Higher polyhedral K-groups

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Abstract

We define higher *polyhedral K*-groups for commutative rings, starting from the stable groups of elementary automorphisms of polyhedral algebras. Both Volodin's theory and Quillen's $+$ construction are developed. In the special case of algebras associated with unit simplices one recovers the usual algebraic K-groups, while the general case of lattice polytopes reveals many new aspects, governed by polyhedral geometry. This paper is a continuation of Bruns and Gubeladze (Polyhedral K_2, Manuscr. Math.) which is devoted to the study of polyhedral aspects of the classical Steinberg relations. The present work explores the polyhedral geometry behind Suslin's well known proof of the coincidence of the classical Volodin's and Quillen's theories. We also determine all K-groups coming from two-dimensional polytopes.
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1. Introduction

In [5] we have initiated a 'polyhedrization' of algebraic K-theory of commutative rings R that is based on the groups of graded automorphisms of *polyhedral* algebras. The algebra associated with a unit simplex is just a polynomial ring over R, its automorphism group is the general linear group, and the resulting theory is nothing else but

usual K-theory (details in Section 6.2). However, for general lattice polytopes (and, further, for lattice polyhedral complexes) many new aspects show up.

The motivation behind such a polyhedrization can be summarized as follows. The geometry of the affine space $\mathbb{A}^d_k$ and polyhedral geometry merge naturally in the concept of toric variety. Linear algebra constitutes part of the geometry of affine spaces, and it admits its own polyhedrization resulting in the study of the category Pol(k) (k a field) of *polytopal algebras* and their graded homomorphisms. The objects of Pol(k) are essentially the homogeneous coordinate rings of projective toric varieties, and they are naturally associated with lattice polytopes. This category contains the category Vect(k) of (finite dimensional) vector spaces over k as a full subcategory and, despite being non-additive, reveals surprising similarities with Vect(k) [4]. This leads to *polyhedral* linear algebra, where the Hom-objects are no longer vector spaces, but certain k-varieties equipped with k^*-equivariant structures. Our general philosophy is that essentially all standard linear algebra facts should have meaningful geometric analogues in Pol(k).

Lower K-theory generalizes linear algebra to the study of projective modules over general rings and the automorphism groups of free modules. By analogy, in the category Pol(k) one studies algebra retractions and the groups of automorphisms. The objects belong to the area of toric varieties and, pursuing this analogy (see, [1–4]), we have gained new insight into well known results and proved several new ones, especially for automorphism groups.

The next natural step in merging K-theory and polyhedral geometry was carried out in [5] where we investigated the Schur multiplier of the stable group of elementary automorphisms of polytopal algebras. The elementary automorphisms play the same rôle in Pol(k) as elementary matrices in Vect(k). For detailed comments and explanations see Section 2.

In this paper we develop higher polyhedral K-theory for commutative rings. From the very beginning it is clear that it cannot be based on additive structures (say, exact categories). There are essentially two classical constructions of higher K-theory, which can be adapted to groups of algebra automorphisms, namely Quillen's $+$ construction [11] and Volodin's theory [16]. We will explore them both, having in view their equivalence in the classical situation. While [5] has been devoted to the polyhedral aspects of the classical Steinberg relations, the present work explores the polyhedral geometry behind Suslin's well known proof of the coincidence of the Volodin and Quillen theories [14,15].

The analysis of the polyhedral Steinberg relations leads one to an appropriate class of lattice polytopes which we have called *balanced*. For them one can proceed similarly to the development of Milnor's K_2. For higher K-groups one has to investigate the extent to which lattice polytopes and their *column structures* support the standard K-theoretical constructions, and this requirement narrows the class of suitable polytopes somewhat further.

In Suslin's approach [14,15] the original Volodin theory is defined in terms of *triangular* linear groups based on abstract finite posets. In the polyhedral interpretation these posets correspond to oriented graphs formed by edges of unit simplices. For a general polytope one considers *column vectors* which, usually, penetrate the interior of

the polytope. These column vectors are roots of the corresponding linear group, namely the group of graded automorphisms of the polytopal algebra, but, as explained in [5], we are working with essentially non-reductive linear groups.

The main results can be summarized as:

(a) Generalization of the triangular subgroups of the general linear groups to subgroups of the stable group of elementary automorphisms, supported by *rigid systems of column vectors*. Their structure is studied in Section 5. Thereafter, in Section 6, we construct polyhedral versions of Volodin's and Quillen's theories. For the index 2 they coincide with the polyhedral version of Milnor's K_2.

(b) The homology acyclicity of the space naturally associated to the Volodin theory. This is shown by a suitable polyhedrization of Suslin's arguments, where several technical difficulties have to be overcome.

(c) Introduction of the class of Col-*divisible* polytopes. For polytopes of this class the theories of Quillen and Volodin agree. In view of (b) this amounts to showing the homotopy simplicity of the relevant spaces. However, the situation is more complicated than in the classical situation [14]. The class of Col-divisible polytopes includes all balanced polygons (two-dimensional polytopes).

(d) Complete computation of all polygonal theories (i.e. those corresponding to lattice polygons). This is accomplished with use of Quillen's [15] and Nesterenko-Suslin's [10] homological computations in linear groups.

Actually two versions of Volodin's theory are defined, one of which is of an auxiliary character—it provides the acyclicity result mentioned in (b) above. Then we prove the coincidence of all the three theories for Col-divisible polytopes.

Sections 2–6 are devoted to the development of the basic notions, including the definition of the polytopal analogues of triangular subgroups and the description of the corresponding K-theories. The K-theoretic calculations are contained in Sections 7–9.

The new K-groups are actually bifunctors in two covariant arguments—a (commutative) ring and a (balanced) polytope. In other words, [5] and the present paper add a polyhedral argument to K-theory, showing up in K_i for $i \geqslant 2$. One is naturally lead to the following two questions:

(1) Are the new groups computable to the same extent as the usual K-groups?
(2) Is there a polyhedral K-theory of schemes?

In connection with the first question we mention that it is not yet clear how far the polyhedral K-groups are from the usual ones. Our expectation, supported by the results in Section 9, is that the polyhedral theory for Col-divisible polytopes is very close to the usual one, namely a finite direct sum of copies. Such a claim should be thought of as a higher stable analogue of the main result of [1] (see Theorem 2.2 below): higher syzygies between elementary automorphisms for Col-divisible polytopes come from unit simplices. Computations that make such results possible are provided by the + construction approach.

However, for general balanced polytopes the theories may well diverge substantially. The simplest hypothetical candidate for such a divergence is the unit pyramid over the unit square, showing up several times in [5] and the present paper. But the corresponding computations remain to be elusive.

In connection with the second question we remark that a more conceptual approach might lead to the theory of *toric bundles* over schemes for which the fibers are no longer affine spaces but general affine toric varieties, equipped with an algebraic action of the one-dimensional torus. The morphisms of such bundles are defined by the requirement that fiberwise they induce morphisms of algebraic varieties, respecting the torus action. One could also try to use 'Jouanolou's device' [9] based on appropriate affinization.

Finally, thanks to the results in [5] and the structural description of triangular groups (Theorems 5.3 and 5.4), we can treat elementary automorphisms in a formal way, i.e. without referring to their action on polytopal algebras. Hence the rather combinatorial flavor of the exposition. However, the paper is really about higher syzygies of elementary automorphisms. For this reason we are restricted to commutative rings of coefficients—for non-commutative rings such automorphisms simply do not exist.

As remarked, all the rings that appear below are commutative.

2. Polytopes, their algebras, and their linear groups

Per suggestion of the referee—to whom we are indeed very grateful for a number of valuable comments, corrections, and suggestions to make the paper more readable–we begin this section with some basic notions related to convex polytopes.

2.1. General polytopes

The interested reader may consult Ziegler [17] for the arguments skipped in this section.

By a *polytope* $P \subset \mathbb{R}^n$, $n \in \mathbb{N}$, we always mean a *finite convex* polytope, i.e. P is the convex hull of a finite subset $\{x_1, \ldots, x_k\} \subset \mathbb{R}^n$:

$$P = \text{conv}(x_1, \ldots, x_k) := \{a_1 x_1 + \cdots + a_k x_k : 0 \leqslant a_1, \ldots, a_k \leqslant 1,$$

$$a_1 + \cdots + a_k = 1\}.$$

Alternatively, a polytope is a bounded subset of $\mathbb{R}^n$ that can be represented as the intersection of a finite system of closed affine halfspaces,

$$P = \bigcap_{j=1}^{s} \mathscr{H}_j, \quad \mathscr{H}_j = \{x \in \mathbb{R}^n : \mathscr{L}_j(x) \geqslant b_j\},$$

where $\mathscr{L}_j : \mathbb{R}^n \to \mathbb{R}$ is a linear mapping and $b_j \in \mathbb{R}$, $j = 1, \ldots, s$.

Polytopes of dimension 1 are called *segments* and those of dimension 2 are called *polygons*.

The *affine hull* $\mathrm{Aff}(X)$ of a subset $X \subset \mathbb{R}^n$ is the smallest affine subspace of $\mathbb{R}^n$ containing X, i.e.

$$\mathrm{Aff}(X) = \{a_1 x_1 + \cdots + a_k x_k : k \in \mathbb{N},\ x_1, \ldots, x_k \in X,\ a_1, \ldots, a_k \in \mathbb{R},$$

$$a_1 + \cdots + a_k = 1\}.$$

If $\dim \mathrm{Aff}(X) = k$ for a subset $X = \{x_1, \ldots, x_k\}$ of cardinality k, then $x_1, \ldots, x_k$ are *affinely independent* and the polytope $P = \mathrm{conv}(x_1, \ldots, x_k)$ is called a *simplex*.

For a halfspace $\mathscr{H} \subset \mathbb{R}^n$ containing P, the intersection $P \cap \partial \mathscr{H}$ of P with the boundary affine hyperplane $\partial \mathscr{H}$ of $\mathscr{H}$ is called a *face* of P. The polytope itself is also considered as a face.

The faces of P are themselves polytopes. Faces of dimension 0 are *vertices* and those of codimension 1 (i.e. of dimension $\dim P - 1$) are called *facets*. A polytope is the convex hull of the set $\mathrm{vert}(P)$ of its vertices. If $\dim P = n$, then there is a unique halfspace $\mathscr{H}$ for each facet $F \subset P$ such that $\partial \mathscr{H} \cap P = F$.

2.2. Lattice polytopes

A polytope $P \subset \mathbb{R}^n$ is called a *lattice polytope* if the vertices of P belong to the integral lattice $\mathbb{Z}^n$. More generally, a lattice in $\mathbb{R}^n$ is a subset $\mathscr{G} = x_0 + \mathscr{G}_0$ with $x_0 \in \mathbb{R}^n$ and an additive subgroup $\mathscr{G}_0$ generated by n linearly independent vectors. A polytope P with $\mathrm{vert}(P) \subset \mathscr{G}$ is called a $\mathscr{G}$-polytope if the vertices of P belong $\mathscr{G}$. However, since all the properties of $\mathscr{G}$-polytopes we are interested in remain invariant under an affine automorphism of $\mathbb{R}^n$ mapping $\mathscr{G}$ to $\mathbb{Z}^n$, we can always assume that our polytopes have vertices in $\mathbb{Z}^n$. More generally, lattice polytopes P and Q that are isomorphic under an integral-affine equivalence of $\mathrm{Aff}(P)$ and $\mathrm{Aff}(Q)$ are equivalent objects or our theory. We simply speak of *integral-affinely equivalent polytopes*.

Faces of a lattice polytope are again lattice polytopes.

For a lattice polytope $P \subset \mathbb{R}^n$ we put $\mathrm{L}_P = P \cap \mathbb{Z}^n$. A simplex Δ is called *unimodular* if $\sum_{z \in \mathrm{vert}(\Delta)} \mathbb{Z}(z - z_0)$ is a direct summand of $\mathbb{Z}^n$ for some (equivalently, every) vertex z_0 of Δ. All unimodular simplices of dimension n are integral-affinely equivalent. Such a simplex is denoted by Δ_n and called a *unit n-simplex*. Standard realizations of Δ_n are $\mathrm{conv}(O, e_1, \ldots, e_n) \subset \mathbb{R}^n$ or $\mathrm{conv}(e_1, \ldots, e_{n+1}) \subset \mathbb{R}^{n+1}$. ($e_i$ is the ith unit vector.)

There is no loss in assuming that a given lattice polytope P is full dimensional (i.e. $\dim P = n$) and that $\mathbb{Z}^n$ is the smallest affine lattice containing L_P. In fact, we choose $\mathrm{Aff}(P)$ as the space in which P is embedded and fix a point $x_0 \in \mathrm{L}_P$ as the origin. Then the lattice $x_0 + \sum_{x \in \mathrm{L}_P} \mathbb{Z}(x - x_0)$ can be identified with $\mathbb{Z}^r$, $r = \dim P$.

Under this assumption let F be a facet of P and choose a point $z_0 \in F$. Then the subgroup

$$F_{\mathbb{Z}} := (-z_0 + \mathrm{Aff}(F)) \cap \mathbb{Z}^n \subset \mathbb{Z}^n$$

is isomorphic to $\mathbb{Z}^{n-1}$. Moreover, there is a unique group homomorphism $\langle F, - \rangle : \mathbb{Z}^n \to \mathbb{Z}$, written as $x \mapsto \langle F, x \rangle$, such that $\mathrm{Ker}(\langle F, - \rangle) = F_{\mathbb{Z}}$, $\mathrm{Coker}(\langle F, - \rangle) = 0$, and on the set L_P, $\langle F, - \rangle$ attains its minimum b_F at the lattice points of F.
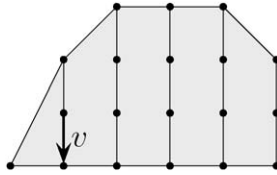
Fig. 1. A column structure.

The $\mathbb{Z}$-linear form $\langle F, - \rangle$ can be extended in a unique way to a linear function on $\mathbb{R}^n$. The description of P as an intersection of halfspaces yields that $x \in P$ if and only if $\langle F, x \rangle \geqslant b_F$ for all facets F of P.

Our blanket assumption throughout the paper is: *all polytopes, considered below, are lattice polytopes.*

2.3. Column structures

Let $P \subset \mathbb{R}^n$ be a polytope. A nonzero element $v \in \mathbb{Z}^n$ is called a *a column vector* for P if there exists a facet $F \subset P$ such that $x + v \in P$ whenever $x \in L_P \backslash F$. In this situation F is uniquely determined and called the *base facet* of v. We use the notation $F = P_v$. The set of column vectors of P is denoted by $\mathrm{Col}(P)$. A *column structure* is a pair of type (P, v), $v \in \mathrm{Col}(P)$. Fig. 1 gives an example of a column structure. Familiar examples of column structures are the unit simplices Δ_n with their edge vectors (i.e. the vectors $z' - z'' \in \mathbb{Z}^n$ for vertices z' and z'' of Δ_n) and the unit square (the convex hull of the set $\{(0,0), (1,0), (0,1), (1,1)\}$) with its edge vectors.

Using the description of P in terms of the functions $\langle F, - \rangle$ it is not hard to see that v is a column vector of P if and only if there exists exactly one facet F with $\langle F, v \rangle = -1$ and $\langle G, v \rangle \geqslant 0$ for all facets $G \neq F$.

2.4. Polytopal semigroups and their rings

To a polytope $P \subset \mathbb{R}^n$ one associates the additive subsemigroup $S_P \subset \mathbb{Z}^{n+1}$, generated by $\{(z, 1) : z \in L_P\} \subset \mathbb{Z}^{n+1}$. Let $C_P \subset \mathbb{R}^{n+1}$ be the cone $\{az : a \in \mathbb{R}_+, \ z \in P\}$. Then C_P is the convex hull of S_P. It is a *finite rational pointed* cone. In other words, C_P is the intersection of a finite system of halfspaces in $\mathbb{R}^{n+1}$ whose boundaries are rational hyperplanes containing the origin $O \in \mathbb{R}^{n+1}$, and there is no affine line contained in C_P.

As in Section 2.2, there is no loss of generality in assuming that $\mathbb{Z}^n$ is the lattice spanned affinely by L_P in $\mathbb{R}^n$. This is equivalent to $\mathrm{gp}(S_P) = \mathbb{Z}^{n+1}$, and the condition $S_P = C_P \cap \mathbb{Z}^{n+1}$ on the polytope P is known as the *normality condition* [1]. This is equivalent to saying that the affine semigroup ring $k[S_P]$ is normal for some (equivalently, every) field k.

All segments and polygons are normal, but in dimensions $\geqslant 3$ this is no longer the case (the interested reader is referred to [7] and [6] for the detailed theory).

It is an easy observation that the normality of P is equivalent to the normality of the facet F if there exists a column structure with base facet F.

While the points $x \in L_P$ are identified with $(x, 1) \in \mathbb{Z}^{n+1}$, a column vector v is to be identified with $(v, 0) \in \mathbb{Z}^{n+1}$.

Let F be a facet of F. We use the function $\langle F, - \rangle$ to define the *height* of $x = (x', x'') \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ above the hyperplane $\mathcal{H}$ through the facet C_F of C_P by setting

$$\mathrm{ht}_F(x) = \langle F, x' \rangle - x'' b_F.$$

For lattice points x the function ht_F counts the number of hyperplanes between $\mathcal{H}$ and x (in the direction of P) that are parallel to, but different from $\mathcal{H}$ and pass through lattice points. If v is a column vector, then ht_v stands for ht_{P_v}. Moreover, we are justified in calling $\mathrm{ht}_F(v, 0) = \langle F, v \rangle$ the *height of v with respect to F*, since v is identified with $(v, 0)$.

Although the semigroup S_P may miss some integral points in the cone C_P this cannot happen on the segments parallel to a column vector v. More precisely, the following holds:

$$z + v \in S_P \quad \text{for all } z \in S_P \setminus C_{P_v}. \tag{1}$$

($C_{P_v} \subset C_P$ is the face subcone, corresponding to P_v.)

Let R be a ring and $P \subset \mathbb{R}^n$ a lattice polytope. The semigroup ring $R[P] := R[S_P]$—the *polytopal R-algebra of P*—carries a graded structure $R[P] = R \oplus R_1 \oplus \cdots$ in which $\deg(x) = 1$ for all $x \in L_P$. By definition of S_P it follows that R_1 generates $R[P]$ over R.

We are interested in the group $\mathrm{gr.aut}_R(P)$ of graded R-algebra automorphisms of $R[P]$. For a field $R = k$ the group $\mathrm{gr.aut}_k(P)$ is naturally a k-linear group. In fact, it is a closed subgroup of $\mathrm{GL}_m(k)$, $m = \#L_P$. We call $\mathrm{gr.aut}_k(P)$ the *polytopal k-linear group* of P. Its structure will be given in Theorem 2.2.

In the special case when P is a unimodular simplex the ring $R[P]$ is isomorphic to a polynomial algebra $R[X_1, \ldots, X_m]$, $m = \#L_P$. Therefore, the category $\mathrm{Pol}(R)$ of polytopal R-algebras and graded homomorphisms between them contains a full subcategory that is equivalent to the category of free R-modules. From this perspective $\mathrm{Pol}(R)$ is a 'polytopal extension' of the category of free R-modules and one might wonder to which extent the basic K-theoretic facts generalize from the smaller category to $\mathrm{Pol}(R)$. As mentioned in the introduction, this direction of investigation is pursued in [4].

However, the motivation for the current paper is somewhat different (and less categorical). This we explain next.

2.5. Polytopal linear groups

Assume R is a ring and P a polytope. Let (P, v) be a column structure and $\lambda \in R$. As pointed out above, we identify the vector v with the degree 0 element $(v, 0) \in \mathbb{Z}^{n+1}$, and further with the corresponding monomial in $R[\mathbb{Z}^{n+1}]$. Then we define a mapping from S_P to $R[\mathbb{Z}^{n+1}]$ by the assignment

$$x \mapsto (1 + \lambda v)^{\mathrm{ht}_v x}.$$

Since ht_v is a group homomorphism $\mathbb{Z}^{n+1} \to \mathbb{Z}$, our mapping is a homomorphism from S_P to the multiplicative monoid of $R[\mathbb{Z}^{n+1}]$. Now it is immediate from (1) in Section

2.4 that the (isomorphic) image of S_P lies actually in $R[P]$. Hence this mapping gives rise to a graded R-algebra endomorphism e_v^λ of $R[P]$. But then e_v^λ is actually a graded automorphism of $R[P]$ because $e_v^{-\lambda}$ is its inverse.

Here is an alternative description of e_v^λ. By a suitable integral change of coordinates we may assume that $v = (0, -1, 0, \ldots, 0)$ and that P_v lies in the subspace $\mathbb{R}^{n-1}$ (thus P is in the upper halfspace). Now consider the standard unimodular n-simplex Δ_n with vertices at the origin and standard coordinate vectors: $\Delta_n = \text{conv}(O, (1, \ldots, 0), \ldots, (0, \ldots, 1))$. It is clear that there is a sufficiently large natural number c, such that P is contained in a parallel translate of $c\Delta_n$ by a vector from $\mathbb{Z}^{n-1}$. Let Δ denote such a parallel translate. Then we have a graded R-algebra embedding $R[P] \subset R[\Delta]$. Moreover, $R[\Delta]$ can be identified with the c-th Veronese subring of the polynomial ring $R[X_0, \ldots, X_n]$ in such a way that $v = X_0/X_1$. Now the automorphism of $R[X_0, \ldots, X_n]$ mapping X_1 to $X_1 + \lambda X_0$ and leaving all the other variables invariant induces an automorphism α of the subalgebra $R[\Delta]$, and α in turn can be restricted to an automorphism of $R[P]$, which is nothing else but e_v^λ.

It is clear from this description of e_v^λ that it becomes an elementary matrix (e_{01}^λ in our notation) in the special case when $P = \Delta_n$, after the identification $\text{gr.aut}_R(P) = \text{GL}_{n+1}(R)$. Accordingly, the automorphisms of type e_v^λ are called *elementary*, and the group they generate in $\text{gr.aut}_R(P)$ is denoted by $\mathbb{E}_R(P)$.

In this way we have generalized the basic building blocks of higher K-theory of rings to the polytopal setting: general linear groups and their elementary subgroups. Actually, the real motivation for us to pursue the analogy has been the main result of [1]. It is the polytopal extension of the fact that an invertible matrix over a field can be diagonalized by elementary transformations on rows (or columns)—or, putting it in different words, the group SK_1 is trivial for fields.

