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ON HYPERBOLIC EQUATIONS WITH GENERAL INTEGRALS REPRESENTABLE BY SUPERPOSITION OF ARBITRARY PARAMETERS, AND CHARACTERISTIC PROBLEMS

As the title of the present article shows, we will touch upon two subjects: the first one is the class of the second order hyperbolic equations with two unknown variables which are connected by a fully definite criterion, and the second are the characteristic problems we start our discussion with. Such problems are, as usual, called partially or entirely characteristic data supports.

There exist several definitions of characteristics themselves, and all of them are, practically, of equal worth. According to one of the definitions, the characteristics are assumed to be the curves which, being the Cauchy data supports, lead to ill-posedness of the problem. In such a case, the Cauchy problem is redefined, and to ensure its well-posedness it is necessary to get rid on the characteristic parts of the support of at least one of the conditions (see, e.g., [1]). Of the above-cited variants, the Goursat problem with the given values of an unknown solution on the support (on two characteristic arcs of different families emanating from one common point) earns special attention by its refinement. This problem generates a rather tempting idea to approximate a support of general type by means of broken lines with characteristic links and hence to prove an impossible, to redefine the Cauchy problem in at least some classes of solutions. Testing of this idea makes us once more sure that the limiting transfer of a characteristic contour onto a support of general type is illegitimate. The equation $u_{xy} = 0$ together with some monotone arc-support representable as the limit of a sequence of broken lines with links, parallel to the coordinate axes, is an obvious to that example. An analogous state of affairs should be taken into account, and this is the case for approximate methods of solving the problems, when characteristic segments of the data support are replaced by close to them contours. For linear equations, there is no need in such a replacement whatsoever. This necessity arises only when characteristic families fail to be defined completely by the principal part of the equation. We are faced with this phenomenon in the case of quasi-linear equations whose major

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coefficients depend already on the lowest derivatives of an unknown solution. Thus the question on the approximation of the data support by means of the characteristics of general type falls away from the very beginning. But there arises another question: in such a case, how one can understand the characteristic, in general. Only after that it will be possible to formulate characteristic problems for quasi-linear equations, or to approximate by their means to some another data supports. Towards this end, we have to take into consideration all details, general both for linear and for quasi-linear equations. Those details are not too many. For clearness, we shall again get back to the above-given equation with two characteristic directions at every point. Along the straight lines with the characteristic direction $dy = 0$, the equation itself can be represented as the relation $d(u_y) = 0$. Thus we come to the conclusion that the expressions $\eta = y$ and $\eta_1 = u_y$, as being given on the plane of variables of the functions x and y, should have one and the same family of level lines. In the case under consideration, this family of straight lines $y = \text{const}$, and the derivative u_y along each of them is constant. Analogously, we can conclude that the expressions $\xi = x$ and $\xi_1 = u_x$ have likewise a general family of level lines. Just this is the way which allows one to construct the well-known representation of a general solution of the given equation. For quasi-linear equations, the two pairs of expressions ξ, ξ_1 and η, η_1 are replaced by certain combinations, depending on the arguments x and y, on an unknown solution $u(x, y)$ and its first order derivatives $u_x(x, y)$ and $u_y(x, y)$. If the existence of a pair ξ, ξ_1 of combinations, constant along the corresponding characteristics, is taken as a basis, then the curved families of level lines of these combinations can be assumed to be characteristics. It is quite rightfull to consider these ξ, ξ_1 combinations as analogues of the well-known invariants introduced by Riemann [3]. The family of level lines of the combinations η, η_1 may, like the foregoing, be considered as characteristics corresponding to the other root of the characteristic equation. The combinations ξ, ξ_1 and η, η_1 will be called characteristic invariants. Thus, unlike linear equations, there comes to light a rather wide choice of correct statements for a number of characteristic problems. For example, in some cases, in the capacity of characteristics can be taken arbitrary curves, arbitrary so far as it permit characteristic invariants. All this can be explained by the fact that they give both a full freedom of characteristics choice and restrict considerably this choice, as well. Explain this by an example. But first of all, we have to point out one important circumstance: in some cases, such an approach allows one even to represent the general integral explicitly with two arbitrary functions, what fully agrees with an order of the equation [4]. For example, in terms of the characteristic invariants

