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CAUCHY PROBLEM FOR A QUASI-LINEAR HYPERBOLIC EQUATION WITH CLOSED SUPPORT OF DATA

G. Baghaturia, J. Gvazava, and M. Menteshashvili UDC 517.9

Abstract. The Cauchy problem with closed support for a quasi-linear hyperbolic equation with a possible parabolic degeneracy is studied. The structure of the domain of solution is studied and initial conditions for which there exist areas of impenetrability of waves are found.

When posing a problem for a hyperbolic equation, it is often necessary to demand the fulfillment of certain conditions regarding initial support. In particular, it refers to the Riemann problem when it is required to find an unknown solution by given values of the solution and its oblique derivative on the initial support. The class of such problems is wide and the known Cauchy problem also belongs to it. Namely, according to the Cauchy problem, we must find the solution of the equation by given initial perturbations:

$$u|_{t=0} = \tau, \quad u_t|_{t=0} = \nu.$$

After posing such problems it is immediately required from the initial support that every characteristic of the equation must not cross it more than in one point. Also it is required from the support that it must not have the characteristic direction in any point. For the above-mentioned Cauchy problem, the initial support is represented by the relation $t = 0$. This initial support must be subjected to the same requirements. The existence of these requirements is due to certain reasons. These reasons are connected with common properties of solutions of the equations. If we consider the equations along the characteristic manifolds, we will see that in the majority of cases these properties themselves emerge from the equations. For obviousness, let us consider the equation of vibration of the string

$$u_{xx} - u_{tt} = 0$$

given on the plane (x, t). In this case, as well as in the case of other linear equations, both families of characteristics are fully determined by the main part of the equation. These one-parameter families are related to the characteristic roots $\lambda_1 = 1$ and $\lambda_2 = -1$, and they are expressed by the following relations:

$$x - t = \text{const}, \quad x + t = \text{const}\,.$$

If we consider the equation of vibration of the string along the first of them, we obtain the relation

$$du_x - du_t = 0$$

and integrating it we obtain

$$u_x - u_t = \text{const}\,.$$

This means that along any characteristic of this family the combination of first derivatives of the solution remains constant. The same fact is also valid for the second family of the characteristic curves. In this case, the combination $u_x + u_t$ remains constant. These combinations are called the characteristic invariants. Now, let us assume that the initial support $t = f(x)$ of the Cauchy or Riemann problem is crossed by the characteristic curve of the first family in two points: (x_0, t_0) and (x_1, t_1). In this case, according to the properties of the characteristic invariants, the following relation is fulfilled:

$$u_x(x_0, t_0) - u_t(x_0, t_0) = u_x(x_1, t_1) - u_t(x_1, t_1).$$

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Here we can make only one conclusion: it turns out that the values of derivatives of the solution are somehow bounded and consequently when posing the problem, we cannot consider the initial data as arbitrary on the support. In this case the data are excessive for the correctness of the problem. So, the existence of the above-mentioned condition posed for the initial support and initial data is quite natural. However, according to Hadamard [4], the Cauchy problem may appear well posed in some cases of closed initial support. Such problems for some linear equations were studied by Sobolev [7], Alexandryan [1], Wolfesdorff [10], Vakhania [8], Nakhushev [6], etc.

All above mentioned facts are valid in the case of linear equations since the characteristic families are determined by the main part of the equation. The situation is quite different when characteristic families cannot be determined in advance since they depend on the unknown solution u and its derivatives u_x and u_t. This occurs in the case of the hyperbolic equation

$$au_{xx} + bu_{xt} + cu_{tt} = f, \tag{1}$$

which is nonlinear and its main coefficients and right-hand side are functions of the five variables x, y, u, u_x, and u_t. In spite of the fact that in such cases the investigation of the problem gets complicated, here at least we are freer when posing the problem. We also have the opportunity to modify and generalize the well-known linear problems [3]. However, there is no need to generalize and modify the Cauchy problem since this always means finding the solution by given values of it and its nontangent direction derivative on the support. We emphasize this fact since in this work we speak of the Cauchy problem posed for the nonlinear equation (1) of special type. As is well known (see [5]), the hyperbolic class of solutions is determined through its characteristic roots by the inequality

$$\lambda_1 \neq \lambda_2.$$

The characteristic directions must also be determined in every point by the following relations:

$$\frac{dt}{dx} = \lambda_1(x, t, u, u_x, u_t), \quad \frac{dt}{dx} = \lambda_2(x, t, u, u_x, u_t).$$