Lemma 2.1. *Let R be a ring, P a polytope, and $v_1, \ldots, v_s$ pairwise different column vectors for P with the same base facet $F = P_{v_i}$, $i = 1, \ldots, s$. Then the mapping*

$$\varphi : (R, +)^s \to \text{gr.aut}_R(P), \quad (\lambda_1, \ldots, \lambda_s) \mapsto e_{v_1}^{\lambda_1} \circ \cdots \circ e_{v_s}^{\lambda_s},$$

is an embedding of groups. In particular, $e_{v_i}^{\lambda_i}$ and $e_{v_j}^{\lambda_j}$ commute for all $i, j \in \{1, \ldots, s\}$, and the inverse of $e_{v_i}^{\lambda_i}$ is $e_{v_i}^{-\lambda_i}$.

In the special case, when R is a field the homomorphism φ is an injective homomorphisms of algebraic groups.

This lemma is proved in [1, Lemma 3.1] for fields $R = k$ (where we use the notation $\Gamma_k(P)$ for $\text{gr.aut}_k(P)$), but the general case makes absolutely no difference.

The image of the embedding φ given by Lemma 2.1 is denoted by $\mathbb{A}(F)$. Of course, $\mathbb{A}(F)$ may consist only of the identity map of $R[P]$, namely if there is no column vector with base facet F. In the case in which P is the unit simplex and $R[P]$ is the polynomial ring, $\mathbb{A}(F)$ is the subgroup of all matrices in $\text{GL}_{\dim P + 1}(R)$ that differ from the identity matrix only in the non-diagonal entries of a fixed column.

For the rest of this subsection we assume that k is a field and set $n = \dim P$.

After $\mathbb{A}(F)$ we introduce some further subgroups of $\text{gr.aut}_k(P)$. First, the $(n+1)$-torus $\mathbb{T}_{n+1} = (k^*)^{n+1}$ acts naturally on $k[P]$ by restriction of its action on $k[\mathbb{Z}^{n+1}]$ that is

given by

$$(\xi_1, \ldots, \xi_{n+1})(e_i) = \xi_i e_i, \quad i \in [1, n+1];$$

here e_i is the ith standard basis vector of $\mathbb{Z}^{n+1}$. This gives rise to an algebraic embedding $\mathbb{T}_{n+1} \subset \mathrm{gr.aut}_k(P)$, and we will identify $\mathbb{T}_{n+1}$ with its image. It consists precisely of those automorphisms of $k[P]$ that multiply each monomial by a scalar from k^*.

Second, the automorphism group $\Sigma(P)$ of the semigroup S_P is in a natural way a finite subgroup of $\mathrm{gr.aut}_k(P)$. It is the group of integral affine transformations mapping P onto itself.

Third, we have to consider a subgroup of $\Sigma(P)$ defined as follows. Assume v and $-v$ are both column vectors. Then for every point $x \in P \cap \mathbb{Z}^n$ there is a unique $y \in P \cap \mathbb{Z}^n$ such that $\mathrm{ht}_v(x, 1) = \mathrm{ht}_{-v}(y, 1)$ and $x - y$ is parallel to v. The mapping $x \mapsto y$ gives rise to a semigroup automorphism of S_P: it 'inverts columns' that are parallel to v. It is easy to see that these automorphisms generate a normal subgroup of $\Sigma(P)$, which we denote by $\Sigma(P)_{\mathrm{inv}}$.

Finally, $\mathrm{Col}(P)$ is the set of column structures on P. Now the main result of [1] is:

Theorem 2.2. *Let P be an n-dimensional polytope and k a field.*

(a) *Every element $\gamma \in \mathrm{gr.aut}_k(P)$ has a (not uniquely determined) presentation*

$$\gamma = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r \circ \tau \circ \sigma,$$

where $\sigma \in \Sigma(P)$, $\tau \in \mathbb{T}_{n+1}$, and $\alpha_i \in \mathbb{A}(F_i)$ such that the facets F_i are pairwise different and $\#(F_i \cap \mathbb{Z}^n) \leqslant \#(F_{i+1} \cap \mathbb{Z}^n)$, $i \in [1, r-1]$.

(b) *For an infinite field k the connected component of unity $\mathrm{gr.aut}_k(P)^0 \subset \mathrm{gr.aut}_k(P)$ is generated by the subgroups $\mathbb{A}(F_i)$ and $\mathbb{T}_{n+1}$. It consists precisely of those graded automorphisms of $k[P]$ which induce the identity map on the divisor class group of the normalization of $k[P]$.*

(c) $\dim \mathrm{gr.aut}_k(P) = \#\mathrm{Col}(P) + n + 1$.

(d) *One has $\mathrm{gr.aut}_k(P)^0 \cap \Sigma(P) = \Sigma(P)_{\mathrm{inv}}$ and*

$$\mathrm{gr.aut}_k(P)/\mathrm{gr.aut}_k(P)^0 \approx \Sigma(P)/\Sigma(P)_{\mathrm{inv}}.$$

Furthermore, if k is infinite, then $\mathbb{T}_{n+1}$ is a maximal torus of $\mathrm{gr.aut}_k(P)$.

3. Stable groups of elementary automorphisms and Polyhedral K_2

3.1. Product of column vectors

The notion of product of two column vectors $u, v \in \mathrm{Col}(P)$ has been introduced in [5, Definition 3.2]: we say that the product uv exists if $u + v \neq 0$ and for every point $x \in L_P \backslash P_u$ the condition $x + u \notin P_v$ holds. In this case, we define the product as $uv = u + v$.

Fig. 2 shows a polytope with all its column vectors and the two existing products $w = uv$ and $u = w(-v)$.
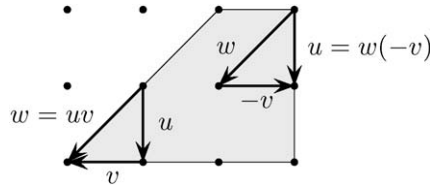
Fig. 2. The product of two column vectors.

In the case of a unimodular simplex the product of two oriented edges, viewed as column vectors, exists if and only if they are not opposite to each other and the end point of the first edge is the initial point of the second edge.

The basic properties of such products are given by Proposition 3.3 and Corollary 3.4 in [5]. For easier reference we summarize these properties in the following

Proposition 3.1. *Let P be a polytope and $u, v, w \in \mathrm{Col}(P)$. Then*:

(a) *uv exists if and only if $u + v \neq 0$ and $\langle P_v, u \rangle > 0$,*
(b) *uv exists if and only if $u + v \in \mathrm{Col}(P)$ and $P_{u+v} = P_u$,*
(c) *if uv exists then $P_{uv} = P_u$ and v is parallel to P_u (i.e. $\langle P_u, v \rangle = 0$),*
(d) *$u + v \in \mathrm{Col}(P)$ if and only if exactly one of the products uv and vu exists,*
(e) *if both uv and vw exist and $u + v + w \neq 0$ then the products $(uv)w$ and $u(vw)$ also exist and, clearly, $(uv)w = u(vw)$,*
(f) *If vw and $u(vw)$ exist and $u + v \neq 0$ then uv also exists. On the other hand, the existence of uv and $(uv)w$ does in general not imply the existence of vw, even if $v + w \neq 0$.*
(g) *The following are equivalent for $v \in \mathrm{Col}(P)$:*
   (1) *$-v \in \mathrm{Col}(P)$;*
   (2) *there exist facets F, G of P such that $\langle F, v \rangle = -1$, $\langle G, v \rangle = 1$, and $\langle H, v \rangle = 0$ for every facet $H \neq F, G$;*
   (3) *there exists $w \in \mathrm{Col}(P)$ and facets F, G such that $\langle F, v \rangle = -1$, $\langle G, v \rangle > 0$, $\langle F, w \rangle > 0$, $\langle G, w \rangle = -1$.*
(h) *If $w = uv$ and $-w \in \mathrm{Col}(P)$, then $-u, -v \in \mathrm{Col}(P)$ as well.*

The basic observation, already mentioned, for the proof of the proposition is that $v \in \mathbb{Z}^n$ belongs to $\mathrm{Col}(P)$ if and only if there exists $F \in \mathbb{F}(P)$ such that

$$\langle F, v \rangle = -1 \quad \text{and} \quad \langle G, v \rangle \geqslant 0 \quad \text{for all } G \in \mathbb{F}(P), \ G \neq F.$$

3.2. Balanced polytopes

A polytope P is called *balanced* if $\langle P_u, v \rangle \leqslant 1$ for all $u, v \in \mathrm{Col}(P)$. One easily observes that P is balanced if and only if $|\langle P_u, v \rangle| \leqslant 1$ for all $u, v \in \mathrm{Col}(P)$.

The reason we introduce balanced polytopes is that the main result of [5] is only proved for this class of polytopes. However, it is not yet excluded that everything generalizes to arbitrary polytopes.

In order to help the reader visualize the class of balanced polytopes we recall the classification result in dimension 2. It uses the notion of projective equivalence: n-dimensional polytopes $P, Q \subset \mathbb{R}^n$ are called *projectively equivalent* if and only if P and Q have the same dimension, the same combinatorial type, and the faces of P are parallel translates of the corresponding ones of Q. An alternative definition in terms of *normal fans* is given in Section 6.5.

Recall the notation $\Delta_n = \mathrm{conv}(O, (1, \ldots, 0), \ldots, (0, \ldots, 1))$ for the unit n-simplex.

Theorem 3.2. *For a balanced polygon P there are exactly the following possibilities* (*up to integral-affine equivalence*):

(a) *P is a multiple of the unimodular triangle $P_a = \Delta_2$. Hence $\mathrm{Col}(P) = \{\pm u, \pm v, \pm w\}$ and the column vectors are subject to the obvious relations,*

(b) *P is projectively equivalent to the trapezoid $P_b = \mathrm{conv}((0,0), (0,2), (1,1), (0,1))$, hence $\mathrm{Col}(P) = \{u, \pm v, w\}$ and the relations in $\mathrm{Col}(P)$ are $uv = w$ and $w(-v) = u$,*

(c) *$\mathrm{Col}(P) = \{u, v, w\}$ and $uv = w$ is the only relation,*

(d) *$\mathrm{Col}(P)$ has any prescribed number of column vectors, they all have the same base edge (clearly, there are no relations between them),*

(e) *P is projectively equivalent to the unit lattice square P_e, hence $\mathrm{Col}(P) = \{\pm u, \pm v\}$ with no relations between the column vectors,*

(f) *$\mathrm{Col}(P) = \{u, v\}$ so that $P_u \neq P_v$ with no relations in $\mathrm{Col}(P)$.*

It turns out that polyhedral K-groups are invariants of the projective equivalence classes of polytopes (in arbitrary dimension); see Lemma 6.8 below.
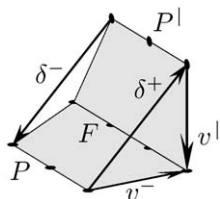
3.3. Doubling along a facet

Let $P \subset \mathbb{R}^n$ be a polytope and $F \subset P$ be a facet. For simplicity we assume that $0 \in F$, a condition that can be satisfied by a parallel translation of P. Denote by $H \subset \mathbb{R}^{n+1}$ the n-dimensional linear subspace that contains F and whose normal vector is perpendicular to that of $\mathbb{R}^n = \mathbb{R}^n \oplus 0 \subset \mathbb{R}^{n+1}$ (with respect to the standard scalar product on $\mathbb{R}^{n+1}$). Then the upper half space $H \cap (\mathbb{R}^n \times \mathbb{R}_+)$ contains a congruent copy of P which differs from P by a $90°$ rotation. Denote the copy by $P^{|F}$, or just by $P^|$ if there is no danger of confusion.

Note that $P^|$ is not always a lattice polytope with respect to the standard lattice $\mathbb{Z}^{n+1}$. However, it is so with respect to the sublattice $(\mathbb{Z}^n)^{|F}$ which is the image of $\mathbb{Z}^n$ under the $90°$ rotation.

The operator of doubling along a facet is then defined by

$$P^{\lrcorner F} = \mathrm{conv}(P, P^|) \subset \mathbb{R}^{n+1}.$$

The doubled polytope is a lattice polytope with respect to the subgroup $(\mathbb{Z}^n)^{\lrcorner F} = \mathbb{Z}^n + (\mathbb{Z}^n)^{|F} \subset \mathbb{R}^{n+1}$. After a change of basis in $\mathbb{R}^{n+1}$ that does not affect $\mathbb{R}^n$ we can replace $(\mathbb{Z}^n)^{\lrcorner F}$ by $\mathbb{Z}^{n+1}$, and consider $P^{\lrcorner F}$ as an ordinary lattice polytope in $\mathbb{R}^{n+1}$. In what follows, whenever we double a lattice polytope $P \subset \mathbb{R}^n$ along a facet F, the

Fig. 3. Doubling along the facet F.

lattice of reference in $\mathbb{R}^{n+1}$ is always $\mathbb{Z}^n + (\mathbb{Z}^n)^{|_F}$. For simplicity of notation this lattice will be denoted by $\mathbb{Z}^{n+1}$.

Sometimes we will refer to P as P^-. For an object z, associated to P (say, a lattice point or a column vector), $z^|$ will denote the corresponding object in $P^|$, presuming the facet F is clear from the context (Fig. 3).

In case $F = P_v$ for some $v \in \mathrm{Col}(P)$ we will use the notation $P^{\lrcorner F} = P^{\lrcorner v}$.

The polytope $P^{\lrcorner F}$ has two distinguished column vectors, which are the lattice unit vectors in $\mathbb{Z}^{n+1}$ parallel to the lines connecting the points $x^- \in \mathrm{L}_{P^-} \backslash F$ with the corresponding points $x^| \in \mathrm{L}_{P^|}$. The column vector of these two, which has $P^|$ as the base facet, will be denoted by δ^+, and δ^- will refer to the other vector. In particular, $P^{\lrcorner F}_{\delta^-} = P^-$. Clearly $\delta^- = -\delta^+$.

Let $\mathbb{F}(P)$ denote the set of facets of P. We have the bijective mapping

$$\Psi : \mathbb{F}(P) \cup \{P\} \to \mathbb{F}(P^{\lrcorner F})$$

defined by

$$\Psi(G) = \begin{cases} \mathrm{conv}(G^-, G^|) & \text{if } G \in \mathbb{F}(P) \backslash \{F\}, \\ P^| & \text{if } G = F, \\ P & \text{if } G = P. \end{cases}$$

The following equations are easily observed:

$$\langle \Psi(G), \delta^+ \rangle = \langle \Psi(G), \delta^- \rangle = 0 \quad \text{for all } G \in \mathbb{F}(P) \backslash \{F\}, \tag{2}$$

$$\langle P^-, \delta^+ \rangle = \langle P^|, \delta^- \rangle = 1, \tag{3}$$

$$\langle G, z \rangle = \langle \Psi(G), z \rangle \quad \text{for all } z \in \mathbb{Z}^n, \ G \in \mathbb{F}(P). \tag{4}$$

(In Eq. (3) the pairings are considered for P and $P^{\lrcorner}$ respectively and $\mathbb{Z}^n$ is thought of as the subgroup $\mathbb{Z}^n \oplus 0 \subset \mathbb{Z}^{n+1}$.)

Lemma 3.3. *Let $F \subset P$ be a facet and $v \in \mathrm{Col}(P)$. Then*:

(a) $v \in \mathrm{Col}(P^{\lrcorner F})$,
(b) *in* $\mathrm{Col}(P^{\lrcorner v})$ *the following equations hold*:

$$v = v^- = \delta^+ v^|, \quad v^| = \delta^- v^-,$$

(c) *if P is balanced then $P^{\lrcorner F}$ is also balanced and*

$$\operatorname{Col}(P^{\lrcorner F}) = \operatorname{Col}(P)^- \cup \operatorname{Col}(P)^| \cup \{\delta^+, \delta^-\}.$$

See [5, Lemmas 4.1, 5.1, Corollary 4.2].

3.4. The stable group of elementary automorphisms

An ascending infinite chain of lattice polytopes $\mathfrak{P} = (P = P_0 \subset P_1 \subset \ldots)$ is called a *doubling spectrum* if the following conditions hold:

 (i) for every $i \in \mathbb{Z}_+$ there exists a column vector $v \subset \operatorname{Col}(P_i)$ such that $P_{i+1} = P_i^{\pm v}$,
 (ii) for every $i \in \mathbb{Z}_+$ and any $v \in \operatorname{Col}(P_i)$ there is an index $j \geqslant i$ such that $P_{j+1} = P_j^{\pm v}$.

Here we use the inclusion $\operatorname{Col}(P_i) \subset \operatorname{Col}(P_{i+1})$, a consequence of Lemma 3.3(a).

One says that $v \in \operatorname{Col}(P_i)$ is *decomposed* at the jth step in $\mathfrak{P}$ for some $j \geqslant i$ if $P_{j+1} = P_j^{\pm v}$. By [5, Lemma 7.2] one has

Lemma 3.4. *Every column vector, showing up in a doubling spectrum, gets decomposed infinitely many times.*

Associated to a doubling spectrum $\mathfrak{P}$ is the 'infinite polytopal' algebra

$$R[\mathfrak{P}] = \lim_{i \to \infty} R[P_i]$$

and the filtered union

$$\operatorname{Col}(\mathfrak{P}) = \lim_{i \to \infty} \operatorname{Col}(P_i).$$

The product of two vectors from $\operatorname{Col}(\mathfrak{P})$ is defined in the obvious way, using the definition for a single polytope. Also, we can speak of systems of elements of $\operatorname{Col}(\mathfrak{P})$ having the same base facets, etc.

Elements $v \in \operatorname{Col}(\mathfrak{P})$ and $\lambda \in R$ give rise to a graded automorphism of $R[\mathfrak{P}]$ as follows: we choose an index i big enough so that $v \in \operatorname{Col}(P_i)$. Then the elementary automorphisms $e_v^\lambda \in \mathbb{E}_R(P_j)$, $j \geqslant i$, constitute a compatible system and, therefore, define a graded automorphism of $R[\mathfrak{P}]$. This automorphism will also be called 'elementary' and it will be denoted by e_v^λ.

The group $\mathbb{E}(R, \mathfrak{P})$ is by definition the subgroup of $\operatorname{gr.aut}_R(R[\mathfrak{P}])$, generated by all elementary automorphisms.

The next result comprises Propositions 7.3 and 7.4 and Theorem 7.6 from [5].

Theorem 3.5. *Let R be a ring and P be a polytope (not necessarily balanced) admitting a column structure. Assume $\mathfrak{P} = (P \subset P_1 \subset P_2 \subset \ldots)$ is a doubling spectrum. Then*:

(a) $\mathbb{E}(R, \mathfrak{P})$ *is naturally isomorphic to* $\mathbb{E}(R, \mathfrak{Q})$ *for any other doubling spectrum* $\mathfrak{Q} = (P \subset Q_1 \subset Q_2 \subset \ldots)$.
(b) $\mathbb{E}(R, \mathfrak{P})$ *is perfect.*

(c) *The center of $\mathbb{E}(R, \mathfrak{P})$ is trivial.*
(d) $e_u^\lambda \circ e_u^\mu = e_u^{\lambda+\mu}$ *for every* $u \in \mathrm{Col}(\mathfrak{P})$ *and* $\lambda, \mu \in R$.
(e) *If P is balanced,* $u, v \in \mathrm{Col}(\mathfrak{P})$, $u + v \neq 0$ *and* $\lambda, \mu \in R$ *then*

$$[e_u^\lambda, e_v^\mu] = \begin{cases} e_{uv}^{-\lambda\mu} & \text{if } uv \text{ exists,} \\ 1 & \text{if } u + v \notin \mathrm{Col}(\mathfrak{P}). \end{cases}$$

The difficult parts of this theorem are the claims (c) and (e), which in the special case $P = \Delta_n$ are just standard facts.

Thanks to Theorem 3.5(a) we can use the notation $\mathbb{E}(R, P)$ for $\mathbb{E}(R, \mathfrak{P})$.

Remark 3.6. Theorem 3.5(e) is the generalization of Steinberg's relations between elementary matrices to balanced polytopes. In order to find the classical Steinberg relation $[e_{ij}^\lambda, e_{jk}^\mu] = e_{ik}^{\lambda\mu}$ in this equality one must observe that in our setting the configuration $e_{ij}e_{jk}$ corresponds to the existence of vu if we associate with e_{ij} the column vector $\varepsilon_i - \varepsilon_j$ where $\varepsilon_1, \ldots, \varepsilon_n$ are the vectors of the canonical basis of $\mathbb{R}^n$, and simultaneously the vertices of Δ_{n-1}.

That we associate $\varepsilon_i - \varepsilon_j$ with e_{ij} (and not $\varepsilon_j - \varepsilon_i$) is forced by our notation in which we add column vectors on the right. Thus the successive addition of first u and then v corresponds to the product uv.

Remark 3.7. One can define the group $\mathbb{E}(R, P)$ using sequences of polytopes $\mathfrak{P}' = (P = P_0' \subset P_1' \subset \cdots)$ that are more general than doubling spectra. In particular, suppose that $\mathfrak{P} = (P = P_0 \subset P_1 \subset \cdots)$ is a doubling spectrum for P and $\mathfrak{P}' = (P_0' \subset P_1' \subset \cdots)$ is a sequence of polytopes for which there exist isomorphisms $\varphi_i : P_i \to P_i'$ of polytopes that commute with the embeddings $P_i \subset P_{i+1}$ and $P_i' \to P_{i+1}'$. Then $\mathfrak{P}'$ need not be a doubling spectrum in the strict sense since condition (ii) is not invariant under isomorphisms as just described. However, there evidently exists a natural isomorphism $\mathbb{E}(R, \mathfrak{P}) \approx \mathbb{E}(R, \mathfrak{P}')$.

For instance, if we start from the unimodular simplex Δ_n, $n \in \mathbb{N}$, and consider the sequence $\mathfrak{P}' = (\Delta_n = P_0' \subset P_1' \subset \cdots)$, in which $P_{i+1}' = P_i'^{\Delta_v}$, $i \in \mathbb{N}$, for the same column vector $v \in \mathrm{Col}(\Delta_n)$, then the resulting sequence of unstable groups is naturally identified with the familiar sequence of groups of elementary matrices

$$E_{n+1}(R) \subset E_{n+2}(R) \subset \cdots, \quad * \mapsto \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}.$$

In particular, $\mathbb{E}(R, \Delta_n) = E(R)$ for all $n \in \mathbb{N}$.

One may ask why the stable group of elementary automorphisms is not defined as a filtered union of the unstable ones. The reason, as explained in the Remark 3.8(b) below, is that such a filtered union is in general *not* possible.