$$\xi = x - y, \quad \xi_1 = (u_x + u_y - 1)\exp u_y$$

and

$$\eta = y - (u_x + u_y)x, \quad \eta_1 = u_x + u_y$$

of equation

$$u_{xx} + (1 + u_x + u_y)u_{xy} + (u_x + u_y)u_{yy} = 0$$

all its hyperbolic solutions can be represented in a general form [5]. But the main point here is that the both characteristic families are representable by straight lines. It is namely they that define an admissible class of characteristic curves. Consequently, when formulating the Goursat problem, we have to choose only the straight line in the capacity of characteristic of the η-family. But in order to avoid parabolic degeneration, we have to choose only one, not parallel to straight lines, i.e., to characteristics of another family. Such a statement goes completely into a general scheme of characteristic problems and gives way to exhaustive investigations. Considerably wide is an admissible class of characteristic curves for the equation

$$(1 + u_y)u_{xx} + (1 - u_x + u_y)u_{xy} - u_x u_{yy} = 0, \tag{1}$$

whose invariants are $\xi = x-y$, $\xi_1 = u_x(1+u_y)^{-1}$ and $\eta = u+y$, $\eta_1 = u_x+u_y$. Despite the outer similarity and identically given $x - y = \text{const}$ characteristics of the ξ-family, the above equations differ significantly. First, in this case, we are able to construct a very laconic general solution, including hyperbolic ones [5],

$$u(x, y) = -y + f\big[x + g(x - y)\big]$$

with arbitrary functions f, g, so smooth for the regularity of a solution to be guaranteed. In this connection, one of arbitrary functions is a component of the argument of another function. Second, there is no need here to restrict an admissible class of characteristic curves as it was the case above. Thus we have now all premises to formulate nonlinear versions of the Goursat problem. Among those variants, we consider the following one:

let the given functions τ and ϑ satisfy the conditions

$$\tau \in C^2[0, a], \quad \vartheta \in C^2[0, b], \quad \vartheta(0) = 0, \quad \vartheta' \neq 1, \quad a > 0, \quad b > 0.$$

A regular solution of equation (1) should be sought together with the domain of its propagation, if it on the characteristic segment $J =: \{(x, y) : y = x,\ 0 \leq x \leq a\}$ takes the values $\tau(x)$, and the arc of the curve $y : \{(x, y) : y = \vartheta(x),\ 0 \leq x \leq b\}$ is the characteristic of the η-family.

A general solution expressed in terms of initial arguments allows one to resolve the above-posed problem by using the D'Alembert method. In the course of its application it becomes clear that under the definite choice of functions τ, ϑ, the domain of propagation of the solution $u(x, y)$ may contain $1 + u_x + u_y = 0$ which indicates parabolic degeneration of equation (1). This fact is also of interest in that any degeneration, no matter how it is, eliminates on the data supports J, y, according to the conditions of the problem.

Greatly richer in such phenomena is another equation, interesting by its practical applications [2]

$$u_y u_{xx} + (u\,u_y - u_x)u_{xy} - u\,u_x u_{yy} = 0 \tag{2}$$

and by a general, of original structure, integral constructed by E. Goursat [1]:

$$u\,f'(u) - f(u) + g\big[x - f(u)\big] = y.$$

Comparing this representation with a general solution of equation (1), we can see that they have much in common. In both of them are connected only three values: arguments with a solution; and this connection is realized by means of two arbitrary functions, one of which is a component of the argument of another function. But there is one essential difference, that is, the appearence of the first order derivative in an arbitrary function. This circumstance does not at all restrict a class of admissible curves as characteristics, but extends a number of well-posed and solvable nonlinear versions of characteristic problems. Moreover, it should be noted that the both representations cover all possible classes of solutions: hyperbolic, parabolic and mixed parabolo-hyperbolic, which is, unconditionally, one more, not lesser important, of their merits. Of special attention is the fact that the correct choice of characteristics allows one to define simultaneously a solution of the characteristic problem and the domains of its regular propagation. Thus, for some nonlinear equations there appears a possibility to consider an inverse problem of characteristic choice in a way for a solution to be defined in a preassigned manner and in preassigned domains. Relying on the above reasoning, the characteristic problem can, in a definite sense, be considered as the problem of control.

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