Let us assume that the initial support is given in the explicit form by the equation $t = f(x)$, where f is a function subject to certain conditions. We speak of these conditions below. In this case, we can determine the direction of any characteristic curve that emanates from an arbitrary point of the support. This direction is described by the following relation:

$$\frac{dt}{dx} = \lambda_1\left(\tau(x_0),\ \frac{\tau'(x_0) - \nu(x_0)f'(x_0)}{1 + (f'(x_0))^2},\ \frac{\nu(x_0) + \tau'(x_0)f'(x_0)}{1 + (f'(x_0))^2}\right).$$

If we consider x_0 as a parameter, we can determine the direction of the characteristic curves everywhere on the support. An analogous situation exists in the case of the second family of characteristics. However, this is not sufficient, because for the solution of the problem we have to write the equations of the characteristic curves of both families. For this purpose we can use the representations of general integrals. Since the construction of general integrals in closed form is not always possible, here we consider equations of special type admitting the construction of general integrals for them [2]. By means of general integrals, we describe both families of characteristics and construct the set of intersection points of different families of characteristic curves. Generally, such sets create the domain inside of which the initial support is situated. Combinations that remain constant in the case of the equation of vibration of a string differ from those for nonlinear equations. The main difference is that for nonlinear equations these combinations depend on an unknown solution and its first derivatives. In the case of aerodynamics, these combinations are called Riemann invariants. Generally, constructing the characteristic invariants and using them in solving problems for quasi-linear equations is very difficult. We try to do this in the case of concrete equations, which somehow are related to the flow of air and liquid in channels [9]:

$$u_t(u_t - 1)u_{xx} + (u_t - u_x - 2u_xu_t + 1)u_{xt} + u_x(u_x + 1)u_{tt} = 0. \tag{2}$$

This is a hyperbolic equation; however, in some cases it admits a parabolic degeneracy. This happens when the characteristic roots of Eq. (2) are equal, i.e.,

$$-\frac{u_x+1}{u_t}=\frac{u_x}{1-u_t}.$$

So, the set of hyperbolic solutions is described as follows:

$$u_x-u_t+1\neq 0.$$

The characteristic invariants are described by the following systems:

$$\begin{cases}\xi_1=u+x,\\ \xi_2=\dfrac{u_x}{u_x-u_t+1},\end{cases}\qquad \begin{cases}\eta_1=u-t,\\ \eta_2=\dfrac{u_x+1}{u_x-u_t+1}.\end{cases}$$

By means of the characteristic method and the Poisson brackets, it is possible to obtain the general integral of Eq. (2)

$$f(u+x)+g(u-t)=t, \tag{3}$$

where $f,g\in C^2(R)$ are arbitrary functions. For Eq. (2), we examine the Cauchy problem posed on the circle:

$$\gamma:\ (x-a)^2+t^2=(\sqrt{2}-a)^2,\quad a<0.$$

Using polar coordinates

$$x=\rho\cos(\alpha),\quad t=\rho\sin(\alpha),$$

we can write this problem as follows:

$$\begin{gathered}u|_\gamma=\tau(\alpha),\quad \alpha\in[0,2\pi],\\ u_\rho|_\gamma=\nu(\alpha),\quad \alpha\in[0,2\pi],\quad \tau,\nu\in C^2[0,2\pi],\end{gathered} \tag{4}$$

where the support of initial data is the following circle:

$$\gamma:\ \rho=a\cos\alpha+\sqrt{a^2\cos^2\alpha+2-2\sqrt{2}a},\quad \alpha\in[0,2\pi]. \tag{5}$$