Remark 3.8. (a) In general the groups $\mathbb{E}_R(P)$ are not perfect. However, after finitely many steps in the doubling spectrum one arrives at a polytope P'' for which this group is perfect.
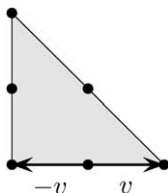
Fig. 4. The polytope $2\Delta_2$.

In fact, after finitely many steps each base facet in P has been used for a doubling, and in the polytope P' then constructed each base facet has an invertible column vector. This property is preserved under further doublings. In this situation it is easily observed that all the vectors δ^+ and δ^- that come up in doublings of P' are automatically decomposed, and after finitely many doublings starting from P' one arrives at a polytope P'' in which all the column vectors of P', and thus all of P'', are decomposed. In view of Theorem 3.5(e) this is enough for the desired perfectness. (The assertion holds for all polytopes.)

(b) The group $\mathbb{E}(R, P)$ is in general not the filtered union of the unstable groups $\mathbb{E}_R(P_i)$. Consider the simple example of the segment $2\Delta_1$. Then the second term in $\mathfrak{P}$ (for an arbitrary doubling spectrum, starting with P) can be identified with the triangle $2\Delta_2 = \mathrm{conv}((0,0),(2,0),(0,2)) \subset \mathbb{R}^2$ so that $2\Delta_1$ is the lower edge (see Fig. 4). Consider the vectors $v = (1,0)$ and $-v = (-1,0)$ from $\mathrm{Col}(2\Delta_1)$. Assume $2 \neq 0$ in R. Then the element $\varepsilon = (e_v^1 \circ e_{-v}^{-1} \circ e_v^1)^2 \in \mathbb{E}(R, \Delta_2)$ is not the identity automorphism of $R[\mathfrak{P}]$ (it switches signs on the second horizontal layer of $2\Delta_2$) whereas the restriction of ε to $R[2\Delta_1]$ is the identity automorphism. In particular there is no natural group homomorphism $\mathbb{E}_R(2\Delta_1) \to \mathbb{E}(R, 2\Delta_2)$.

(c) On the other hand, every element $e \in \mathbb{E}_R(P_i)$, $i \in \mathbb{Z}_+$ is a restriction to $R[P_i]$ of some element of $\mathbb{E}(R, P)$ and every element $\varepsilon \in \mathbb{E}(R, P)$ restricts to an element of $E_R(P_i)$ whenever i is big enough. Clearly, we have the following approximation principle: if two elements $\varepsilon, \varepsilon' \in \mathbb{E}(R, P)$ restrict to the same elements of $\mathbb{E}_R(P_i)$ for all sufficiently large i then $\varepsilon = \varepsilon'$.

(d) Unlike the group $\mathbb{E}(R, P)$, the Steinberg group $\mathbb{St}(R, P)$, to be introduced in Section 3.5, is the direct limit of the corresponding unstable groups.

3.5. The Schur multiplier

Let P be a balanced polytope and $\mathfrak{P} = (P \subset P_1 \subset P_2 \subset \ldots)$ be a doubling spectrum. Then for a ring R we define the *stable polytopal Steinberg group* $\mathbb{St}(R, P)$ as the group generated by symbols x_v^λ, $v \in \mathrm{Col}(\mathfrak{P})$, $\lambda \in R$, which are subject to the relations

$$x_v^\lambda x_v^\mu = x_v^{\lambda+\mu}$$

and

$$[x_u^\lambda, x_v^\mu] = \begin{cases} x_{uv}^{-\lambda\mu} & \text{if } uv \text{ exists,} \\ 1 & \text{if } u + v \notin \mathrm{Col}(\mathfrak{P}) \cup \{0\}. \end{cases}$$

The use of the notation $\mathbb{St}(R, P)$ is justified by the fact that, like in Theorem 3.5(a), the stable Steinberg groups are determined by the underlying doubling spectra (with the same initial polytope) up to canonical isomorphism. Clearly, $\mathbb{St}(R, P)$ is a perfect group.

It follows from Remark 3.6 that for every $n \in \mathbb{N}$ we have $\mathbb{St}(R, \varDelta_n) = \mathrm{St}(R)$—the usual Steinberg group.

Lemma 3.9. *Let $v_1, \ldots, v_s$ be pairwise different elements of $\mathrm{Col}(\mathfrak{P})$ with the same base facet. Then the mapping*

$$R^s \to \mathbb{St}(R, P), \quad (\mu_1, \ldots, \mu_s) \mapsto x_{v_1}^{\mu_1} \cdots x_{v_s}^{\mu_s}$$

is a group isomorphism.

This follows from the Claim in the proof of [5, Proposition 8.2].

Remark 3.10. One can introduce (as we do in [5]) the notion of 'unstable' Steinberg polytopal group. The sequence of such groups, associated to the members of the spectrum $\mathfrak{P}$, then forms an inductive system of groups whose limit is $\mathbb{St}(R, P)$.

The central result of [5] is the following

Theorem 3.11. *For a ring R and a balanced polytope P the natural surjective group homomorphism $\mathbb{St}(R, P) \to \mathbb{E}(R, P)$ is a universal central extension whose kernel coincides with the center of $\mathbb{St}(R, P)$.*

See Proposition 8.2 and Theorem 8.4 in [5].

The group $\mathrm{Ker}(\mathbb{St}(R, P) \to \mathbb{E}(R, P))$ is called the *polyhedral Milnor group*. We denote it by $K_2(R, P)$.

In [5] we have developed the notion of polytopal Steinberg group for arbitrary lattice polytopes. However, one should note that the proof of Theorem 3.11, as presented in [5], uses the fact that P is balanced in a crucial way.

4. Rigid systems of column vectors

The triangular subgroups of $\mathrm{GL}(n, R)$ on which Volodin's K-theory is based are defined in terms of partial orders on $\{1, \ldots, n\}$ (see Section 6.2). In this section we develop a polyhedral analogue in terms of column vectors, called rigid and Y-rigid systems. We start with basic properties of long products of column vectors.

It follows from Proposition 3.1(e) that we can speak of the product $\prod_{i=1}^{m} v_i$ of elements $v_i \in \mathrm{Col}(P)$ whenever the following two conditions are satisfied:

 (i) the products $v_i v_{i+1}$ exist for all $i \in [1, m-1]$,
(ii) $\sum_{i=r}^{s} v_i \neq 0$ for all $1 \leqslant r < s \leqslant m$.

In this case every bracket structure on the sequence $v_1 v_2 \dots v_m$ yields pairs of column vectors whose product exist.

Proposition 3.1(c) implies that $v_j \| P_{v_i}$ whenever $i < j$, and so $\mathrm{rank}_{\mathbb{Q}}(v_1, \dots, v_m) = m$:

Lemma 4.1. *If $\prod_{i=1}^{m} v_i$ exists, then $v_1, \dots, v_m$ are linearly independent. In particular, $\sum_I v_i \neq 0$ for all subsets $I \subset [1, m]$, and $v_1, \dots, v_m$ are pairwise different column vectors.*

Obviously, if $\prod_{i=1}^{m} v_i$ exists, then $\prod_{i=r}^{s} v_i$ exists as well for all r, s, $1 \leqslant r < s \leqslant m$. For a system of column vectors $V \subset \mathrm{Col}(P)$ we set

$$[V] = \{v \in \mathrm{Col}(P) : \text{ there exist } v_1, \dots, v_m \text{ with } v = v_1 \cdots v_m\}.$$

It is useful to have another, weaker notion of product. We say that $\prod_{i=1}^{m} v_i$ exists *weakly* if there is a bracket structure on the sequence

$$v_1 v_2 \cdots v_m$$

such that all the recursively defined products of pairs of column vectors exist. Since $v_1 \cdots v_n = v_1 + \cdots + v_n$ in the case of weak existence, the value of the product does not depend on the bracket structure.

It follows from Proposition 3.1(f) that $\prod_{i=1}^{m} v_i$ exists if and only if

$$(v_1(v_2(v_3 \dots (v_{m-3}(v_{m-2}(v_{m-1}v_m)))\dots))).$$

exists and $\sum_{i=r}^{s} v_i \neq 0$ for all r, s, $1 \leqslant r < s \leqslant m$.

By $\langle V \rangle$ we denote the hull of V in $\mathrm{Col}(P)$ under products (of two column vectors). One has $v \in \langle V \rangle$ if and only if there exist $v_1, \dots, v_m \in V$ such that $v = v_1 \cdots v_m$ is their weak product.

Clearly $\langle \langle V \rangle \rangle = \langle V \rangle$, but in general $[[V]] \neq [V]$. In fact, $[[V]] = [V]$ if and only $[V] = \langle V \rangle$. (A simple example for $[[V]] \neq [V]$ will be discussed in Remark 4.3(b).)

Both $\langle V \rangle$ and $[V]$ carry an *associative* partial product structure. However, the partial product structure on $[V]$ is not always the restriction of that on $\mathrm{Col}(P)$. For $w_1, w_2 \in [V]$ the product may exist in $\mathrm{Col}(P)$, but it need not belong to $[V]$ if $[V] \neq \langle V \rangle$.

For simplicity we introduce the following convention: $v_1 \cdots v_m \in [V]$ means that the product of $v_1, \dots, v_m$ exists (in the strong sense), whereas $v_1 \cdots v_m \in \langle V \rangle$ means that the product of $v_1, \dots, v_m$ exists in the weak sense.

We will represent certain partial product structures on sets of column vectors by equivalence classes of directed paths in graphs. The *graphs* considered by us will always be finite directed graphs **G** satisfying the following conditions:

(i) **G** has no isolated vertices;
(ii) **G** has no multiple edges and no edges from a vertex to itself;
(iii) if vertices a and b are connected by an edge, then there is no other directed path connecting a and b.

Condition (iii) implies that there are no directed cycles in **G** (but the existence of non-directed cycles is not excluded). A *path* is always assumed to be oriented.

Fig. 5.

By definition, a Y-*graph* is a graph **F** that at each vertex a satisfies the following condition:

(Y) a is the end point of at most one edge of **F**.

In other words, if we direct all edges upwards, then branching is only allowed in the form of a Y (with any numbers of 'arms').

The set of nonempty paths in a graph **F** carries a natural partial product structure—ll' exists if the end point of the path l is the initial point for l'. The set of all paths in **F** is denoted by path(**F**). There is an equivalence relation on path **F**: two paths are considered to be equivalent if they have the same initial and the same end point. We let $\overline{\text{path}}\,\textbf{F}$ denote the corresponding quotient set. Thus for Y-graphs (or more generally, for graphs without non-oriented cycles) we have $\overline{\text{path}}\,\textbf{F} = \text{path}\,\textbf{F}$. The aforementioned partial product operation on path **F** induces a partial product operation on $\overline{\text{path}}\,\textbf{F}$. We write $\overline{\text{path}}\,\textbf{F} = \text{path}\,\textbf{F}$ in order to indicate that every equivalence class contains exactly one path.

In the following a vertex a of **F** is called *terminal* if there is no edge with initial vertex a.

Definition 4.2. A system of column vectors $V \subset \text{Col}(P)$ is called *rigid* if the following conditions are satisfied:

(a) $[V]$ does not contain a subset of type $\{v, -v\}$, $v \in \text{Col}(P)$;
(b) $[V] = \langle V \rangle$;
(c) there exist a graph **F** and an isomorphism $[V] \approx \overline{\text{path}}\,\textbf{F}$ of partial product structures.

Furthermore, V is called a Y-*rigid system* if **F** is a Y-graph.

The graph **F** and the isomorphism $[V] \approx \overline{\text{path}}\,\textbf{F}$ in Definition 4.2(c) are *not* part of the data defining a rigid system. We only require their existence. In general, V does not uniquely determine the graph **F** (see Remark 4.3(a)). A graph satisfying condition (c) will be referred to as a graph that *supports* the rigid system V, or a graph *associated* to V. Moreover, whenever a graph **F** is associated to a rigid system V it is implicitly assumed that we have also fixed an isomorphism $[V] \approx \overline{\text{path}}\,\textbf{F}$ as above.

Remark 4.3. (a) It is easy to find examples of rigid systems for which the associated graph **F** is not uniquely determined by V, not even if V happens to be Y-rigid. For instance, when V consists of two vectors u and v such that $u+v \neq 0$ and $u+v \notin \text{Col}(P)$, then V is a Y-rigid system, and the graphs **G**, **F** and **F**′ in Fig. 5 support it. Two of them, **F** and **F**′, are non-isomorphic Y-graphs. On the other hand, if V is Y-rigid, then the corresponding Y-graph is unique if we additionally require that there is only one
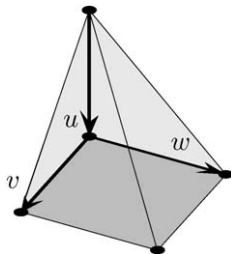
Fig. 6. The pyramid over the unit square.

edge leaving each of its roots (vertices without an entering edge). However, we will not make such a requirement.

(b) Although a set of vectors can be arranged geometrically as a graph, this does not imply the rigidity of the system. Consider the balanced polytope

$$P = \text{conv}((0,0,0),(1,0,0),(0,1,0),(1,1,0),(0,0,1)) \subset \mathbb{R}^3$$

and the system V of its column vectors $u = (0,0,-1)$, $v = (1,0,0)$ and $w = (0,1,0)$; see Fig. 6. Let $\mathbf{F}$ be the Y-graph determined by the three vectors u, v and w as shown in the figure. Then $[V] \approx \text{path} \, \mathbf{F}$ but $(uv)w \in \langle V \rangle \setminus [V]$.

Let $V \subset \text{Col}(P)$. The elements w of V that have no decomposition $w = uv$ with $u, v \in \langle V \rangle$ are called the *irreducible* or *indecomposable* elements of V.

In the next proposition we collect some properties of rigid systems, among them the decomposition into irreducible elements.

Proposition 4.4. *Let* $V \subset \text{Col}(P)$ *be a rigid system and* $\mathbf{F}$ *a graph associated to* V. *Then the following hold*:

(a) $[[V]] = [V]$.
(b) *The product* $v_1 \cdots v_n$ *of elements* $v_i \in [V]$ *exists if* (*and only if*) *it exists weakly.*
(c) V *is* Y-*rigid if and only if* $\mathbf{F}$ *satisfies the condition* (Y) *at all its nonterminal vertices* c, *and* $\text{path} \, \mathbf{F} = \overline{\text{path} \, \mathbf{F}}$.
(d) *Every* $w \in [V]$ *has a decomposition into irreducible elements. The irreducible elements are those represented by edges of* $\mathbf{F}$. *The decomposition is unique if* V *is* Y-*rigid.*
(e) *Let* $v_1, \ldots, v_n$, $n \geqslant 1$, *be arbitrary elements of* $[V]$. *Then* $\sum_{i=1}^{n} v_i \neq 0$.

Proof. (a) follows immediately from $[V] = \langle V \rangle$.

(b) For the partial product structures $\text{path} \, \mathbf{F}$ and $\overline{\text{path} \, \mathbf{F}}$ we can define strongly and weakly existing products as for column vectors. In contrast to $\text{Col}(P)$, strong and weak existence are evidently equivalent in these partial product structures. Therefore they must be equivalent in a partial product structure isomorphic to $\overline{\text{path} \, \mathbf{F}}$. (Note that (b) does not necessarily hold for an arbitrary subset V of $\text{Col}(P)$ that just satisfies the equation $\langle V \rangle = [V]$: for example, it is in general false for $V = \text{Col}(P)$ itself.)

(c) Suppose that $\mathbf{F}$ satisfies (Y) at all nonterminal vertices. Two equivalent paths with the same initial points and endpoints would necessarily end in a terminal vertex c of $\mathbf{F}$.

If such paths do not exist, we can replace the edges ending in c by edges with separate endpoints without changing the path structure. The resulting graph is a Y-graph. Conversely, if such paths exist, then it is impossible to find a Y-graph $\mathbf{F}'$ with $\overline{\mathrm{path}\,\mathbf{F}'} \approx \overline{\mathrm{path}\,\mathbf{F}}$.

(d) The existence of the decomposition is clear since every element in $[V]$ has a longest representation as a product $v_1 \cdots v_n$. Its factors must be irreducible. The irreducible elements correspond to the edges of $\mathbf{F}$ since (in our class of graphs) no edge is equivalent to a path of length $\geqslant 2$. The uniqueness of the decomposition in Y-rigid systems follows from the fact that a path in a Y-graph is uniquely determined by its initial and endpoint.

(e) Suppose that $\sum_{i=1}^n v_i = 0$. We can assume that $v_1 \cdots v_m$ is a longest possible product that can be formed from the vectors $v_1, \ldots, v_n$. Then $w = v_1 \cdots v_m$ is a column vector. Let F be its base facet. Among the remaining vectors $v_{m+1}, \ldots, v_n$ there must be one, say v_{m+1}, with $\langle F, v_{m+1} \rangle > 0$: note that $\sum_{i=1}^n \langle F, v_i \rangle = 0$, but $\sum_{i=1}^m \langle F, v_i \rangle = \langle F, v_1 \cdots v_m \rangle < 0$.

It is impossible that $v_{m+1} = -w$. So $v_{m+1}(v_1 \cdots v_m)$ exists weakly. By (b) it exists also in the strong sense, and we obtain a contradiction to the choice of $v_1, \ldots, v_m$. $\square$

The basic examples of rigid systems are provided by the unit simplices and their edge vectors:

Example 4.5. The set of column vectors $\mathrm{Col}(\varDelta_n)$ of the unit n-simplex $\varDelta_n$ coincides with the set $\{a - b : a$ and b are different vertices of $\varDelta_n\}$. We can think of these column vectors as oriented edges of $\varDelta_n$ from b to a.

(a) Then uv exists for column vectors u and v if and only if they form a broken line of length 2. More generally, for every system $\{v_1, \ldots, v_m\} \subset \mathrm{Col}(\varDelta_n)$ the following conditions are equivalent:

(i) $\prod_{i=1}^m v_i$ exists,
(ii) $\{v_1, \ldots, v_m\}$ is a Y-rigid system and its underlying Y-graph is the linear directed graph of length m:

$$\bullet \xrightarrow{1} \bullet \xrightarrow{2} \bullet \cdots \bullet \xrightarrow{m} \bullet$$
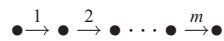
(iii) $[v_1 \ldots, v_m] \subset \partial\varDelta_n$ is a broken line without self-intersections (∂ denotes the boundary).

Observe that the weak existence of $\prod_{i=1}^m v_i$ together with $\sum_I v_i \neq 0$ for every subset $I \subset [1, m]$ is equivalent to the existence of $\prod_{i=1}^m v_i$ in the strong sense.

(b) The rigid systems in $\mathrm{Col}(\varDelta_n)$ are exactly the non-empty subsets V for which $\langle V \rangle$ does not contain a pair $\{v, -v\}$. By induction on m it follows easily from (a) that all products $v_1 \cdots v_m \in \langle V \rangle$ exist also in the strong sense, and if we take the longest
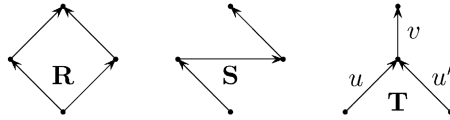
Fig. 7.

possible representation $w = v_1 \cdots v_m$ of $w \in \langle V \rangle$ with $v_i \in V$, $i \in [1, m]$, then the v_i must be irreducible. The graph $\mathbf{F}$ formed by the irreducible elements in the boundary of Δ_n then satisfies the condition $\overline{\mathrm{path}\, F} \approx [V]$. (If V is Y-rigid, then the graph just produced need not be Y-rigid; see Remark 4.3(a).)

(c) In preparation of Section 6.2 we note that the rigid systems V in $\mathrm{Col}(\Delta_n)$ can be identified with those partial orders on subsets X of the vertex set $\mathrm{vert}(\Delta_n)$ for which no $x \in X$ is simultaneously maximal and minimal. The set X corresponding to V consists of all vertices x of Δ_n such that there exists $y \in \mathrm{vert}(\Delta_n)$ for which $x - y \in [V]$ or $y - x \in [V]$, and it is partially ordered by the condition $x \leqslant y \Leftrightarrow y - x \in [V]$.

Among these partially ordered sets the Y-rigid systems in $\mathrm{Col}(\Delta_n)$ are characterized by the following two conditions:

$$a < c \text{ and } b < c \text{ for some } c \notin \max(X) \Rightarrow a \leqslant b \text{ or } b \leqslant a.$$

and

$$a < b, c \text{ and } b, c < d \in \max(X) \Rightarrow b \leqslant c \text{ or } c \leqslant b.$$

These conditions reflect Proposition 4.4(c).

(d) Since any finite poset can be augmented to a linear order, we conclude that for any rigid system $V \subset \mathrm{Col}(\Delta_n)$ there is a 'linear' Y-rigid system $W \subset \mathrm{Col}(\Delta_n)$ as in (ii) above such that $[V] \subset [W]$. Namely, we augment the partial order on $X = \{x_1, \ldots, x_m\}$ (as defined in (b)) to a linear one. we may assume the vertices are labelled such that $x_i < x_j$ if $i < j$, and choose $W = \{x_{i+1} - x_i : i = 1, \ldots, m - 1\}$. See Fig. 7 where $\mathbf{R}$ is the graph of a rigid system V and $\mathbf{S}$ is the graph of a 'linear' Y-rigid system W containing it.

Remark 4.6. In general it is not possible to embed a rigid system in $\mathrm{Col}(P)$ into a Y-rigid system. The pyramid P over the unit square (see Fig. 6) serves again as an example. The column vectors $u, u' = u + w, v$ form a rigid system V with associated graph $\mathbf{T}$ as shown in Fig. 7. A Y-rigid system $V' \supset V$ would have to contain a column vector t with $u' = tu$ or $u = tu'$, in other words, $t = w = u' - u \in V'$ or $t = -w \in V'$. However, $u' = uw$ and $u = u'(-w)$—the products exist in the wrong order.

As far as the partial product structure is concerned, every rigid system can be realized in a unit simplex:

Lemma 4.7. *Let P be a polytope and $V \subset \mathrm{Col}(P)$ a rigid system with associated graph* **F**. *Set $n = \#\mathrm{vert}(\mathbf{F}) - 1$.*

(a) *Then $[V]$ is isomorphic to a rigid system W in $\mathrm{Col}(\Delta_n)$ as a partial product structure.*
(b) *Every subset $U \subset [V]$ is a rigid system.*

Proof. (a) We identify the vertices of **F** with those of Δ_n, and let W' be the set of edge vectors of Δ_n corresponding to the edges of **F**. Then $W = \langle W' \rangle$ is the set of edge vectors in Δ_n that as a path are equivalent to some broken line formed by the elements of W'. None such broken line intersects itself: **F** has no directed cycles. Moreover it is impossible that $\{w, -w\} \subset W$ for some $w \in \mathrm{Col}(\delta_n)$—again we would have a directed cycle in **F**. Clearly W is a rigid system with associated graph **F**.

