlinear equations. For nonlinear equations we suggested the variants of characteristic problems with the given supports of data which allowed one to perceive a connection with the linear theory and a degree of its generalization ([14-19]).

In the case of equation (2.1) we dealt with the characteristic problem with partially unknown supports of data. This problem for equation (3.1) is characteristic with an oblique derivative and entirely free support of data.

Noteworthy is the fact that the complete system of characteristic invariants of equations (1.1), (2.1) and (3.1) made it possible to combine and put to general frames the problems, remote from each other in a sense of their statement as well as of their investigation.

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