If we subject the general integral (3) to the initial conditions (4), we obtain the implicit solution for the problem (2), (4):

$$\int\limits_{T_2(u-t)}^{T_1(u+x)} H(\alpha)\,d\alpha+(2-a)\sin\left(T_2(u-t)\right)=t, \tag{6}$$

where

$$H(\alpha)=\frac{(2-a)\left[((a-1)\sin\alpha-\nu(\alpha))\cos\alpha+\tau'(\alpha)\sin\alpha\right]}{((a-2)\nu(\alpha)+\tau'(\alpha))\cos\alpha+((2-a)\nu(\alpha)+\tau'(\alpha))\sin\alpha+a-2}\times\left[\tau'(\alpha)+(a-2)\sin\alpha\right]$$

and the functions $\alpha=T_1(z)$ and $\alpha=T_2(\omega)$ are inverse functions of the following expressions, respectively:

$$z=\tau(\alpha)+(2-a)\cos\alpha+a,\quad \omega=\tau(\alpha)+(2-a)\sin\alpha. \tag{7}$$

Based on (6), it is easy to describe both families of characteristic curves. They are one-parameter families, which emanate from an arbitrary point of support (5):

$$t = \int\limits_{T_1(c-t-x)}^{c} H(\alpha)\,d\alpha + (2-a)\sin\left(T_2(c-t-x)\right), \tag{8}$$

$$t = \int\limits_{c}^{T_1(c+t+x)} H(\alpha)\,d\alpha + (2-a)\sin\left(T_2(c)\right). \tag{9}$$

As is known, the discriminant curve for the family (8) exists when the following system is solvable in regard to variables x and t:

$$\begin{cases} \int\limits_{T_1(c-t-x)}^{c} H(\alpha)\,d\alpha + (2-a)\sin\left(T_2(c-t-x)\right) = t, \\ H(c) - H\left(T_1(c-t-x)\right)T_1'(c-t-x) + (2-a)\cos\left(T_2(c-t-x)\right)T_2'(c-t-x) = 0, \end{cases} \tag{10}$$

where the second equation of the system is obtained by differentiation of (8) with respect to the parameter c. Similarly, we can write a similar system in the case of family (9):

$$\begin{cases} \int\limits_{c}^{T_1(c+t+x)} H(\alpha)\,d\alpha + (2-a)\sin\left(T_2(c)\right) = t, \\ H(T_1(c+t+x))T_1'(c+t+x) - H(c) + (2-a)\cos\left(T_2(c)\right)T_2'(c) = 0. \end{cases} \tag{11}$$

It is obvious that both systems (10) and (11) include the functions τ and ω and their derivatives. They also include the functions T_1 and T_2 and their derivatives. Note that according to (7), T_1 and T_2 are inverse functions of expressions that depend only on τ.

Among multiple variants of concrete initial functions τ and ω, we have found an example where for the families (8) and (9), the discriminant curve is the same circle line:

$$\partial: \ x^2 + t^2 = 1. \tag{12}$$

The initial conditions (4) in this case are as follows:

$$u|_\gamma = u_0 + \left(a\cos\alpha + \sqrt{a^2\cos^2\alpha + 2 - 2\sqrt{2}a}\right)\sin\alpha, \quad \alpha \in [0, 2\pi],$$
$$u_\rho|_\gamma = \sin\alpha, \quad \alpha \in [0, 2\pi],$$

and the characteristic curves are the straight lines

$$x\cos\alpha + t\sin\alpha = 1, \quad \alpha \in [0, 2\pi].$$

Each of these straight lines touches the circle (12) at one point. This point divides each straight line into two rays. Each of these rays belongs to different families of characteristics. Each line of one family of characteristics crosses the line of another family only once.

We have proved that the Cauchy problem (2), (4) is well posed. None of the straight lines penetrate inside the circle area. So, we found exactly the area of definition of the solution of the problem (2), (4) posed on the closed curve (5). The solution is defined everywhere except the inside area of the circle (12). At the same time the circle (12) is the envelope for both families of characteristics. Thus, we have the parabolic degeneration on the line of the circle. Generally, parabolic degeneration is one of the reasons for the formation of areas impenetrable for nonlinear waves. Such area is the set of inner points of circle (12) in our case.

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G. Baghaturia
N. Muskhelishvili Institute of Computational Mathematics,
Georgian Technical University, Tbilisi, Georgia
E-mail: nogela@gmail.com
J. Gvazava
Georgian Technical University, Tbilisi, Georgia
E-mail: jgvaza@rmi.ge
M. Menteshashvili
N. Muskhelishvili Institute of Computational Mathematics,
Georgian Technical University, Tbilisi, Georgia
E-mail: menteshashvili_m@mail.ru