(b) Since $\{v, -v\} \not\subset [U]$ for any $v \in \mathrm{Col}(P)$, the rest is only a condition on the partial product structure of $[U]$. By (a) we can therefore assume $V \subset \Delta_n$, and then the claim follows from the observation in Example 4.5(b). $\quad\square$

Note that part (b) of Lemma 4.7 has no Y-version: in general a subset of a Y-rigid system need not be Y-rigid, as is clear from Example 4.5(d).

The construction of stable groups uses the doubling of polytopes. Therefore we must analyze how rigid systems can be extended to the doubled polytope. The result will be described in Lemma 4.9. First we formulate an auxiliary lemma that sheds more light on the structure of rigid (and somewhat weaker) systems V. It shows that the table of values $\langle F, v \rangle$ for such a system looks like that of a system of column vectors in a unit simplex, at least if one pays attention only to those facets F that appear as base facets of elements of V.

Lemma 4.8. *Let P be a polytope, $V \subset \mathrm{Col}(P)$, and $\mathscr{B}(V)$ be the set of the base facets of the vectors $v \in V$.*

(a) *Then the following are equivalent:*
  (i) *Whenever $v_1 \cdots v_n \in \langle V \rangle$ with $n \in \mathbb{N}$ and $v_1, \ldots, v_n \in V$, then $v_i v_{i+1}$ exists or $v_{i+1} = -v_i$ for each $i \in [1, n-1]$.*
  (ii) *For each $v \in V$ one has $\langle F, v \rangle = 0$ for all $F \in \mathscr{B}(V)$, $F \neq P_v$, with at most one exception G for which $\langle G, v \rangle = 1$.*
(b) *Suppose one of the conditions in (a) holds. For every product $w = v_1 \cdots v_n \in \langle V \rangle$ with $v_1, \ldots, v_n \in V$ there exists at most one facet $G \in \mathscr{B}(V)$ with $\langle G, w \rangle > 0$. Moreover, in the case of existence one has $\langle G, w \rangle = \langle G, v_n \rangle = 1$.*

The face $G \in \mathscr{B}(V)$ with $\langle G, w \rangle = 1$ (if it exists) is in some sense opposite to the base facet of w. Therefore we denote it by P_V^w.

Proof. In this proof and that of Lemma 4.9 we use Proposition 3.1 and Lemma 4.1 freely, without mentioning each application explicitly.

Note that the property in (ii) is automatically (and 'globally') satisfied for all column vectors v with $-v \in \mathrm{Col}(P)$; see Proposition 3.1(g).

We first prove (i) $\Rightarrow$ (ii). More generally than necessary, let $v \in \langle V \rangle$, $v = v_1 \cdots v_n$ with $v_1, \ldots, v_n \in V$, and suppose first that $\langle F, v \rangle \geqslant 2$ for some $F \in \mathcal{B}(V)$. Then $-v \notin \mathrm{Col}(P)$, and so $vw = (v_1 \cdots v_n)w$ exists for every $w \in V$ with $F = P_w$. Moreover, $(vw)w$ also exists (since $vw = -w$ is evidently impossible), but ww does not. This shows $\langle F, v \rangle \leqslant 1$ for all $F \in \mathcal{B}(V)$.

Next suppose that there exist different $F_1, F_2 \in \mathcal{B}(V)$ with $\langle F_1, v \rangle = \langle F_2, v \rangle = 1$ and choose $w_i \in V$, $i = 1, 2$, with $F_i = P_{w_i}$. (Again $-v \notin \mathrm{Col}(P)$.) Both vw_1 and vw_2 exist. One has $\langle F_2, vw_1 \rangle = \langle F_2, v \rangle + \langle F_2, w_1 \rangle \geqslant 1$ since $\langle F_2, v \rangle = 1$ and $\langle F_2, w_1 \rangle \geqslant 0$. We have already seen that $\langle F_2, vw_1 \rangle \leqslant 1$, and so $\langle F_2, w_1 \rangle = 0$. Thus $w_1 \neq -w_2$ and $w_1 w_2$ does not exist, but $(vw_1)w_2$ exists: it is impossible that $vw_1 = -w_2$, since $-v \in \mathrm{Col}(P)$ otherwise.

For (ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (b) we use induction on n. For $n = 1$ there is nothing to show in (i), and (b) is identical to (ii). Let $n > 1$. Then

$$w = v_1 \cdots v_n = w_1 w_2, \quad w_1 = v_1 \cdots v_m, \ w_2 = v_{m+1} \cdots v_n,$$

with $1 \leqslant m < n$ and $w_1, w_2 \in \langle V \rangle$. We can apply induction to each of the shorter products w_1 and w_2.

Clearly $F = P_{w_2} = P_{v_{m+1}} \in \mathcal{B}(V)$, and $\langle F, w_1 \rangle > 0$. By the induction hypothesis for (b) we have $F = P_V^{w_1} = P_V^{v_m}$ and $\langle F, w_1 \rangle = \langle F, v_m \rangle = 1$. By an application of the induction hypothesis to w_2 it follows immediately that among all the facets $G \in \mathcal{B}(V)$ there is at most one with $\langle G, w_2 \rangle > 0$, namely $P_V^{v_n}$ if it exists. This completes (b).

If $v_{m+1} \neq -v_m$, then the product $v_m v_{m+1}$ exists since $\langle F, v_m \rangle > 0$, and (i) is also complete. $\square$

Now we can describe how rigid systems can be extended under doublings of balanced polytopes along facets.

Lemma 4.9. *Suppose $V \subset \mathrm{Col}(P)$ is a rigid system and $v \in V$ is one of its irreducible elements. Then the system $V' = V \cup \{\delta^+, v^|\} \subset \mathrm{Col}(P^{\downarrow v})$ is also rigid. It is Y-rigid if V is so.*

Proof. Let $\mathbf{F}$ be V's underlying graph, and $E \in E(\mathbf{F})$ be the edge corresponding to the column vector v. According to the equation $v = \delta^+ v^|$ we replace E by a path $\bullet \to \bullet \to \bullet$ where the left edge represents δ^+, and the right $v^|$. Let $\mathbf{F}'$ be the extended graph.

The rigid system V has property (i) of Lemma 4.8(a) (in which the possibility $v_{i+1} = -v_i$ is excluded for V rigid). Let W consist of all the irreducible elements $w \neq v$ of V and the 'new' column vectors δ^+ and $v^|$. We have $\mathcal{B}(W) = \mathcal{B}(V) \cup \{P^-\}$, identifying each facet of P with its extension to $P^{\downarrow v}$ as given by the mapping Ψ in Section 3.3. Since V satisfies (ii) in Lemma 4.8(a), one sees immediately that W also satisfies it. In fact, each $u \in V$ is parallel to P^-, δ^+ is an invertible column vector, and if $\langle F, v^| \rangle > 0$, then $\langle F, v \rangle > 0$.

Thus Lemma 4.8 allows us to control the pairs $w_i w_{i+1}$ in weakly existing products $w_1 \cdots w_n \in \langle W \rangle = \langle V' \rangle$. By inspection of the base facets and the 'positive' facets with respect to W one sees that only the following types can occur for w_i, w_{i+1}:

$$w_i, w_{i+1} \in V \quad \text{and} \quad w_i w_{i+1} \text{ exists in } V,$$

$$w_i \in V, \ w_{i+1} = \delta^+ \quad \text{and} \quad w_i v \text{ exists in } V,$$

$$w_i = v^|, w_{i+1} \in V \quad \text{and} \quad v w_{i+1} \text{ exists in } V,$$

$$w_i = \delta^+, w_{i+1} = v^|.$$

Therefore each product $w_1 \cdots w_n \in \langle W \rangle$ represents a path in $\mathbf{F}'$.

Now we choose a product $w_1 \cdots w_n$ 'along' a path in $\mathbf{F}'$ and show that it exists strongly. It is enough to consider a maximal path since strong existence is inherited by segments. We must verify the existence of

$$w_i(w_{i+1} \cdots w_n), \quad i = 1, \ldots, n-1$$

and show that $\sum_{i=r}^{s} w_i \neq 0$ for all $r, s, \ 1 \leqslant r < s \leqslant m$.

If $w_1, \ldots, w_n \in V$, the strong existence follows from that in V. Otherwise $w_{i-1} = \delta^+$ and $w_i = v^|$ for exactly one i, whereas $w_j \in V$ for $j \neq i-1, i$.

Let us first take care of the conditions on sums over segments. If the segment contains none or both of δ^+ and $v^|$, then we are summing column vectors in V along a path of $\mathbf{F}$, and such a sum is necessarily non-zero. If the segment contains δ^+, but not $v^|$, then $\langle P^-, \sum_{i=r}^{s} w_i \rangle = 1$, and if it contains $v^|$, but not δ^+, then $\langle P^-, \sum_{i=r}^{s} w_i \rangle = -1$. In any case $\sum_{i=r}^{s} w_i \neq 0$. If we have to check the existence of $w_i(w_{i+1} \cdots w_n)$ in the following, then we can use that $w_i \neq -w_{i+1} \cdots w_n$, as just shown.

Clearly $w_{i+1} \cdots w_n \in [W]$ since it represents a path in $\mathbf{F}$ (unless it is empty).

If $i = n$, there is nothing to show for the existence of $w_i \cdots w_n$. If $i < n$, then $v w_{i+1}$ exists, since $v w_{i+1}$ is a path in $\mathbf{F}$. The base facet of $w_{i+1} \cdots w_n$ is $P_W^v = P_W^{v^|}$ and so the product $w_i(w_{i+1} \cdots w_n)$ exists.

Next we must attach $w_{i-1} = \delta^+$ at the left side of $w_i \cdots w_n$. One has $\langle P^-, \delta^+ \rangle = 1$ for the base facet P^- of $w_i = v^|$ which is also the base facet of $w_i \cdots w_n$. Again we are done.

After having shown the existence of $w_{i-1}(w_i \cdots w_n)$, we can replace $w_{i-1}w_i = \delta^+ v^|$ by v, and from now on the product $v w_{i+1} \cdots w_n$ and the succeeding ones are prefixed by elements from V, a harmless operation.

To sum up: we have shown that all the weakly existing products represent paths in $\mathbf{F}'$, and that each such path yields a strongly existing product. That the equivalence classes of the paths in $\mathbf{F}'$ represent the elements of $[W]$ follows immediately from the corresponding property of $\mathbf{F}$ for V.

It remains to show that $[W] = [V']$ does not contain a column vector u and its inverse $-u$. The only critical pairs of products are those in which one element contains $v^|$, but not δ^+, and the other contains δ^+, but not $v^|$. The first product must end in δ^+ and the second must start with $v^|$. But then we can concatenate them to a path in $\mathbf{F}'$, and the sum over such paths is nonzero, as shown already. $\square$

The last lemma of this section will be used in Section 7 in the context of Mayer-Vietoris sequences.

Lemma 4.10. *Assume $U, V \subset \mathrm{Col}(P)$ are rigid systems. Then the intersection $[U] \cap [V]$ is also a rigid system. Moreover, if U and V are Υ-rigid, then $[U] \cap [V]$ is Υ-rigid, too.*

Proof. Set $W = [U] \cap [V]$. It follows from Lemma 4.7(b) that W is a rigid system.

Only the claim on Y-rigidity has yet to be proved. Suppose U and V are Y-rigid, and let $\mathbf{F}_U$ and $\mathbf{F}_V$ be the corresponding Y-graphs. Then we construct a graph $\mathbf{F}$ associated with W as in the proof of Lemma 4.7(b) (say from $\mathbf{F}_U$). Let W' be the set of irreducible elements in W.

First we have to make sure that there is no triple of distinct elements $w_1, w_2, w_3 \in W'$ such that the products $w_1 w_3$ and $w_2 w_3$ exist—this will show that all nonterminal vertices c of $\mathbf{F}$ satisfy the condition (Y). But if both $w_1 w_3$ and $w_2 w_3$ existed, then the corresponding paths $l(w_1), l(w_2) \in \mathrm{path}\,\mathbf{F}_U$ would have the same terminal point and either $l(w_1) \subset l(w_2)$ or $l(w_2) \subset l(w_1)$. We may assume $l(w_1) \subset l(w_2)$; then $w_2 = w'w_1$ for some $w' \in [U]$. Since the vector $w' \in \mathrm{Col}(P)$ is uniquely determined, the same arguments applied to V show that $w' \in [V]$, that is, w_2 is decomposable within W—a contradiction.

By Proposition 4.4(c) the only remaining obstruction to the Y-rigidity of W is the existence of a (non-directed) cycle in $\mathbf{F}$ that is the union of two oriented paths with the same initial point and the same terminal point.

Assume such a cycle exists. Then there are elements $w_1, \ldots, w_r, w'_1, \ldots, w'_s \in W'$, $r, s \in \mathbb{N}$ such that

$$w_1 \cdots w_r = w'_1 \cdots w'_s.$$

We have to show that $r = s$ and $w_i = w_j$ for $i \in [1, r]$. Consider the corresponding paths $l(w_i), l(w'_j) \in \mathrm{path}\,\mathbf{F}_U$. Since $\overline{\mathrm{path}\,\mathbf{F}_U} = \mathrm{path}\,\mathbf{F}_U$ we get

$$l(w_1) \cdots l(w_r) = l(w'_1) \cdots l(w'_s) \in \mathrm{path}\,\mathbf{F}_U,$$

where the multiplication is the concatenation of paths. We may assume that $l(w_r) \neq l(w'_s)$. Since $\mathbf{F}_U$ is a Y-graph either $l(w_r) \subset l(w'_s)$ or $l(w'_s) \subset l(w_r)$. Then the same arguments as in checking the condition (Y) for nonterminal vertices show that w'_s or w_r is decomposable in W—a contradiction. $\quad\square$

We conclude this section by a further examination of products $v_1 \cdots v_n$. It has been observed in Example 4.5(a) that the existence of $v_1 \cdots v_m$ in $\mathrm{Col}(\Delta_n)$ implies the rigidity of $\{v_1, \ldots, v_m\}$. This is not true for all polytopes:

Example 4.11. Let P be the 3-simplex $\mathrm{conv}((0,0,0), (2,0,0), (0,2,0), (0,0,1))$ and consider its column vectors $u = (1, 0, -1), v = (-1, 0, 0), w = (0, 1, 0)$ (see Fig. 8). Then uvw and uw exist. This excludes the rigidity of $\{u, v, w\}$.

This example and many other observations in [5] and in this paper, have naturally lead us to the class of balanced polytopes. Their combinatorial properties allow one to develop polyhedral K-theory. The next proposition shows that the phenomenon just observed is indeed impossible in balanced polytopes.

Proposition 4.12. *Let P be a balanced polytope. Assume $\prod_{i=1}^{m} v_i$ exists for $V = \{v_1, \ldots, v_m\} \subset \mathrm{Col}(P)$. Then V is a Y-rigid system whose associated Y-graph is the directed linear graph $\bullet \to \bullet \cdots \bullet \to \bullet$ of length m.*
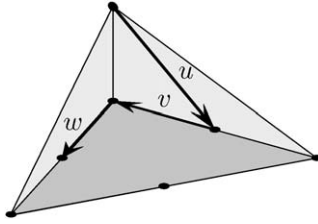
Fig. 8.

Proof. In this proof we use Proposition 3.1 heavily. Set $F_i = P_{v_i}$. We have to show that

$$
(\langle F_j, v_i \rangle) = \begin{pmatrix}
-1 & 1 & 0 & \cdots & 0 \\
0 & -1 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
\vdots & & \ddots & -1 & 1 \\
0 & \cdots & \cdots & 0 & -1
\end{pmatrix}
$$

where row i corresponds to v_i and column j to F_j. The case $j = i$ is clear by definition. Clearly $\langle F_j, v_i \rangle = 0$ for all $i > j$ by the existence of $v_j \cdots v_n$ (independently of the fact that P is balanced). But also $w = v_1 \cdots v_{j-1}$ exists, and since P is balanced, we must have $\langle F_j, w \rangle = \sum_{i=1}^{j-1} \langle F_j, v_i \rangle \leqslant 1$. Since $\langle F_j, v_{j-1} \rangle > 0$ and $\langle F_j, v_i \rangle \geqslant 0$ for $i < j$, this implies $\langle F_j, v_{j-1} \rangle = 1$ and $\langle F_j, v_i \rangle = 0$ for $i < j-1$.   □

Remark 4.13. If the product $\prod_{i=1}^{m} v_i$ exists only in the weak sense and $\sum_{i \in I} v_i \neq 0$ for every subset $I \subset [1, m]$, then $\{v_1, \ldots, v_n\}$ need not be a Y-rigid system. Consider the balanced polytope P of Remark 4.3(b) and the same column vectors $u = (0, 0, -1)$, $v = (1, 0, 0)$ and $w = (0, 1, 0)$.

5. Triangular subgroups in $\mathbb{E}(R, P)$ and $\mathbb{S}t(R, P)$

In this section we generalize the notion of a triangular group of matrices to the polyhedral setting. These groups play a crucial rôle in the definition of Volodin simplicial sets (Section 6).

Let R be a ring and P a balanced polytope admitting a column structure. We fix a doubling spectrum $\mathfrak{P} = (P \subset P_1 \subset \cdots)$. Thanks to Theorem 3.5(a) (and its straightforward analogue for polyhedral Steinberg groups) all the objects defined below are independent of the fixed spectrum.

We say that $V \subset \mathrm{Col}(\mathfrak{P})$ is a rigid (Y-rigid) system if there exists an index $j \in \mathbb{N}$ such that $V \subset \mathrm{Col}(P_j)$ and is rigid (Y-rigid) in the sense of Definition 4.2.

Definition 5.1. (a) A subgroup $G \subset \mathbb{E}(R, P)$ is called *triangular* if there exists a rigid system $V \subset \text{Col}(\mathfrak{P})$ such that G is generated by the elementary automorphisms e_v^λ, where λ runs through R and v through V. The triangular subgroup corresponding to a rigid system V is denoted by $G(R, V)$, and $\mathbf{T}(R, P)$ is the family of all triangular subgroups of $\mathbb{E}(R, P)$.

(b) The triangular subgroups of $\mathbb{S}\mathrm{t}(R, P)$ are defined similarly, and $G'(R, V)$ and $\mathbf{T}'(R, P)$ denote the corresponding objects.

(c) The Y-triangular subgroups in $\mathbb{E}(R, P)$ and $\mathbb{S}\mathrm{t}(R, P)$ are those supported by Y-rigid systems. Their families are $\mathbf{T}(R, P)^{\mathsf{Y}}$ and $\mathbf{T}'(R, P)^{\mathsf{Y}}$.

Let $\mathbf{F}$ be a Y-graph underlying a Y-rigid system $V \subset \text{Col}(\mathfrak{P})$ and let $E(\mathbf{F})$ be the set of edges of $\mathbf{F}$. There is a natural partial order on $E(\mathbf{F})$ defined as follows: for $f, g \in E(\mathbf{F})$ we put $f \leqslant g$ if there is $l \in \text{path}\,\mathbf{F}$ with first edge f and last edge g. We have the disjoint partition

$$E(\mathbf{F}) = E_1 \cup E_2 \cup \cdots \cup E_t,$$

where each of the E_r consists of those elements f of $E(\mathbf{F})$ that admit sequences of type $f_1 < f_2 < \cdots < f_r = f$ and do not admit sequences $f_0 < f_1 < f_2 < \cdots < f_r = f$. ($E_t$ is the set of maximal elements.) Edges in E_r have *degree* r.

We get the partition

$$\text{path}\,\mathbf{F} = \bigcup_{r,s} \text{path}_{rs}\,\mathbf{F}$$

into disjoint sets where $1 \leqslant r \leqslant s \leqslant t$ and

$$\text{path}_{rs}\,\mathbf{F} = \{[f_r, \ldots, f_s] \mid f_r \in E_r, \ldots, f_s \in E_s\}.$$

Let $v_l \in [V]$ denote the column vector corresponding to a path $l \in \text{path}(\mathbf{F})$. Then we have the analogous disjoint partition

$$[V] = \bigcup_{rs} [V]_{rs}$$

where $1 \leqslant r \leqslant s \leqslant t$ and

$$[V]_{rs} = \{v_l \mid l \in \text{path}_{rs}\mathbf{F}\}.$$

We introduce the following notation:

- $[V]^r = \{v_{r1}, \ldots, v_{rN_r}\} = \bigcup_{s=r}^t [V]_{rs}$, for $r \in [1, t]$ ($N_r = \#\bigcup_{s=r}^t [V]_{rs}$). That is, $[V]^r$ consists of the column vectors which correspond to paths with initial edges of degree r.
- $[V]_1 = [V]$ and $[V]_r = [V]_{r-1} \setminus [V]^{r-1}$ for $r \in [2, t]$. That is, $[V]_r$ consists of the column vectors corresponding to the paths with initial edges of degree $\geqslant r$.
- $G_r(R, V)$ (resp. $G_r'(R, V)$) is the subgroup of $G(R, V)$ (resp. $G_r'(R, V)$) generated by e_v^λ (resp. x_v^λ) with $\lambda \in R$ and $v \in [V]_r$, where $r \in [1, t]$.

Observe that $[V]_r$ is a rigid system for each $r \in [1, t]$. We have the following ascending sequence of triangular subgroups:

$$G_t(R, V) \subset G_{t-1}(R, V) \subset \cdots \subset G_1(R, V) = G(R, V).$$

Consider the mappings

$$\tau_r : R^{N_r} \to G_r(R,P), \quad r \in [1,t],$$

given by

$$(\lambda_1,\dots,\lambda_{N_r}) \mapsto e_{v_{r1}}^{\lambda_1} \circ \cdots \circ e_{v_{rN_r}}^{\lambda_{N_r}}, \quad \{v_{r1},\dots,v_{rN_r}\} = [V]^r.$$

and set $A_r(R,V)=\mathrm{Im}(\tau_r) \subset G(R,V)$, $r \in [1,t]$. Then $A_r(R,V) \subset G_r(R,V)$ for $1 \leqslant r \leqslant t$ and $A_t(R,V) = G_t(R,V)$.

Remark 5.2. The name 'triangular' is explained by the following observation. Let $T_m(R) \subset E(R)$ denote the usual triangular subgroup of upper $m \times m$-matrices over R with diagonal entries 1. Then it is a Y-triangular subgroup of $E(R)$ (viewed as $\mathbb{E}(R,\Delta_n)$ for some $n \in \mathbb{N}$), supported by the simplest graph $\bullet \to \bullet \cdots \bullet \to \bullet$. In this situation $t = m - 1$ and $G_r(R,V)$ becomes the subgroup of $T_m(R)$ consisting of those matrices which admit non-diagonal entries only in the rows of index $i \geqslant r$. The subset $A_r(R,V)$ is just the abelian subgroup of the matrices $\prod_{j=r+1}^{t} e_{rj}^{\lambda_j}$, $\lambda_j \in R$.

The next two theorems generalize the properties of these objects to the polyhedral situation.

Theorem 5.3. *Let V be a rigid system.*

(a) *The mappings τ_r, $r \in [1,t]$, are injective group homomorphisms (defined on $(R^{N_r},+)$).*
(b) *Every element $\varepsilon \in G_r(R,V)$ admits a unique representation of type*

$$\varepsilon = \varepsilon_r \circ \cdots \circ \varepsilon_t, \quad \varepsilon_s \in A_s(R,V), \quad s \in [r,t],$$

which we will call the canonical representation.

Proof. (a) We know that $[V]^r$ consists of those column vectors whose corresponding paths in **F** have an initial edge of degree r. No two such paths can be multiplied in path **F**. Therefore, by Definition 4.2 the product ww' does not exist for any elements $w,w' \in [V]^r$. The definition of a rigid system also excludes that $w + w' = 0$ for $w,w' \in [V]^r$. So by Theorem 3.5(e) the mappings τ_r are in fact group homomorphisms. The injectivity follows from the second part of the proof of (b).

(b) Choose $\varepsilon \in G_r(R,V)$ and fix a representation $\varepsilon = e_{v_1}^{\lambda_1} \circ e_{v_2}^{\lambda_2} \circ \dots$ where $v_1,v_2,\dots \in [V]_r$. Clearly, there is no loss of generality in assuming that $v_i \in [V]^r$ for some i. We pick the minimal such index. Assume $i \neq 1$. Then, using Theorem 3.5(e) (and Definition 4.2), we see that, by commuting $e_{v_i}^{\lambda_i}$ successively with the elementary automorphisms $e_{v_{r-1}}^{\lambda_{r-1}}$, $e_{v_{r-2}}^{\lambda_{r-2}}$, we can draw the factor $e_{v_i}^{\lambda_i}$ to the left end of the new representation of ε. It is of course essential that $e_{v_i}^{\lambda_i}$ commutes with all the commutators, produced along the way it moves towards the initial position in the representation. In fact, all of these commutators belong to $A_r(R,V)$, an abelian group. In particular, they all commute with $e_{v_i}^{\lambda_i}$ and with each other. Next we apply the same procedure to that element of $A_r(R,V)$ showing up first from the left in the obtained representation etc. The crucial observation is that the iteration of the process terminates after finitely many steps since

each nontrivial commutator corresponds to a column vector with a path longer than those of each of its factors. But the length of paths is bounded.

The uniqueness is shown by induction on $\#V$. For $\#V = 0$ there is nothing to prove. Assume we have shown the uniqueness for every rigid system in $\mathrm{Col}(\mathfrak{P})$ with $< m = \#V$ elements. Replacing the original polytope P by some element of the doubling spectrum, we may assume $V \subset \mathrm{Col}(P)$.

Consider an element $\varepsilon \in G_r(R, V)$ and fix a representation

$$\varepsilon = e_{v_1}^{\lambda_1} \circ \cdots \circ e_{v_N}^{\lambda_N}, \quad N = \sum_{s=r}^{t} N_s$$

in which the column vectors in $[V]^r$ appear first, then those of $[V]^{r+1}$, etc. For simplicity we consider a fully expanded representation in the sense that all the factors $e_{v_i}^{\lambda_i}$, $i \in [1, N]$ are present, what we can achieve by choosing $\lambda_i = 0$ if necessary.

For an element $v \in [V]^r$ either $v_1 + v \notin \mathrm{Col}(P)$ or the product $v_1 v$ exists. By Proposition 3.1 (and since $\langle P_{v_1}, v \rangle < 0$ is obviously equivalent to $P_{v_1} = P_v$) there are only two possibilities: either $\langle P_{v_1}, v \rangle = 0$ or $P_{v_1} = P_v$. Let $1 = i_1 < i_2 < \cdots$ and $j_1 < j_2 < \cdots$ be the indices determined correspondingly by the conditions: $P_{v_1} = P_{v_{i_2}} = P_{v_{i_3}} = \cdots$ and $\langle P_{v_1}, v_{j_1} \rangle = \langle P_{v_1}, v_{j_2} \rangle = \cdots = 0$. (The sequence of the j may be empty.) Let us show that

$$\varepsilon = \varepsilon' \circ \varepsilon'', \quad \varepsilon' = e_{v_1}^{\lambda_1} \circ e_{v_{i_2}}^{\lambda_{i_2}} \circ e_{v_{i_3}}^{\lambda_{i_3}} \circ \dots, \quad \varepsilon'' = e_{v_{j_1}}^{\lambda_{j_1}} \circ e_{v_{j_2}}^{\lambda_{j_2}} \circ \cdots.$$

By Theorem 3.5(e) it is enough to show that $v_{i_k} + v_{j_l} \notin \mathrm{Col}(\mathfrak{P})$ for every pair of indices $j_l < i_k$. Assume to the contrary $v_{i_k} + v_{j_l} \in \mathrm{Col}(\mathfrak{P})$ for some $j_l < i_k$. Then by the definition of the sets $[V]^s$ and by Proposition 3.1(d) a product of type $v_j v_i$, $j < i$, exists. But this is a contradiction to Proposition 3.1(a) because $\langle P_{v_i}, v_j \rangle = 0$.

The proper subset $V' = \{v \in [V]_r \mid \langle P_{v_1}, v \rangle = 0\} \subset [V]_r$ is a rigid system. This follows from the fact that $V' = [U]$ for a certain subset $U \subset V$ of irreducible elements of V. In fact, assume $\langle P_{v_1}, u_1 \cdots u_k \rangle = 0$ for some $u_1, \dots, u_k \in [V]$ corresponding to the edges of our graph. As observed above, each of the $u_1, \dots, u_k$ is either parallel to P_{v_1} or has P_{v_1} as the base facet. What we claim is that the latter case is impossible. Assume to the contrary that one of the $u_1, \dots, u_k$ has the base facet P_{v_1}. By Proposition 3.1(a) this can only be u_1. But then Proposition 3.1(c) implies $P_{u_1 \cdots u_k} = P_{v_1}$—a contradiction with the assumption.

The restrictions of ε and ε'' to the polytopal ring $R[P_{v_1}]$ coincide and $\#V' < \#V$ (maybe $\#V' = 0$). Therefore, by the induction hypothesis the elements $\lambda_{j_1}, \lambda_{j_2}, \dots \in R$ are uniquely determined by ε. Observe that we can apply the induction hypothesis because the factors of ε'' are already ordered in the right way:

$$\{v_{j_1}, v_{j_2}, \dots\} = \{v_{j_1}, \dots, v_{j_a}\} \cup \{v_{j_{a+1}}, \dots, v_{j_{a+b}}\} \cup \{v_{j_{a+b+1}}, \dots, v_{j_{a+b+c}}\} \cup \cdots$$

where successive subsets belong to $A_{s'}(R, V')$ and $A_{s''}(R, V')$ for indices $s' < s''$.

By Lemma 2.1(a) $\varepsilon \circ (\varepsilon'')^{-1}$—and, therefore, ε itself—determine uniquely the elements $\lambda_1, \lambda_{i_2}, \lambda_{i_3}, \dots \in R$ as well. □

After the hard work has been done, we draw some consequences.

Theorem 5.4. *Let V be a rigid system supported by the graph* **F**.

(a) *For each $r \in [1, t-1]$ we have the exact sequence*

$$0 \to A_r(R, V) \to G_r(R, V) \to G_{r+1}(R, V) \to 0$$

where the mapping $G_r(R, V) \to G_{r+1}(R, V)$ is determined by $\varepsilon \mapsto \varepsilon_r^{-1}\varepsilon$ (notation as in Theorem 5.3(b)). This surjective homomorphism is split by the identity embedding $G_{r+1}(R, V) \to G_r(R, V)$.

(b) *If Q is another polytope and $W \subset \mathrm{Col}(Q)$ is a rigid system supported by the same graph* **F**, *then the assignment $e_w^\lambda \mapsto e_v^\lambda$, where w and v correspond to the same path in* **F**, *gives rise to a group isomorphism $G_r(R, W) \to G_r(R, V)$.*

(c) *Let $U \subset \mathrm{Col}(\mathfrak{P})$ be another rigid system. Then*

$$G(R, U) \cap G(R, V) = G(R, [U] \cap [V]).$$

(d) *The natural surjective mappings $G'_r(R, V) \to G_r(R, V)$, $r \in [1, t]$ are isomorphisms, and the assertions of Theorem 5.3 and (a)–(c) hold analogously.*

Proof. (a) follows from Theorem 5.3(b) once one observes that the same arguments as in the first half of its proof imply the following: for arbitrary elements $\varepsilon, \varepsilon' \in A_r(R, V)$ and $\rho, \rho' \in G_{r+1}(R, V)$ there exists $\varepsilon'' \in A_r(R, V)$ such that

$$\varepsilon \circ \rho \circ \varepsilon' \circ \rho' = \varepsilon'' \circ \rho \circ \rho'.$$

(b) We have the bijective mapping $G_r(R, W) \to G_r(R, V)$ defined in a natural way via the canonical representations. It restricts to the assignment $e_w^\lambda \mapsto e_v^\lambda$ as above. (That this mapping is a bijection follows from Theorem 5.3.) In order to see that it is a group homomorphism one notices that only the structure of the underlying graph and the commuting rules of Theorem 3.5(e) are used in deriving the canonical representation of an element of $G(V, r)$ or $G(W, r)$ from an arbitrary representation (see the proof of Theorem 5.3(b)).

(c) Without loss of generality we can assume $U, V \subset \mathrm{Col}(P)$. We use induction on $\#U + \#V$ starting from the $\#U + \#V = 0$ for which there is nothing to prove.

Assume we have shown the claim when $\#U + \#V < m$ and consider the case $\#U + \#V = m$. Pick an element $\varepsilon \in G(R, U) \cap G(R, V)$ and consider canonical representations $\varepsilon = e_{u_1}^{\lambda_1} \circ e_{u_2}^{\lambda_2} \circ \cdots$ and $\varepsilon = e_{v_1}^{\mu_1} \circ e_{v_2}^{\mu_2} \circ \cdots$ with respect to U and V. Let $U' \subset U$ and $V' \subset V$ be the subsets determined correspondingly by the conditions $\langle P_{v_1}, - \rangle = 0$ and $\langle P_{u_1}, - \rangle = 0$. Then U' and V' are proper rigid subsystems.

We claim that $P_{u_1} = P_{v_1}$. As we know (from the proof of Theorem 5.3(b)) every vector from U is either parallel to P_{u_1} or has the base facet P_{u_1}. For $x \in L_P$ this implies that $\varepsilon(x)$ is an R-linear combination of the points $\{y \in L_P \mid \langle P_{u_1}, y \rangle \leqslant \langle P_{u_1}, x \rangle\}$. Since the same is true with respect to the facet P_{v_1} the claim follows.

Now consider the representations $\varepsilon = \varepsilon'_U \circ \varepsilon''_U$ and $\varepsilon = \varepsilon'_V \circ \varepsilon''_V$ with respect to U and V, similar to those in the proof of Theorem 5.3(b). We see that

$$\varepsilon|_{R[P_{u_1}]} = \varepsilon''_U|_{R[P_{u_1}]} = \varepsilon''_V|_{R[P_{v_1}]} \in G_r(R, U') \cap G_r(R, V').$$

By the induction hypothesis one concludes $\varepsilon|_{R[P_{u_1}]} \in G(R, [U'] \cap [V']) \subset G(R, [U] \cap [V])$. (The same arguments, as at the end of the proof of Theorem 5.3(b), show that we can use the induction hypothesis.)

The equation $\varepsilon'_U = \varepsilon'_V$ and Lemma 2.1(b) imply that $\varepsilon'_U = G(R, [U] \cap [V])$. Therefore, $\varepsilon \in G(R, [U] \cap [V])$.

(d) That the mappings are isomorphisms follows from Theorem 5.3(a) and part (a) of this theorem, with the use of Lemma 3.9 (and the induction on $\#V$). The rest is clear. $\square$

Remark 5.5. It follows from Lemma 4.7(a) and Theorem 5.4(b) that the triangular groups associated with rigid systems are just usual triangular matrix groups used in the construction of the Volodin theory (see Section 6.2). The essential point is that in general it is not possible to realize the whole set $\mathrm{Col}(P)$ with its partial product structure in $\mathrm{Col}(\Delta_n)$ for any n. Even if this is the case (as for the class of Col-divisible polytopes discussed in Section 8) we do not have *all* triangular matrix groups used in the classical theory.

6. Higher polyhedral *K*-groups

We now present the polytopal versions of the standard K-theoretical constructions. Despite the fact that this paper exclusively treats the case of single polytopes we use the attribute 'polyhedral' in order to indicate the possibility of a further generalization to *polyhedral algebras*, defined in terms of *lattice polyhedral complexes* [2]. (See also Remark 8.3.)

6.1. Volodin's theory

Let R be a ring and P a balanced polytope, admitting a column structure.

Definition 6.1. (a) The d-simplices of the Volodin simplicial set $\mathbb{V}(\mathbb{E}(R, P))$ are those sequences $(\varepsilon_0, \ldots, \varepsilon_d) \in (\mathbb{E}(R, P))^{d+1}$ for which there exists a triangular group $G \in \mathbf{T}(R, P)$ such that $\varepsilon_k \varepsilon_l^{-1} \in G$, $k, l \in [0, d]$. The ith face (resp. degeneracy) of $\mathbb{V}(\mathbb{E}(R, P))$ is obtained by omitting (resp. repeating) ε_i.

(b) The Volodin Y-simplicial set $\mathbb{V}(\mathbb{E}(R, P))^{\mathsf{Y}}$ is defined similarly using the Y-triangular subgroups of $\mathbb{E}(R, P)$. (In particular, $\mathbb{V}(\mathbb{E}(R, P))^{\mathsf{Y}} \subset \mathbb{V}(\mathbb{E}(R, P))$ as simplicial sets.)

(c) The simplicial sets $\mathbb{V}(\mathbb{St}(R, P))$ and $\mathbb{V}(\mathbb{St}(R, P))^{\mathsf{Y}}$ are defined analogously.

(d) The higher Volodin polyhedral K-groups of R are defined by

$$K_i^{\mathrm{V}}(R, P) = \pi_{i-1}(|\mathbb{V}(\mathbb{E}(R, P))|, (\mathbf{Id})), \quad i \geqslant 2,$$

and

$$K_i^{\mathrm{V}}(R, P)^{\mathsf{Y}} = \pi_{i-1}(|\mathbb{V}(\mathbb{E}(R, P))^{\mathsf{Y}}|, (\mathbf{Id})), \quad i \geqslant 2,$$

where $|-|$ refers to the geometric realization of a simplicial set.

It is clear that the Volodin complexes are connected. Also, the definition of the Volodin simplicial set is independent of the choice of $\mathfrak{P}$ (see Theorem 3.5(a) and the corresponding remarks in Section 3.5).

Below, for Volodin simplicial sets, we will usually omit $|-|$, i.e. we will use the same notation for a simplicial set and its geometric realization. Also, the base points will be omitted since they are always assumed to be the unit elements.

6.2. The case of a unimodular simplex

We work out the case of a unimodular simplex in detail. Volodin's K-theory for a ring R is defined as follows (see [14–16]):

Suppose σ is a partial order on $\{1,\ldots,n\}$. Define $\mathbf{T}_n^\sigma(R)$ to be the subgroup of $\mathrm{GL}_n(R)$ consisting of those matrices $\alpha = (a_{ij})$ for which $a_{ii} = 1$ and $a_{ij} = 0$ if $i \not\leqslant^\alpha j$. In particular, for the natural order $\sigma := (1 < \cdots < n)$ the corresponding group $\mathbf{T}_n^\sigma(R)$ is just the group of upper triangular matrices with 1s on the diagonal.

The simplicial complex $V_n(R) := \mathrm{V}(\mathrm{GL}_n(R), \{\mathbf{T}_n^\alpha(R)\}_\alpha)$ is defined in the same way as in Definition 6.1 where we now take $\mathrm{GL}_n(R)$ instead of $\mathbb{E}(R,P)$ and the $\mathbf{T}_n^\alpha(R)$ instead of the triangular subgroups of Section 5. The embeddings $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$, $* \mapsto \left(\begin{smallmatrix} * & 0 \\ 0 & 1 \end{smallmatrix}\right)$, together with the induced embeddings $\mathbf{T}_n^\sigma(R) \to \mathbf{T}_{n+1}^{\sigma'}(R)$ where σ' is the extension of σ to the partial order of $\{1,\ldots,n,n+1\}$ under which $n+1$ is the biggest element, define embeddings of simplicial complexes $V_n(R) \to V_{n+1}(R)$.

Finally, for $i \geqslant 1$ we put $K_{i,n}^\mathrm{V}(R) = \pi_{i-1}(\mathrm{V}_n(R))$ and $K_i^\mathrm{V}(R) = \pi_{i-1}(\mathrm{V}(R)) = \varinjlim(K_{i,n}^\mathrm{V}(R))$ where $\mathrm{V}(R) := \varinjlim \mathrm{V}_n(R)$. The base point for the homotopy groups is the identity matrix.

Using the same construction we could define another simplicial complex, based on the group $\mathrm{E}_n(R)$ instead of $\mathrm{GL}_n(R)$. Denote this complex by $\mathrm{V}^\mathrm{E}(R)$. Then it is clear that $\mathrm{V}^\mathrm{E}(R)$ is the connected component of $\mathrm{V}(R)$ containing the identity element of $\mathrm{GL}(R)$. Therefore, $K_i^\mathrm{V}(R) = \pi_{i-1}(\mathrm{V}^\mathrm{E}(R))$ for $i \geqslant 2$ and, of course, we have the natural identification of $K_1(R) = \mathrm{GL}(R)/E(R)$ with the set $\pi_0(\mathrm{V}(R))$.

In order to see that $\mathrm{V}(R)$ is the same as $\mathbb{V}(R, \varDelta_m)$ (for every natural number m) we have to recognize the triangular subgroups of Section 5 in the groups $\mathbf{T}_n^\sigma$ after the natural identification $E(R) = \mathbb{E}(R, \varDelta_m)$. We use the sequences $\mathfrak{P}' = (\varDelta_m = P_0' \subset P_1' \subset \cdots)$ in Remark 3.7 for the identification. Then the vertices of a polytope in $\mathfrak{P}$ can be identified with the indices used to enumerate them. Moreover, we can even assume that $\{1,\ldots,m+1\} = \mathrm{vert}(\varDelta_m)$ and that the new vertex of P_{l+1} is larger (as a natural number) than the vertices of P_l for $l \in \mathbb{N}$.

Pick a natural number n and a partial order σ on $\{1,\ldots,n\}$. Then $\{1,\ldots,n\} \subset \mathrm{vert}(P_l)$ for a sufficiently large index l. Next we consider the *reverse* partial order σ^op on $\{1,\ldots,n\}$. As observed in Example 4.5(c), an arbitrary partial order on a subset of the vertices of a unimodular simplex gives rise to a rigid system of column vectors. In particular, so does the order σ^op. For the resulting rigid system V we have the equality $\mathbf{T}_n^\sigma(R) = G(R,V)$. It is also clear that this process of assigning the triangular groups $G(R,V)$ to the groups $\mathbf{T}_n^\sigma(R)$ can be reversed. Thus in the special case of a unimodular simplex we recover Volodin's usual K-groups.

The reason that we need to pass to the reverse partial orders in the assignment between the triangular groups of automorphisms and the triangular groups of matrices is explained by Remark 3.6 in Section 3.4.

Finally, we remark that the analogous claim for Quillen's theory, introduced in Section 6.4 below, is just obvious and needs no detailed explanation.

6.3. Connection with the Milnor group

The group $K_2(R,P)$ acts freely both on $\mathbb{V}(\mathbb{S}t(R,P))$ and $\mathbb{V}(\mathbb{S}t(R,P))^\mathsf{Y}$ by multiplication on the right and we have

$$\mathbb{V}(\mathbb{S}t(R,P))/K_2(R,P) = \mathbb{V}(\mathbb{E}(R,P)),$$

$$\mathbb{V}(\mathbb{S}t(R,P))^\mathsf{Y}/K_2(R,P) = \mathbb{V}(\mathbb{E}(R,P))^\mathsf{Y}.$$

In fact, the equality on vertex sets follows from the very definition of $K_2(R,P)$ and, hence, the equality for higher dimensional simplices follows from Theorem 5.4(d).

As in the classical case we have

Lemma 6.2. *Both* $\mathbb{V}(\mathbb{S}t(R,P))$ *and* $\mathbb{V}(\mathbb{S}t(R,P))^\mathsf{Y}$ *are simply connected.*

Proof. We consider $\mathbb{V}(\mathbb{S}t(R,P))$. The arguments are completely similar for the Y-theory.

Clearly, we have only to show that any loop l through $1 \in \mathbb{V}(\mathbb{S}t(R,P))$, consisting of *edges* of the simplicial complex $\mathbb{V}(\mathbb{S}t(R,P))$, is contractible. Thus we can assume that there are a natural number k, vectors $v_i \in \mathrm{Col}(\mathfrak{P})$ and elements $\lambda_i \in R$, $i \in [1,k]$ such that $\prod_{i=1}^{k} x_{v_i}^{\lambda_i} = 1$ and

$$l = [[1,s_1],[s_1,s_2],\ldots,[s_k,1]],$$

where $s_1 = x_{v_1}^{\lambda_1}$ and $s_i = x_{v_i}^{\lambda_i} s_{i-1}$ for $i \in [1,k-1]$. For simplicity denote this loop by $(x_1,\ldots,x_k)$ where $x_i = x_{v_i}^{\lambda_i}$.

Let $\mathfrak{F}$ denote the free (non-commutative) monoid generated by

$$\{|x|: x \text{ a standard generator of } \mathbb{S}t(R,P)\}.$$

Moreover, we define the group $\mathfrak{G}$ as the quotient of the free group generated by these elements modulo the relations

$$|x^{-1}| = |x|^{-1}.$$

For a word $w = |y_1| \cdots |y_t| \in \mathfrak{F}$ put $w^* = |y_t^{-1}| \cdots |y_1^{-1}|$. For words $w', w'' \in \mathfrak{F}$ we say that w' is obtained from w'' by *elementary cancellation* if $w' = w_1 w_2$ and $w'' = w_1 w w^* w_2$ for some (maybe empty) words $w, w_1, w_2 \in \mathfrak{F}$. Further, w' is obtained from w'' by *cancellation* if there is a finite sequence of elements $w_1, \ldots, w_t$ such that w' is obtained from w_t by elementary cancellation, w_t is obtained from w_{t-1} by elementary cancellation, $\ldots, w_1$ is obtained from w'' by elementary cancellation.

The natural monoid homomorphism $\mathfrak{F} \to \mathfrak{G}$ satisfies the following condition:

- $w_1, w_2 \in \mathfrak{F}$ map to the same element in $\mathfrak{G}$ if and only if there is $w_3 \in \mathfrak{F}$ such that both w_1 and w_2 are obtained from w_3 by cancellation.

Let $\mathfrak{W}$ denote the smallest submonoid of $\mathfrak{F}$ determined by the following conditions:

(i) the words of types

$$|x_u^{-\lambda-\mu}\|x_u^\lambda\|x_u^\mu|, \qquad |x_{uv}^{\lambda\mu}\|x_u^\lambda\|x_v^\mu\|x_u^{-\lambda}\|x_v^{-\mu}| \text{ if } uv \text{ exists,}$$

$$|x_u^\lambda\|x_v^\mu\|x_u^{-\lambda}\|x_v^{-\mu}|, \quad \text{ if } u+v \notin \mathrm{Col}(\mathfrak{P})\cup\{0\},$$

and their $-^*$ versions are in $\mathfrak{W}$,

(ii) if $w \in \mathfrak{W}$ then $AwA^* \in \mathfrak{W}$ for arbitrary $A \in \mathfrak{F}$.

By the observation above the equation $x_1 \cdots x_k = 1$ in $\mathbb{St}(R,P)$ is equivalent to the existence of words $w' \in \mathfrak{W}$ and $w'' \in \mathfrak{F}$ such that $|x_1| \cdots |x_k|$ and w' are obtained from w'' by cancellation.

Let $w = |y_1| \cdots |y_t|$ be a word in $\mathfrak{F}$ such that $y_1 \cdots y_t = 1$ in $\mathbb{St}(R,P)$. In a natural way it defines a loop in $\mathbb{V}(\mathbb{St}(R,P))$ consisting of edges of $\mathbb{V}(\mathbb{St}(R,P))$. We denote this loop by $l(w)$. Furthermore, if a word $w \in \mathfrak{F}$ is a obtained from another word $w' \in \mathfrak{F}$ by cancellation then they define the same element in $\mathbb{St}(R,P)$.

Summing up all these observations, we see that it is enough to show the following two claims:

(i) if a word $w_1 \in \mathfrak{F}$ is obtained from a word $w_2 \in \mathfrak{W}$ by cancellation, then $l(w_1)$ is homotopic to $l(w_2)$;

(ii) for every word $w \in \mathfrak{W}$ the loop $l(w)$ is contractible.

Now Claim (i) follows from the fact that the loop of w_2 only differs from that of w_1 by finitely many attached paths, consisting of edges of $\mathbb{V}(\mathbb{St}(R,P))$—for each of these "tails" we perform forward and backward movements when we go along $l(w_2)$. By a similar argument for Claim (ii) we only need that the loops of types

$$l(|x_u^{-\lambda-\mu}\|x_u^\lambda\|x_u^\mu|), \qquad l(|x_{uv}^{\lambda\mu}\|x_u^\lambda\|x_v^\mu\|x_u^{-\lambda}\|x_v^{-\mu}|) \text{ if } uv \text{ exists}$$

$$l(|x_u^\lambda\|x_v^\mu\|x_u^{-\lambda}\|x_v^{-\mu}|) \quad \text{ if } u+v \notin \mathrm{Col}(\mathfrak{P})\cup\{0\}$$

and those corresponding to the $-^*$ versions are contractible. But due to Proposition 3.1(d), systems in $\mathrm{Col}(\mathfrak{P})$ of the types

$$\{u,v,uv\} \quad \text{and} \quad \{u,v \mid u+v \notin \mathrm{Col}(\mathfrak{P}),\ u+v \neq 0\}$$

are Y-rigid. Therefore, in view of Definition 6.1(c) these loops are boundaries of simplices of $\mathbb{V}(\mathbb{St}(R,P))$. $\quad\square$

By Lemma 6.2 $\mathbb{V}(\mathbb{St}(R,P))$ (resp. $\mathbb{V}(\mathbb{St}(R,P))^Y$) is a universal cover of $\mathbb{V}(\mathbb{E}(R,P))$ (resp. $\mathbb{V}(\mathbb{E}(R,P))^Y$). Therefore we have the following

Proposition 6.3.

(a) $K_2(R,P) = K_2^{\mathbb{V}}(R,P) = K_2^{\mathbb{V}}(R,P)^{\curlyvee}$,
(b) $K_i^{\mathbb{V}}(R,P) = \pi_{i-1}(\mathbb{V}(\mathbb{St}(R,P)))$ and $K_i^{\mathbb{V}}(R,P)^{\curlyvee} = \pi_{i-1}(\mathbb{V}(\mathbb{St}(R,P))^{\curlyvee})$ for all $i \geqslant 3$.

As usual, BG denotes the classifying space of a group G. By Theorem 5.4(c) we have the following formula for rigid systems $U, V \subset \mathrm{Col}(\mathfrak{P})$:

$$\mathrm{B}G(R,U) \cap \mathrm{B}G(R,V) = \mathrm{B}G(R,[U] \cap [V]). \tag{5}$$

The group $\mathbb{E}(R,P)$ acts freely both on $\mathbb{V}(\mathbb{E}(R,P))$ and $\mathbb{V}(\mathbb{E}(R,P))^{\curlyvee}$ by multiplication on the right, and the corresponding quotient spaces admit the following representations:

$$\mathrm{X}(R,P) = \bigcup_{G \in \mathbf{T}(R,P)} \mathrm{B}G \quad \text{and} \quad \mathrm{X}(R,P)^{\curlyvee} = \bigcup_{G \in \mathbf{T}(R,P)^{\curlyvee}} \mathrm{B}G \tag{6}$$

—a general observation valid for abstract Volodin simplicial sets associated to an arbitrary group H and a system of subgroups $\{H_\alpha\}$. Similarly, the quotient spaces of the action of $\mathbb{St}(R,P)$ by multiplication on the right on $\mathbb{V}(\mathbb{St}(R,P))$ and $\mathbb{V}(\mathbb{St}(R,P))^{\curlyvee}$ admit the representations

$$\mathrm{X}'(R,P) = \bigcup_{G' \in \mathbf{T}'(R,P)} \mathrm{B}G' \quad \text{and} \quad \mathrm{X}'(R,P)^{\curlyvee} = \bigcup_{G' \in \mathbf{T}'(R,P)^{\curlyvee}} \mathrm{B}G'. \tag{7}$$

Proposition 6.4. *We have*

(a) $\mathrm{X}(R,P) = \mathrm{X}'(R,P)$ and $\mathrm{X}(R,P)^{\curlyvee} = \mathrm{X}'(R,P)^{\curlyvee}$,
(b) $\pi_1(\mathrm{X}(R,P)) = \pi_1(\mathrm{X}(R,P)^{\curlyvee}) = \mathbb{St}(R,P)$,
(c) $\pi_{i-1}(\mathrm{X}(R,P)) = K_i^{\mathbb{V}}(R,P)$ and $\pi_{i-1}(\mathrm{X}(R,P)^{\curlyvee}) = K_i^{\mathbb{V}}(R,P)^{\curlyvee}$ for $i \geqslant 3$.

Proof. (a) follows from Theorem 5.4(d). As shown above, the spaces $\mathbb{V}(\mathbb{St}(R,P))$ and $\mathbb{V}(\mathbb{St}(R,P))^{\curlyvee}$ are simply connected. This implies the rest of the proposition. $\square$

6.4. Quillen's theory

We define *Quillen's higher polyhedral K-groups* by

$$K_i^{\mathrm{Q}}(R,P) = \pi_i(\mathrm{B}\mathbb{E}(R,P)^+), \quad i \geqslant 2,$$

where $\mathrm{B}\mathbb{E}(R,P)^+$ refers to Quillen's + construction applied to $\mathrm{B}\mathbb{E}(R,P)$ with respect to the whole group $\mathbb{E}(R,P) = [\mathbb{E}(R,P), \mathbb{E}(R,P)]$ (Theorem 3.5(b)). As remarked in Section 6.2, Quillen's polyhedral K-groups coincide with the ordinary K-groups [11] when P is a unimodular simplex.

By a well known argument (see [8]) we have the equations

$$K_i^{\mathrm{Q}}(R,P) = \pi_i(\mathrm{B}\mathbb{St}(R,P)^+), \quad i \geqslant 3, \tag{8}$$

where the + construction is considered with respect to the whole group $\mathbb{St}(R,P)$.

We need the following general fact (see [14,15]).

Proposition 6.5. *For a group G and a perfect subgroup $H \subset G$ the homotopy fiber Y of Quillen's $+$ construction $\mathrm{B}G \to \mathrm{B}G^+$ has the following properties*:

(a) *Y has the homotopy type of a CW-complex,*
(b) *Y is simple in dimension $\geqslant 2$ (i.e. the fundamental group acts trivially on the higher homotopy groups),*
(c) *the (reduced) singular integral homology $\tilde{H}_*(Y) = 0$,*
(d) *Y is connected and $\pi_1(Y)$ is a universal central extension of H,*
(e) *$\pi_i(Y) = \pi_{i+1}(\mathrm{B}G^+)$ for $i \geqslant 2$.*

The properties (a)–(d) *characterize Y up to homotopy equivalence.*

By Theorem 3.11, Propositions 6.3 and 6.5(d), (e) we obtain

Proposition 6.6. $K_2^{\mathrm{Q}}(R,P) = K_2(R,P) = K_2^{\mathbb{V}}(R,P)^{\mathrm{Y}} = K_2^{\mathbb{V}}(R,P).$

In the next sections we will establish the coincidence of Quillen's and Volodin's theories for all higher groups for a certain class of balanced lattice polytopes. However we do not know whether these theories coincide for all balanced polytopes. As mentioned in the introduction, the strategy is as follows: Volodin's Y-theory has an auxiliary function—we are able to obtain certain acyclicity results for the spaces associated to this theory, while the expectation is that the right theory is the one based on arbitrary rigid systems. The acyclicity for this theory remains an open question. Fortunately there are many polytopes for which Volodin's complexes coincide with their Y-subcomplexes. As it will become clear in Section 8, this problem is related to certain subtle properties of column vectors in lattice polytopes.

Lemma 6.7. *Quillen's and Volodin's theories coincide if the space $\mathrm{X}(R,P)$ is acyclic (i.e. the reduced integral homologies are trivial) and simple in dimension $\geqslant 2$.*

Proof. In view of Theorem 3.11 and the homotopy uniqueness of Y in Proposition 6.5, the acyclicity and simplicity of $\mathrm{X}(R,P)$ in dimension $\geqslant 2$ identify the groups $K_i^{\mathrm{V}}(R,P) = \pi_{i-1}(\mathrm{X}(R,P))$ (Proposition 6.4(c)) and $K_i^{\mathrm{Q}}(R,P) = \pi_{i-1}(Y)$ (Proposition 6.5(e)) for $i \geqslant 3$, the case $i = 2$ being settled by Proposition 6.6. $\square$

6.5. Functorial properties

Let P and Q be balanced polytopes and R a ring. If there exists a mapping $\mu : \mathrm{Col}(P) \to \mathrm{Col}(Q)$, such that the conditions

   (i) $\langle P_w, v \rangle = \langle Q_{\mu(w)}, \mu(v) \rangle$ and   (ii) $\mu(vw) = \mu(v)\mu(w)$ if vw exists,

hold for all $v, w \in \mathrm{Col}(P)$, then the assignment $x_v^\lambda \mapsto x_{\mu(v)}^\lambda$ induces a homomorphism

   $\mathbb{St}(R, \mu) : \mathbb{St}(R, P) \to \mathbb{St}(R, Q).$

This has been proved in [5, Proposition 9.1]. Moreover, if μ is bijective, then

$$\mathbb{S}\mathrm{t}(R,P) \approx \mathbb{S}\mathrm{t}(R,Q), \quad \mathbb{E}(R,P) \approx \mathbb{E}(R,Q), \quad K_2(R,P) \approx K_2(R,P).$$

This observation allows one to study polyhedral K-theory as a functor also in the polytopal argument. The map μ is called a *K-theoretic morphism* from P to Q. Though we cannot prove K_2-functoriality for all maps μ (see, however, [5, Proposition 9.1] for partial results) it is useful to note the $\mathbb{S}\mathrm{t}$-functoriality, since it implies bifunctoriality of the higher polyhedral K-groups with covariant arguments:

$$K_i^{\mathrm{Q}}(-,-), K_i^{\mathrm{V}}(-,-) : \underline{Commutative\ Rings} \times \underline{Balanced\ Polytopes} \to$$
$$\to \underline{Abelian\ Groups}, \quad i \geqslant 3.$$

For Quillen's theory this follows from equation (8) in Section 6.4. For Volodin's theory one observes that the mapping μ as above extends naturally to the column vectors in doubling spectra $\mathrm{Col}(\mathfrak{P}) \to \mathrm{Col}(\mathfrak{Q})$ so that the analogous conditions are satisfied. But then the extended mapping sends rigid systems to rigid systems. In fact, thanks to Proposition 3.1, for every balanced polytope P one can decide from the matrix

$$(\langle P_u, v \rangle)_{u,v \in \mathrm{Col}(P)}$$

when a subset $\{v_1, \ldots, v_n\} \subset \mathrm{Col}(P)$ defines the product $v_1 \cdots v_n$ in the strong or in the weak sense. It is, of course, also important that $v = -w$ if and only if $\imath(v) = -\imath(w)$. For details we refer the reader to [5, Section 9]. Now the functoriality of the groups $K_i^{\mathrm{V}}(-,-)$, $i \geqslant 3$ follows from Proposition 6.4(a,c).

In particular,

$$K_i(R, P \times Q) = K_i(R,P) \oplus K_i(R,Q), \quad i \geqslant 2,$$

for both theories because the analogous equations hold for $\mathbb{S}\mathrm{t}$ and $\mathbb{E}$ ([5, Section 9]).

Finally, we want to point out that the K-theoretic groups only depend on the projective toric variety associated with a polytope P.

The *normal fan* $\mathcal{N}(P)$ of a finite convex (not necessarily lattice) polytope $P \subset \mathbb{R}^n$ is defined as the complete fan in the dual space $(\mathbb{R}^n)^* = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ given by the system of cones

$$\left(\left\{ \varphi \in (\mathbb{R}^n)^* \,|\, \max_P(\varphi) = F \right\}, \ F \text{ a face of } P \right).$$

Two polytopes $P, Q \subset \mathbb{R}^n$ are called *projectively equivalent* if $\mathcal{N}(P) = \mathcal{N}(Q)$.

Next, we recall the relationship with projective toric varieties. Let P and Q be *very ample* polytopes in the sense of [5, Section 5]. This means that for every vertex $v \in P$ the affine semigroup $-v + (C_v \cap \mathbb{Z}^n) \subset \mathbb{Z}^n$ is generated by $-v + \mathrm{L}_P$, where C_v is the cone in $\mathbb{R}^n$ spanned by P at v (we assume that $P \subset \mathbb{R}^n$ and $\mathrm{gp}(S_P) = \mathbb{Z}^{n+1}$), and similarly for Q. Then $\mathcal{N}(P) = \mathcal{N}(Q)$ if and only if the projective toric varieties $\mathrm{Proj}(k[P])$ and $\mathrm{Proj}(k[Q])$ are naturally isomorphic for some field k. These varieties are normal, but not necessarily projectively normal [1, Example 5.5].

Projectively equivalent polytopes P and Q have the same set of column vectors: $\mathrm{Col}(P) = \mathrm{Col}(Q)$ (see [1]), and the identity map on this set is a K-theoretic morphism $P \to Q$. Therefore, we have

Proposition 6.8. *If P and Q are projectively equivalent balanced polytopes, then $K_i^Q(R,P) \approx K_i^Q(R,Q)$ and $K_i^V(R,P) \approx K_i^V(R,Q)$ for $i \geqslant 2$.*

7. Acyclicity of $X(R, P)^V$

In this section we follow Suslin [14]. However, a number of changes in Suslin's arguments [14] are necessary. Actually, the polyhedral constructions below do *not* specialize to those from [16] in the classical situation of unit simplices. In fact, a direct analogue of [14] seems to be impossible for general balanced polytopes.

As usual, P will denote a balanced polytope, admitting a column structure, and $\mathfrak{P}$ denote a doubling spectrum, starting with P.

Definition 7.1. Let $U, V \subset \mathrm{Col}(\mathfrak{P})$ be rigid systems and $k \in \mathbb{N}$. We say that U is *k-decomposable* in V if every irreducible vector $u \in U$ admits a representation $u = v_1 \cdots v_k$ with $v_1, \ldots, v_k \in V$ and, moreover, the sets of irreducible elements of V that appear in the V-decomposition of two different irreducible elements of U are disjoint.

Clearly, if U is k-decomposable in V, then $[U]$ is k-decomposable in V.

Lemma 7.2. *Let $k \geqslant 2$ be a natural number and $U_1, \ldots, U_m \subset \mathrm{Col}(P)$ be rigid systems for some $m \in \mathbb{N}$. Then there exist rigid systems $V_1, \ldots, V_m \subset \mathrm{Col}(\mathfrak{P})$ such that $U_i \subset V_i$ for $i \in [1,m]$ and $\bigcap_{i=1}^m [U_i]$ is k-decomposable in $\bigcap_{i=1}^m [V_i]$. If the U_i are V-rigid, then also the V_i can be chosen to be V-rigid.*

Proof. Obviously, the lemma follows by an iterated use of the following

Claim. *For an irreducible element $u \in \bigcap_{i=1}^m [U_i]$ there exist rigid systems $V_i \subset \mathrm{Col}(\mathfrak{P})$, $i \in [1,m]$, satisfying the conditions*:

- $U_i \subset V_i$ for all i,
- the irreducible elements in $\bigcap_{i=1}^m [U_i]$, except u, remain irreducible in $\bigcap_{i=1}^m [V_i]$,
- there are exactly k new irreducibles, say $v_1, \ldots, v_k$, in $\bigcap_{i=1}^m [V_i]$, belonging neither to $\bigcap_{i=1}^m [U_i]$ nor to the affine hull of P, such that $u = v_1 \cdots v_k$.

(The condition that $v_1, \ldots, v_k$ do not belong to the affine hull of P yields the separation property of irreducible elements, required in the second half of Definition 7.1.)

Consider the (uniquely determined) factorizations

$$u = u_{i1} u_{i2} \cdots u_{ir_i} \quad i \in [1,m]$$

where the $u_{i1}, \ldots, u_{ir_i}$ are irreducible elements in U_i. By Proposition 3.1(c) we have

$$P_{u_{11}} = P_{u_{21}} = \cdots = P_{u_{m1}}.$$

Consider the polytope

$$Q = P^{\lrcorner u_{11}} \quad (= P^{\lrcorner u_{21}} = \cdots = P^{\lrcorner u_{m1}}).$$

We have $\mathrm{Col}(Q) \subset \mathrm{Col}(\mathfrak{P})$ (Lemma 3.3(a)) and

$$u_{i1} = \delta^+ u_{i1}^|, \quad i \in [1, m].$$

By Lemma 4.9 the systems

$$W_i = U_i \cup \{\delta^+, u_{i1}^|\} \subset \mathrm{Col}(\mathfrak{P}), \quad i \in [1, m]$$

are rigid. In particular, the products $u_{i1}^| u_{i2} \cdots u_{ir_j}$, $i \in [1, m]$, exist and, clearly, they are equal. Let w denote this product.

Since neither δ^+ nor $u_{i1}^|$ is in the affine hull of P, W_i has the same irreducibles as U_i, except that u_{i1} is replaced by the pair of new irreducibles δ^+ and $u_{i1}^|$. By an iterative application of Lemmas 3.4 and 4.9 to δ^+ we can produce vectors $\delta_1, \ldots, \delta_{k-1}$ such that

- $\delta_j \notin$ (the affine hull of $Q \cup \{\delta_1, \ldots, \delta_{j-1}\}$) for $1 \leqslant j \leqslant k - 1$,
- the sets $V_i = U_j \cup \{\delta_1, \ldots, \delta_{k-1}, u_{i1}^|\} \subset \mathrm{Col}(\mathfrak{P})$, $i \in [1, m]$ are rigid systems,
- $\delta^+ = \delta_1 \cdots \delta_{k-1}$.

We have

$$\delta_1, \ldots, \delta_{k-1}, \quad w \in \bigcap_{j=1}^{l} V_j \quad \text{and} \quad u = \delta_1 \cdots \delta_{k-1} \cdot w.$$

By definition of the V_i, what remains to show is the irreducibility of the elements $\delta_1, \ldots, \delta_{k-1}$ and w in $\bigcap_{i=1}^{m} [V_i]$. But for the vectors of type δ this is obvious, and the irreducibility of w is an easy consequence of the irreducibility of u in $\bigcap_{i=1}^{m} [U_i]$—one argues in terms of supporting graphs, the irreducibility being interpreted as the condition that only the initial and terminal points belong simultaneously to all the corresponding paths coming from different rigid systems. $\square$

The only place in the paper where we use Υ-rigid systems essentially is the proof of the next lemma. It is a polyhedral translation of Suslin's arguments [14]. The possibility of such a translation depends heavily on Theorems 5.3 and 5.4. We do not know how to apply these arguments to rigid systems in general.

Lemma 7.3. *Let $U, V \subset \mathrm{Col}(\mathfrak{P})$ be Υ-rigid systems such that U is $(k+1)$-decomposable in V for some $k \in \mathbb{N}$. Then the homomorphisms*

$$H_i(G(R, U), \mathbb{Z}) \to H_i(G(R, V), \mathbb{Z}), \quad i \in [1, k]$$

of integral homologies, induced by the embedding $[U] \subset [V]$, are zero-maps.

Proof. It is sufficient to prove that the homomorphisms

$$H_i(G(R, U), F) \to H_i(G(R, V), F), \quad i \in [1, k]$$

are zero-maps for arbitrary field F (acted trivially by the triangular groups). We use induction on the pairs $(k, \#[U])$ with respect to the lexicographical order (implicitly used in [14].)
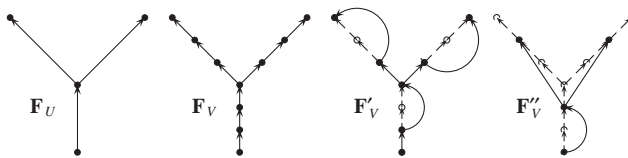
Fig. 9. The construction of $\mathbf{F}'_V$ and $\mathbf{F}''_V$.

Because of Theorem 3.5(b), $\mathrm{Im}(G(R,U) \to G(R,V))$ dies in the abelianization of $G(R,V)_{\mathrm{ab}}$ whenever U is 2-decomposable in V. In other words, our statement is true for $k = 1$ and arbitrary $\#[U]$. Therefore, we can assume $k \geqslant 2$. Observe that the claim is vacuously true for arbitrary k when $\#[U] = 0$–we assume $G(R, \emptyset) = 0$ by convention.

By the induction hypothesis we only need to show that the kth homology homomorphism is zero.

Let $\mathbf{F}_U$ and $\mathbf{F}_V$ denote the corresponding underlying Y-graphs. Since the sets of irreducible elements of V that appear in the V-decomposition of two different irreducible elements of U are disjoint (by Definition 7.1), one easily observes that there is no loss of generality in assuming that $\mathbf{F}_V$ arises from $\mathbf{F}_U$ by subdivision of each edge into $k + 1$ edges: first one considers the Y rigid subsystem of V consisting of these irreducible elements and then one changes certain subsets of irreducibles (corresponding to suitable paths in $\mathbf{F}_V$) by the corresponding products. It is clear that the new system will be again Y-rigid. For an edge $f \in E(\mathbf{F}_U)$ the corresponding path in $\mathbf{F}_V$ will be denoted by $[\varphi_1^f, \ldots, \varphi_{k+1}^f]$. We will use the notation introduced in Section 5.

The graph $\mathbf{F}'_V$ is defined as follows: For every $f \in E(\mathbf{F}_U)$ we replace the path $[\varphi_k^f, \varphi_{k+1}^f] \in \mathrm{path}\,\mathbf{F}_V$ of length 2 by a new edge φ^f from the initial to the end point of $[\varphi_k^f, \varphi_{k+1}^f]$ and omit the endpoint of $[\varphi_k^f]$. These new edges together with the remaining original edges of $\mathbf{F}_V$ define the graph $\mathbf{F}'_V$.

We now construct a further graph $\mathbf{F}''_V$ which contains the omitted vertices. Every path $l = [f_1, \ldots, f_s] \in \mathrm{path}_{1s}\mathbf{F}_U$ gives rise to a system of paths

$$\Phi^l = \{\Phi_1^l, \ldots, \Phi_s^l\} \subset \mathrm{path}\,\mathbf{F}_V$$

where

$$\Phi_1^l = [\varphi_1^{f_1}, \ldots, \varphi_k^{f_1}], \quad \Phi_2^l = [\varphi_{k+1}^{f_1}, \varphi_1^{f_2}, \ldots, \varphi_k^{f_2}],$$

$$\Phi_3^l = [\varphi_{k+1}^{f_2}, \varphi_1^{f_3}, \ldots, \varphi_k^{f_3}], \ldots, \Phi_s^l = [\varphi_{k+1}^{f_{s-1}}, \varphi_1^{f_s}, \ldots, \varphi_k^{f_s}].$$

Observe that each path ends in a vertex of $\mathbf{F}_V$ that was omitted in the construction of $\mathbf{F}'_V$. We take these omitted vertices and the initial points of the paths Φ_1^l as the vertices of $\mathbf{F}''_V$ and insert edges for each of the paths Φ_i^l. Clearly, both $\mathbf{F}'_V$ and $\mathbf{F}''_V$ are Y-graphs. (The reader should observe that the Y-property of the involved graphs is used crucially in the definition of $\mathbf{F}''_V$.) We illustrate the construction in Fig. 9.

There are natural embeddings $\mathrm{path}\,\mathbf{F}'_V, \mathrm{path}\,\mathbf{F}''_V \subset \mathrm{path}\,\mathbf{F}_V$. Let V' and V'' be the Y-rigid subsystems of $[V]$ supported by $\mathbf{F}'$ and $\mathbf{F}''$, and denote the natural embedding $G(R,U) \subset G(R,V')$ by $\imath$.

By construction every edge $\bar{\varphi} \in E(\mathbf{F}_V'')$ corresponds to a unique element $f \in E(\mathbf{F}_U)$ and, clearly, this correspondence is even an isomorphism of the graphs $\mathbf{F}_V'' \approx \mathbf{F}_U$.

By Theorem 5.4(b) we have the natural group isomorphism $G(R, U) \approx G(R, V'')$, denoted by ψ.

It follows from Theorem 3.5(e) that the elements of $G(R, U) \subset G(R, V)$ and Im $\psi = G(R, V'') \subset G(R, V)$ commute with each other. Hence we have the group homomorphism

$$\imath \cdot \psi \colon G(R, U) \to G(R, V), \quad (\imath \cdot \psi)(g) = g \cdot \psi(g).$$

Since the homologies are taken with coefficients in a field, by the Künneth formula the induced homomorphism of the kth homology groups can be decomposed as follows:

$$H_k(G(R, U), F) \overset{\Delta_*}{\to} H_k(G(R, U) \times G(R, U), F)$$

$$= \bigoplus_{i+j=k} H_i(G(R, U), F) \otimes H_j(G(R, U), F) \overset{\oplus_{i+j=k} H_i(\imath, F) \otimes H_j(\psi, F)}{\xrightarrow{\hspace{3cm}}}$$

$$\bigoplus_{i+j=k} H_i(G(R, V'), F) \otimes H_j(G(R, V''), F)$$

$$= H_k(G(R, V') \times G(R, V''), F)$$

$$\overset{m_*}{\to} H_k(G(R, V), F).$$

where Δ_* is the homomorphism induced by the diagonal mapping and m_* is induced by the multiplication in $G(R, V)$.

Now U is k-decomposable in V'. Therefore, by the induction hypothesis the homomorphisms $H_i(G(R, U), F) \to H_i(G(R, V'), F)$ are zero for $1 \leqslant i \leqslant k - 1$ and the decomposition above implies $H_k(\imath \cdot \psi) = H_k(\imath) + H_k(\psi)$.

Consider the element

$$e = \prod_{f \in E(\mathbf{F}_U)} e_{v(f)}^1 \in G_2(R, V),$$

where $v(f)$ is the element of $[V]$ that corresponds to the edge $\varphi_{k+1}^f \in E(\mathbf{F}_V)$ (and the product is understood as composition).

The following equation is proved by a routine verification on generators (using Theorem 3.5(e)):

$$\imath \cdot \psi = (\psi^e) \cdot ((\imath')^e \circ \pi),$$

where

- $\imath'$ is the restriction of $\imath$ to $G_2(R, U)$,
- for any element $\varepsilon \in G(R, V)$ we put $\psi^e(\varepsilon) = e \circ \psi(\varepsilon) \circ e^{-1}$ and similarly for $(\imath')^e$,
- $\pi \colon G(R, U) \to G_2(R, U)$ is the surjection from Theorem 5.4(a) ($r = 1$),
- the dot between the two homomorphisms on the right has the same meaning as that in $\imath \cdot \psi$. (The images again commute, since they the e-conjugates of commuting sets.)

We have

$$H_k(\imath) + H_k(\psi) = H_k(\imath \cdot \psi) = H_k(\psi) + H_k(\imath') \circ H_k(\pi)$$

where the second equation is proved similarly to the first. Since $\#[U]_2 < \#[U]$ the induction hypothesis implies $H_k(\imath') = 0$. Therefore, $H_k(\imath) = 0$. $\square$

By Lemmas 4.10, 7.2 and 7.3 we get

Corollary 7.4. *Let* $U_1, \ldots, U_m \subset \mathrm{Col}(\mathfrak{P})$ *be* Y*-rigid systems and* k *be a natural number. Then there are* Y*-rigid systems* $V_1, \ldots, V_m \subset \mathrm{Col}(\mathfrak{P})$ *such that* $U_i \subset V_i$, $i \in [1, m]$ *and the homomorphisms*

$$H_i\left(G\left(R, \bigcap_{i=1}^{m}[U_i]\right), \mathbb{Z}\right) \to H_i\left(G(R, \bigcap_{i=1}^{m}[V_i]), \mathbb{Z}\right), \quad i \in [1, k]$$

are zero-maps.

It is clear that if $V_1, \ldots, V_m$ satisfy the condition of the corollary then arbitrary rigid systems $W_1, \ldots, W_m \subset \mathrm{Col}(\mathfrak{P})$ with $V_i \subset W_i$, $i \in [1, m]$ do also.

Theorem 7.5. $X(R, P)^{\mathsf{Y}}$ *is acyclic.*

Proof. In view of the equality (6) in Section 6.1 we have to show that for arbitrary Y-rigid systems $U_i \subset \mathrm{Col}(\mathfrak{P})$, $i \in [1, m]$ the natural homomorphism

$$H_k\left(\bigcup_{i=1}^{m}BG(R, U_i), \mathbb{Z}\right) \to H_k\left(X(R, P)^{\mathsf{Y}}, \mathbb{Z}\right)$$

is zero for every $k \in \mathbb{N}$.

We show the following stronger claim in which we use the follow notation: for a family of sets Z_i, $i \in [1, m]$ and a subset $I \subset [1, m]$ we let Z_I denote the intersection $\bigcap_{i \in I} Z_i$.

Claim. Let s and k be natural numbers and $I_1, \ldots, I_s \subset [1, m]$ be nonempty subsets. Then there exist Y-rigid systems $W_1, \ldots, W_m \subset \mathrm{Col}(\mathfrak{P})$ such that $U_i \subset W_i$, $i \in [1, m]$ and the homomorphisms

$$H_i\left(\bigcup_{j=1}^{s}BG(R, [U]_{I_j}), \mathbb{Z}\right) \to H_i\left(\bigcup_{j=1}^{s}BG(R, [W]_{I_j}), \mathbb{Z}\right), \quad i \in [1, k]$$

are zero.

The claim gives the acyclicity as in the theorem when $s = m$ and $I_1 = \{1\}, \ldots, I_m = \{m\}$. On the other extreme, Corollary 7.4 implies the claim when $s = 1$.

We will use induction on s, the case $s = 1$ being already proved.

Assume $s \geqslant 2$. By the induction hypothesis there exist Y-rigid systems $U_i \subset V_i \subset W_i$, $i \in [1, m]$ such that the upper-right vertical and lower left vertical homomorphisms in the commutative diagram below are zero:

$$H_k \left(\bigcup_{j=1}^{s-1} \mathrm{B}G([U]_{I_j}) \right) \oplus H_k(\mathrm{B}G([U]_{I_s})) \;\to\; H_k \left(\bigcup_{j=1}^{s} \mathrm{B}G([U]_{I_j}) \right) \;\to\; \tilde{H}_{k-1} \left(\bigcup_{j=1}^{s-1} \mathrm{B}G([U]_{I_j} \cap [U]_{I_s}) \right)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$H_k \left(\bigcup_{j=1}^{s-1} \mathrm{B}G([V]_{I_j}) \right) \oplus H_k(\mathrm{B}G([V]_{I_s})) \;\to\; H_k \left(\bigcup_{j=1}^{s} \mathrm{B}G([V]_{I_j}) \right) \;\to\; \tilde{H}_{k-1} \left(\bigcup_{j=1}^{s-1} \mathrm{B}G([V]_{I_j} \cap [V]_{I_s}) \right)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$H_k \left(\bigcup_{j=1}^{s-1} \mathrm{B}G([W]_{I_j}) \right) \oplus H_k(\mathrm{B}G([W]_{I_s})) \;\to\; H_k \left(\bigcup_{j=1}^{s} \mathrm{B}G([W]_{I_j}) \right) \;\to\; \tilde{H}_{k-1} \left(\bigcup_{j=1}^{s-1} \mathrm{B}G([W]_{I_j} \cap [W]_{I_s}) \right)$$

Here the rows represent Mayer-Vietoris sequences in which the $\tilde{H}_{k-1}$ terms are being identified according to equation (5). (For typographical reasons we have omitted the ring R and the coefficient group $\mathbb{Z}$.)

It follows that $W_1, \ldots, W_m$ are the desired Y-rigid systems. $\square$

Question 7.6. Is $X(R, P)$ acyclic for every balanced polytope P?

8. On the coincidence of Quillen's and Volodin's theories

All polytopes are assumed to be balanced and to admit a column vector, unless specified otherwise.

In this section we single out the class of Col-*divisible* polytopes and prove that Quillen's and Volodin's K-theories coincide for them. The reason for the introduction of this class of polytopes is that a rigid system of column vectors should be embeddable into a Y-rigid system. Fortunately, Col-divisibility persists in doubling spectra. Moreover, it is closely related with the desired homotopy properties (simplicity) of the relevant spaces, thus yielding the coincidence of all the three theories $K^V(-, -)$, $K^V(-, -)^Y$ and $K^Q(-, -)$. These polytopes are not so rare. For instance, all balanced polygons are such (see Section 9).

Definition 8.1. A (balanced) polytope P is Col-*divisible* if its column vectors satisfy the conditions following:

$(\mathrm{CD})_1$ if ac and bc exist and $a \neq b$, then $a = db$ or $b = da$ for some d;
$(\mathrm{CD})_2$ if $ab = cd$ and $a \neq c$, then there exists t such that $at = c$, $td = b$, or $ct = a$, $tb = d$.
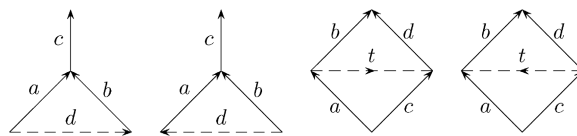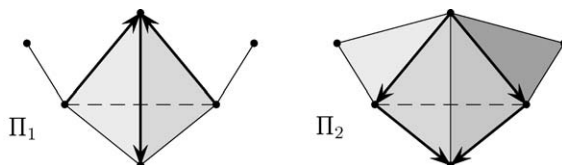
(See Fig. 10.)

Fig. 10. Col-divisibility.

Fig. 11. The violation of (CD_1) and (CD_2).

Remark 8.2. (a) It is enough in (CD_2) that $a = ct$. Then the product tb exists and is necessarily equal to d. For the existence we note that $\langle F, t \rangle > 0$ for all facets F with $\langle F, a \rangle > 0$, and so $\langle P_b, t \rangle > 0$ since $\langle P_b, a \rangle > 0$. (Compare Proposition 3.1.) Moreover, $t = -b$ is evidently impossible.

Similarly one sees that $d = tb$ is sufficient.

(b) The vector d required for (CD_1) exists as soon as a (or b) is invertible. Then $\langle P_c, -a \rangle = -\langle P_c, a \rangle < 0$, and $P_{-a} = P_c$. But $\langle P_c, b \rangle > 0$ by the existence of bc. So $b(-a)$ exists ($b = a$ is excluded by hypothesis), and also $(b(-a))a = b$ exists. (Again one uses Proposition 3.1.)

The same argument (in conjunction with (a)) shows that the invertibility of b or d is enough for the existence of t in (CD_2). But also the invertibility of a or c implies the existence of t, as the reader may check. (One obtains $a = c((-c)a)$ if c is invertible.)

Remark 8.3. (a) Not all polytopes are Col-divisible. Consider the polytope P from Remark 4.3(b)—the unit pyramid over the unit square. It violates (CD_1) and (CD_2) simultaneously as one can easily check by listing the column vectors and their products.

(b) The conditions (CD_1) and (CD_2) are independent of each other. This is illustrated by the following examples. As already remarked, the whole theory generalizes to lattice polyhedral complexes (in the sense of [2]) whose opposite extreme cases are single polytopes, treated here, and simplicial complexes, viewed as lattice polyhedral complexes. (The corresponding algebras are *Stanley-Reisner* rings of simplicial complexes.)

For simplicial complexes, for instance, a column vector is just an oriented edge such that the facets of the complex that contain the terminal point of the edge contain also the initial point. It is not difficult to see that the complex Π_1 of Fig. 11 satisfies (CD_2), but not (CD_1), whereas Π_2 satisfies (CD_1) and violates (CD_2). Both complexes contain a three-dimensional tetrahedron, and the additional triangles of Π_2 are two-dimensional cells. (Perhaps there are similar examples in the class of single balanced polytopes.) On

the other hand, it is clear that every simplicial complex admits a K-theoretic morphism $\imath$ to a unit simplex as in Proposition 8.5.

The next lemma contains the crucial combinatorial properties of Col-divisible polytopes. A column vector v is called *terminal* if there exists no base facet F with $\langle F, v \rangle > 0$.

Lemma 8.4. *Let P be a* Col-*divisible polytope and $u, v, w \in \mathrm{Col}(P)$.*

(a) *Suppose that $\langle P_u, w \rangle = 0$ and that the product vw exists. Then $\langle P_u, v \rangle \leqslant 0$.*
(b) *If v is not terminal, then there exists exactly one base facet F with $\langle F, v \rangle = 1$.*
(c) *Suppose that $v, w \in \mathrm{Col}(P)$ have the property that there exist base facets F, G, $F \neq G$ with*

$$\langle F, v \rangle = \langle F, w \rangle = -1 \quad and \quad \langle G, v \rangle = \langle G, w \rangle = 1.$$

Then $v = w$.

Proof. (a) We have to exclude $\langle P_u, v \rangle > 0$. If this were the case, then the products vu and $(vw)u$ would exist by Proposition 3.1(a). In fact, if $v + u = 0$ or $v + w + u = 0$, then $-v \in \mathrm{Col}(P)$ (see Proposition 3.1(h) for the second case), and P_{-v} is the only facet over which v has positive height. Since $\langle P_w, v \rangle > 0$, one concludes that $P_w = P_u$, a contradiction to $\langle P_u, w \rangle = 0$. So we can assume that vu and $(vw)u$ exist.

By (CD$_1$) there exists $x \in \mathrm{Col}(P)$ such that either $xv = vw$ or $x(vw) = v$. In the first case $x = w$, and in the second case $x = -w$. Now the first equality is excluded by Proposition 3.1(d) and, in view of Proposition 3.1(f), the second equality implies the existence of the product $(-w)v$. In particular, $\langle P_v, -w \rangle > 0$ and $\langle P_v, w \rangle < 0$, in contradiction with the existence of vw.

(b) Suppose first that $-v \in \mathrm{Col}(P)$. Then $F = P_{-v}$ is the only base facet with $\langle F, v \rangle = 1$.

If $-v \notin \mathrm{Col}(P)$, $u \neq -v$ for each column vector u such that $F = P_u$, and the product vu exists. Clearly, $\langle P_u, v \rangle = 1$. Consider another base facet $P_w \neq P_u$. We can also assume $P_w \neq P_v$. If $\langle P_w, u \rangle = 1$, then $\langle P_w, v \rangle = 0$ since P is. So assume that $\langle P_w, u \rangle = 0$. Then $\langle P_w, v \rangle = 0$ by (a).

(c) Again we consider the case that $-v \in \mathrm{Col}(P)$ first. Then $-v = -w$, and so $v = w$, follows from Proposition 3.1(g).

By symmetry we can assume that $-v \notin \mathrm{Col}(P)$ and $-w \notin \mathrm{Col}(P)$. Then vu exists for u with $G = P_u$, and wu also exists. By (CD$_1$) one of v, w is divisible by the other. But this is a contradiction because if, say, $v = tw$ then v and w have different base facets according to Proposition 3.1(c). □

Lemma 8.4 makes it easy to identify column vectors and to control the partial product structure. We extend our notation as follows (similarly as in connection with Lemma 4.8): if $v \in \mathrm{Col}(P)$ is not terminal, then P^v denotes the unique base facet of P with $\langle F, v \rangle = 1$. When we write $G \neq P^v$, this should not be interpreted as including the existence of P^v. One has $\langle F, v \rangle = 0$ for all facets $F \neq P_v, P^v$. Note the following

product rule:

- The existence of the product uv is equivalent to $P^u = P_v$ and $P_u \neq P^v$, as follows immediately from Proposition 3.1. Moreover $P_{uv} = P_u \neq P^u = P_v$ and $P^{uv} = P^v$, provided the latter exists.

This simple rule will save us many references to Proposition 3.1.

Next we want to show that for a Col-divisible polytope P there exist a unit simplex Δ_n and an embedding $\imath : \mathrm{Col}(P) \to \mathrm{Col}(\Delta_n)$ which induces an isomorphism of partial product structures between $\mathrm{Col}(P)$ and the image of $\imath$.

The critical vectors for the construction are the terminal ones. Therefore we need some preparation. Terminal elements u, v of $\mathrm{Col}(P)$ are called *neighbors* if there exists $t \in \mathrm{Col}(P)$ with $u = tv$ or $v = tu$, and the classes of the finest equivalence relation on the set of terminal column vectors that respects neighbors are called *cliques*. We claim:

(i) If $u, v, u \neq v$, belong to the same clique, then they are neighbors, or there exist $a, b \in \mathrm{Col}(P)$ and a terminal w such that $u = aw$ and $v = bw$.
(ii) Two different members of the same clique have different base facets.

Observe that (ii) follows readily from (i): if $u = tv$, then u and v have different base facets by the rule above. In the second case, $u = aw$ and $v = bw$, we must use (CD$_1$): we have $P_u = P_a$ and $P_v = P_b$ (again by our rule) and the divisibility of one of a, b by the other forces these base facets to be different.

For (i) it is enough to show that every chain $v_1, \ldots, v_n$ of successive, pairwise different neighbors $v_i \neq v_{i+1}$ can be shortened if one of the following conditions is satisfied for some i:

$$(1)\ v_{i-1} = t_1 v_i,\ v_i = t_2 v_{i+1},\quad (2)\ v_{i+1} = t_1 v_i,\ v_i = t_2 v_{i-1},\quad (3)\ v_i = a v_{i-1},\ v_i = b v_{i+1}.$$

In case (1) $v_{i-1} = t_1 v_i = t_1(t_2 v_{i+1}) = (t_1 t_2) v_{i+1}$: the last product exists by Proposition 3.1(f) since $t_1 + t_2 \neq 0$. Case (2) is symmetric to (1). In case (3) we have to invoke (CD$_2$): there exists t such that $v_{i-1} = t v_{i+1}$ or $v_{i+1} = t v_{i-1}$.

Now we choose a unit simplex Δ with enough facets, namely such that it has a facet E_F for each base facet F of P, and a facet E_C for each clique C of terminal column vectors v of P. (We assume that all these facets are pairwise different.)

Define $\imath : \mathrm{Col}(P) \to \mathrm{Col}(\Delta)$ as follows:

- If v is not terminal then we choose $\imath(v)$ to be $e \in \mathrm{Col}(\Delta)$ with $\Delta_e = E_{P_v}$ and $\Delta^e = E_{P^v}$.
- If v is terminal, we choose $\imath(v)$ as the column vector e of Δ such that $\Delta_e = E_{P_v}$ and $\Delta^e = E_C$ where C is the clique of v.

By Lemma 8.4(c) and claim (ii) above the map $\imath$ is injective. Moreover, $\imath(v) = -\imath(-v)$ if $v, -v \in \mathrm{Col}(P)$, and if $\imath(v) = -\imath(w)$, then $v = -w$. Next, if $\langle P_w, v \rangle = 1$, then $\langle P_{\imath(w)}, \imath(v) \rangle = 1$, and conversely. This shows already that vw exists if and only if $\imath(v)\imath(w)$ exists. The reader may check that indeed $\imath(vw) = \imath(v)\imath(w)$ if vw exists: it is enough that $\langle E, \imath(vw) \rangle = \langle E, \imath(v) \rangle + \langle E, \imath(w) \rangle$ for all facets E of Δ_n. In view of Section 6.5 we get

Proposition 8.5. *Let P be a Col-divisible polytope. Then there exist a unit simplex Δ_n and an embedding $\imath : \mathrm{Col}(P) \to \mathrm{Col}(\Delta_n)$ which defines a K-theoretic morphism $\imath : P \to \Delta_n$. In particular, a system $V \subset \mathrm{Col}(P)$ is rigid (Y-rigid) if and only if $\imath(V)$ is rigid (Y-rigid).*

We want to show that Col-divisibility persists under doubling.

Proposition 8.6. *Assume P is a Col-divisible polytope and $v \in \mathrm{Col}(P)$. Then $P^{\dashv v}$ is Col-divisible as well.*

Proof. It is important for the proof that $P^{\dashv v}$ has the properties stated in Lemma 8.4(b) and (c). This follows easily from Eqs. (2)–(4) in Section 3.4. We are justified to use the notation Q^v, and the product rule formulated above holds accordingly.

First observe that the conditions (CD$_1$) and (CD$_2$) are satisfied if all the relevant vectors belong to either $\mathrm{Col}(P^-)$ or $\mathrm{Col}(P^|)$. By symmetry between P^- and $P^|$ the cases listed below cover all the other possible situations. To simplify the notation we set $Q = P^{\dashv v}$.

Condition (CD$_1$). We have to show that a is divisible by b or vice versa if $a \neq b$ and the products ac and bc exists. By Remark 8.2 we can stop checking (CD$_1$) if a turns out to be invertible in the case under consideration.

Case (i): $c = \delta^+$. Then $Q^a = Q^b = Q_{\delta^+} = P^|$. Since $Q_a, Q_b \neq Q_{\delta^+} = P^-$, both a and b are column vectors of $P^- = P$, and $P^a = P^b = P^| \cap P^- = P_v$. So av and bv exist if $a, b \neq -v$, and the property (CD$_1$) of P can be applied. We may assume that $a = -v$ and can stop.

Case (ii): $c \in \mathrm{Col}(P^-)$ and $P^c = P_v$. Then $Q_a, Q_b, Q^a, Q^b \neq Q^c = P^|$, and so $a, b \in \mathrm{Col}(P^|)$.

We can further assume $a \in \mathrm{Col}(P^|) \backslash \mathrm{Col}(P^-)$. Then Lemma 3.3(c) implies $ac = \delta^-$. In particular, a is an invertible column vector, and we are done. (Use Proposition 3.1(h).)

Case (iii): $c \in \mathrm{Col}(P_v)$. We can assume $a \in \mathrm{Col}(P^-)$. If $a \in \mathrm{Col}(P_v)$, then all three vectors a, b, c belong to P^- or $P^|$, and (CD$_1$) for P applies. One has $P_v \neq P^a$ since $P^a = P_c \neq P_v$.

Thus we are left with the case $P_a = P_v$. If $b = a^|$ then δ^+ does the job: $a = \delta^+ b$. So we can further assume $b \neq a^|$. We have $b^- \in \mathrm{Col}(P)$, $b^- \neq a$ and the products ac, $b^- c$ both exist. By (CD$_1$) there exists $x \in \mathrm{Col}(P)$ such that either $a = xb^-$ or $b^- = xa$. By symmetry we can assume that the first equation holds. Then $P_a \neq P_{b^-}, P^{b^-}$, i.e. $b^- \in \mathrm{Col}(P_v)$. But this is equivalent to $b \in \mathrm{Col}(P_v)$ and, thus, all three vectors a, b, c belong to $\mathrm{Col}(P)$. Done.

Case (iv): $c \in \mathrm{Col}(P^-)$ and $P_c = P_v$. In this situation $Q^a = Q^b = P^|$. If neither of a and b is δ^-, then $a, b, c \in \mathrm{Col}(P^-)$ by Lemma 3.3(c). So we can assume that $a = \delta^-$, and are done since a is invertible.

Condition (CD$_2$). We have to show that $ab = cd$, $a \neq c$, implies the existence of t such that $a = ct$, $d = tb$, or $c = at$, $b = td$. By Remark 8.2 we can stop the discussion of a case if one of a, b, c, d turns out to be invertible.

Case (i): $ab = \delta^+$. Then all vectors a, b, c, d are invertible, and we are done. (See Proposition 3.1(h).)

Case (ii): $ab \in \mathrm{Col}(P)$ and $P^| = Q^{ab}$. Without loss of generality we can assume that $a \in \mathrm{Col}(P^|)$, $P^- = Q^a$ and $b = \delta^-$. All the other possible situations are either symmetric to this one or reduce to the case in which all the four vectors belong to $\mathrm{Col}(P)$. Since b is invertible, we are again done.

Case (iii): $ab \in \mathrm{Col}(P)$ and $P^| = Q_{ab}$. Similarly to the previous case we can assume that $a = \delta^+$, and are done.

Case (iv): $ab \in \mathrm{Col}(P_v) = \mathrm{Col}(P^-) \cap \mathrm{Col}(P^|)$. Without loss of generality we can assume $a, b \in \mathrm{Col}(P)$ such that $P^a = P_v$, $P_b = P_v$, and $c, d \in \mathrm{Col}(P^|)$ such that $(P^|)^c = (P^|)_{v|}$ and $(P^|)_d = (P^|)_{v|}$. (If, say, $a \in \mathrm{Col}(P_v)$, then also $b \in \mathrm{Col}(P_v)$, and all the four vectors belong to $\mathrm{Col}(P^-)$ or $\mathrm{Col}(P^|)$).

It follows that $(P^|)_c = P_a$ and $(P^|)^c = (P_a)^|$. The only vector c satisfying these two conditions is $c = a^| = a\delta^+$, and we are done by Remark 8.2. $\square$

Now we are ready to present the polyhedral version of Suslin's argument for the desired simplicity.

Proposition 8.7. *Suppose P is a Col-divisible polytope. Then the space $\mathrm{X}(R, P)$ is simple in dimension $\geqslant 2$.*

Proof. By Proposition 6.4(a,b) we have to show that $\pi_1(\mathrm{X}'(R,P)) = \mathbb{S}\mathrm{t}(R,P)$ acts trivially on the higher homotopy groups of the universal cover $\mathbb{V}(\mathbb{S}\mathrm{t}(R,P)) \to \mathrm{X}'(R,P)$. To this end we show that for any $z \in \mathbb{S}\mathrm{t}(R,P)$ the mapping $\mu_z : \mathbb{V}(\mathbb{S}\mathrm{t}(R,P)) \to \mathbb{V}(\mathbb{S}\mathrm{t}(R,P))$, determined by right multiplication with $z \in \mathbb{S}\mathrm{t}(R,P)$, is homotopic to the identity mapping.

In view of Proposition 8.6 (and the equations $x_v^\lambda = [x_{\delta^+}^1, x_{v|}^\lambda]$) it is enough to show that for a Col-divisible polytope Q, a vector $v \in \mathrm{Col}(Q)$ and an element $\lambda \in R$ the simplicial mappings $\mathbb{V}(\mathbb{S}\mathrm{t}(R,Q)) \to \mathbb{V}(\mathbb{S}\mathrm{t}(R,Q^{\lrcorner v}))$, induced by

  (i) the natural group homomorphism $f_v : \mathbb{S}\mathrm{t}(R,Q) \to \mathbb{S}\mathrm{t}(R,Q^{\lrcorner v})$,
 (ii) $f_v \cdot x_{\delta^+}^1 : \mathbb{S}\mathrm{t}(R,Q) \to \mathbb{S}\mathrm{t}(R,Q^{\lrcorner v})$, $x \mapsto xx_{\delta^+}^1$,
(iii) $f_v \cdot x_{\delta^+}^{-1} : \mathbb{S}\mathrm{t}(R,Q) \to \mathbb{S}\mathrm{t}(R,Q^{\lrcorner v})$, $x \mapsto xx_{\delta^+}^{-1}$,
 (iv) $f_v \cdot x_{v|}^\lambda : \mathbb{S}\mathrm{t}(R,Q) \to \mathbb{S}\mathrm{t}(R,Q^{\lrcorner v})$, $x \mapsto xx_{v|}^\lambda$,
  (v) $f_v \cdot x_{v|}^{-\lambda} : \mathbb{S}\mathrm{t}(R,Q) \to \mathbb{S}\mathrm{t}(R,Q^{\lrcorner v})$, $x \mapsto xx_{v|}^{-\lambda}$

are homotopy equivalent.

We denote the induced mappings between the Volodin simplicial sets by the same names f_v, $f_v \cdot x_{\delta^+}^1$, $f_v \cdot x_{\delta^+}^{-1}$, $f_v \cdot x_{v|}^\lambda$ and $f_v \cdot x_{v|}^{-\lambda}$. Then the desired homotopies $f_v \sim f_v \cdot x_{\delta^+}^1$, $f_v \sim f_v \cdot x_{\delta^+}^{-1}$, $f_v \sim f_v \cdot x_{v|}^\lambda$, and $f_v \sim f_v \cdot x_{v|}^{-\lambda}$ are given by

$$\left(\left(\underbrace{0,\dots,0}_{s}, \underbrace{1,\dots,1}_{t} \right) \times (z_1, \dots, z_{s+t}) \right) \mapsto (z_1, \dots, z_s, z_{s+1}x, \dots, z_{s+t}x),$$

where x is correspondingly $x_{\delta^+}^1$, $x_{\delta^-}^{-1}$, $x_{v|}^\lambda$, or $x_{v|}^{-\lambda}$.

We only need to make sure that these homotopies do actually exist. But this holds since the systems $V \cup \{\delta^+\}$ and $V \cup \{v^|\}$ in $\mathrm{Col}(Q^{\lrcorner v})$ are rigid for every rigid system $V \subset \mathrm{Col}(Q)$. In fact, by Proposition 8.5 these sets with partial products are isomorphic to subsets of $\mathrm{Col}(\Delta_{n+1})$ respectively of the type $W \cup \{\delta^+\}$ and $W \cup \{v^|\}$ for some rigid system $W \subset \mathrm{Col}(\Delta_n)$. (Here $n \in \mathbb{N}$ and $\delta^+ \notin \mathrm{Col}(\Delta_n)$). But the latter sets are obviously rigid (Example 4.5(b)). $\square$

Remark 8.8. (a) We have used only the following consequence of Col-divisibility for all members Q of a doubling spectrum starting from P: $V \cup \{\delta^+\}$ and $V \cup \{v^|\}$ in $\mathrm{Col}(Q^{\lrcorner v})$ are rigid for every rigid system $V \subset \mathrm{Col}(Q)$. It may hold in a larger class of polytopes, but we have not found a more general natural sufficient condition for it than Col-divisibility.

It is not hard to give an example for which $V \cup \{\delta^+\}$ is not a rigid system. In fact, suppose there exist column vectors u, v, w such that $\langle P_v, u \rangle = \langle P_w, u \rangle = 1$, but $P_v \neq P_w$. (This is the case for the unit pyramid over the unit square; see Example 4.3(b).) Take $V = \{u, v\}$. Then uv exists, and both $(uv)\delta_w^+$ and $u\delta_w^+$ exist. This is impossible if $V \cup \{\delta_w^+\}$ is rigid.

(b) It is also worth noticing that the proof above does not work for the Y-theory: the enlarged systems $V \cup \{\delta^+\}$ and $V \cup \{v^|\}$ may no longer be Y-rigid even if the original system V was. This and the difficulty in extending Lemma 7.3 to arbitrary rigid systems make it necessary to use both versions of Volodin's theory in order to establish the coincidence with Quillen's theory.

Next we introduce the notion of λ-*complexity* of a graph $\mathbf{F}$. (Recall that the graphs we deal with are without multiple edges, loops and they are assumed to be oriented.) The *height* $\mathrm{ht}\, x$ of an element $x \in \mathrm{vert}\mathbf{F}$ is by definition the maximal possible length of an element $l \in \mathrm{path}\,\mathbf{F}$ having x as its terminal point. We assume $\mathrm{ht}\, x = 0$ if such a path does not exist.

Consider the two element subsets $\{l, l'\} \subset \mathrm{path}\,\mathbf{F}$ such that l and l' meet only at their initial and terminal points. These couples will be called *regular cycles* and we will use the notation $l \Diamond l'$ for them. Put

$$\mathrm{ht}(l \Diamond l') = \mathrm{ht}(\text{the terminal point of } l).$$

Now consider the triples:

$$\{\alpha_1, \alpha_2, \beta \in E(\mathbf{F}) \mid \alpha_1 \neq \alpha_2, \ [\alpha_1, \beta], [\alpha_2, \beta] \in \mathrm{path}\,\mathbf{F}\}.$$

The point $\lambda(\alpha_1, \beta, \alpha_2)$ where such edges meet will be called a *meeting point* of α_1, α_2 and β.

Definition 8.9. The λ-complexity $\mathrm{comp}_\lambda \mathbf{F}$ of the graph $\mathbf{F}$ is defined by

$$\mathrm{comp}_\lambda \mathbf{F} = (A, B)$$

where $A = \max(\mathrm{ht}(l \Diamond l')), \mathrm{ht}(\lambda(\alpha_1, \beta, \alpha_2))$ for $l, l', \alpha_1, \alpha_2, \beta$ as above and B is the number of those vertices of $\mathbf{F}$ where this maximum is achieved.

If there are no regular cycles and no meeting points the λ-complexity is $(0, 0)$.

Lemma 8.10. *If a rigid system $V \subset \mathrm{Col}(P)$ is supported by a graph whose λ-complexity is $(0,0)$ then V is a Y-rigid system.*

Proof. If $\mathbf{F}$ is a graph supporting V and satisfying the condition $\mathrm{comp}_\lambda \mathbf{F} = (0,0)$ then we can form a new graph $\mathbf{F}_Y$ by disconnecting edges of $\mathbf{F}$ whenever they meet at their terminal points—we split the terminal points of meeting edges. Using the fact that there are no edges starting form such meeting points, it is easily seen that $\mathbf{F}_Y$ is a Y-graph supporting V. $\square$

Proposition 8.11. *Assume P is a Col-divisible polytope. Then for any rigid system $V \subset \mathrm{Col}(P)$ there is a Y-rigid system $W \subset \mathrm{Col}(P)$ such that $[V] \subset [W]$.*

Proof. Let $\mathbf{F}$ be a graph supporting the system V. Assume $l \Diamond l'$ is a regular cycle of the maximal possible height for some paths $l = [e_1, \ldots, e_n]$ and $l' = [e'_1, \ldots, e'_m]$ $(e_i, e'_j \in E(\mathbf{F}))$.

If $m = 1$ or $n = 1$ then we delete the corresponding edge of $\mathbf{F}$. The obtained smaller graph still supports V. Therefore, without loss of generality we can assume that $m, n \geqslant 2$. Consider the column vectors

$$a = v_{e_1} \cdots v_{e_{n-1}}, \quad b = v_{e_n}, \quad c = v_{e'_1} \cdots v_{e'_{m-1}}, \quad d = v_{e'_m}.$$

(v_e is the column vector that corresponds to the edge $e \in E(\mathbf{F})$.) By condition (CD_2) we can assume $at = c$ and $td = b$ for some $t \in \mathrm{Col}(P)$.

If $t \in [V]$ then the subgraph $\mathbf{G} \subset \mathbf{F}$, obtained from $\mathbf{F}$ by deleting the edge e_n, supports V. Clearly, the maximal possible height of a regular cycle in $\mathbf{G}$ is at most $\mathrm{ht}(l \Diamond l')$ and the number of regular cycles in $\mathbf{G}$ of this height is strictly less than the corresponding number for the graph $\mathbf{F}$.

Now assume $t \notin [V]$. There are two cases: either $W = V \cup \{t\}$ is a rigid system or it is not such.

First consider the case when $W = V \cup \{t\}$ is a rigid system. In this situation W is supported by the graph $\mathbf{H}$ which is obtained by deleting the edge e_n and adding a new oriented edge that connects the terminal point of e_{n-1} with the initial point of e'_m. Again, any regular cycle in $\mathbf{H}$ has height at most $\mathrm{ht}(l \Diamond l')$ and the number of such cycles has strictly decreased.

In the remaining case when W is not a rigid system, Proposition 8.5 (together with the description of rigid systems in a unit simplex, Example 4.5(b),(c)) implies that $\{w, -w\} \subset [W]$ for some $w \in \mathrm{Col}(P)$. Since V is rigid, the latter condition is equivalent to the condition $-t \in [V]$. But then $d = (-t)b$. (We use the properties of a unit simplex). Therefore, V is supported by the subgraph $\mathbf{E} \subset \mathbf{F}$ obtained by deleting the edge e'_m. Again, we can decrease the number of the highest regular cycles.

By continuation of this process we finally reach a rigid system $V' \subset \mathrm{Col}(P)$ supported by a graph without regular cycles and $[V] \subset [V']$.

Thereafter we carry out a similar procedure to eliminate the meeting points. This is possible due to condition (CD_1). It is essential that we do not create regular cycles during the process. The final result will be a rigid system W supported by a graph $\mathbf{K}$ such that $[V] \subset [W]$ and $\mathrm{comp}_\lambda \mathbf{K} = (0,0)$. By Lemma 8.10 we are done. $\square$

Theorem 8.12. *Suppose P is a* Col-*divisible polytope. Then*

$$K_i^Q(R, P) = K_i^{\mathbb{V}}(R, P)^{\mathsf{Y}} = K_i^{\mathbb{V}}(R, P), \quad i \geqslant 2.$$

Proof. By Propositions 8.6 and 8.11 we have $\mathrm{X}(R, P) = \mathrm{X}(R, P)^{\mathsf{Y}}$. By Proposition 8.7(c) we obtain the equality of Volodin's two theories (for the case of K_2 one has Proposition 6.6), and by Lemma 6.7, Theorem 7.5 and Proposition 8.7 we obtain their coincidence with Quillen's theory. $\square$

A balanced polytope P could be called *K-theoretic relative to R* if the equations in Theorem 8.12 are satisfied for it.

Question 8.13. Is the property of being K-theoretic absolute, i.e. independent of the ring? Is the polytope from Remark 4.3(b) K-theoretic? What are the corresponding groups? Recall, that there is no nice matrix theoretical representation available for the corresponding stable group of elementary automorphisms (see [5, Example 10.3]). It is exactly such representations that in the polygonal case allow us to perform the computations in the next section.

9. Polygonal *K*-theories

The class of Col-divisible polytopes may at first glance seem rather restricted. However, it follows immediately from Theorem 3.2 that *all* balanced polygons are Col-divisible. In particular, balanced polygons are K-theoretic. Classification of the Col-divisible polytopes is an interesting problem already in dimension 3.

In Theorem 3.2 we have grouped all balanced polygons in six infinite series which, according to [5, Theorem 10.2], give rise to the following isomorphism classes of stable elementary automorphism groups:

(a) $\mathbb{E}_a = E(R),$

(b) $\mathbb{E}_b = \begin{pmatrix} E(R) & \mathrm{End}_R(\oplus_{\mathbb{N}} R) \\ 0 & E(R) \end{pmatrix},$

(c) $\mathbb{E}_c = \begin{pmatrix} E(R) & \mathrm{End}_R(\oplus_{\mathbb{N}} R) & \mathrm{Hom}_R(\oplus_{\mathbb{N}} R, R) \\ 0 & E(R) & \mathrm{Hom}_R(\oplus_{\mathbb{N}} R, R) \\ 0 & 0 & 1 \end{pmatrix},$

(d) $\mathbb{E}_{d,t} = \begin{pmatrix} E(R) & \mathrm{Hom}_R(\oplus_{\mathbb{N}} R, R^t) \\ 0 & \mathbf{Id}_t \end{pmatrix}, \quad t \in \mathbb{N},$

(e) $\mathbb{E}_e = E(R) \times E(R),$

(f) $\mathbb{E}_f = \begin{pmatrix} E(R) & \mathrm{Hom}_R(\oplus_{\mathbb{N}} R, R) \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} E(R) & \mathrm{Hom}_R(\oplus_{\mathbb{N}} R, R) \\ 0 & 1 \end{pmatrix}.$

In view of the remarks above the following theorem identifies all possible polygonal K-groups (under a technical restriction on rings).

Definition 9.1 (Nesterenko and Suslin [10]). A not necessarily commutative ring A is an $S(n)$-*ring* if there are $a_1, \ldots, a_n \in A^*$ such that the sum of each nonempty subfamily is a unit. If A is an $S(n)$-ring for all $n \in \mathbb{N}$, then A has *many units*.

The class of rings with many units includes local rings with infinite residue fields and algebras over rings with many units.

Theorem 9.2. *For every* (*commutative*) *ring* R *and every index* $i \geqslant 2$ *we have*:

(a) $\pi_i(\mathrm{B}\mathbb{E}_a^+) = K_i(R)$,
(b) $\pi_i(\mathrm{B}\mathbb{E}_b^+) = K_i(R) \oplus K_i(R)$,
(c) $\pi_i(\mathrm{B}\mathbb{E}_c^+) = K_i(R) \oplus K_i(R)$ *if* R *has many units*,
(d) $\pi_i(\mathrm{B}\mathbb{E}_{d,t}^+) = K_i(R)$ *if* R *has many units*,
(e) $\pi_i(\mathrm{B}\mathbb{E}_e^+) = K_i(R) \oplus K_i(R)$,
(f) $\pi_i(\mathrm{B}\mathbb{E}_f^+) = K_i(R) \oplus K_i(R)$ *if* R *has many units*.

Proof. Let $\mathbb{G}_a, \mathbb{G}_b, \ldots, \mathbb{G}_f$ denote the groups, obtained by the corresponding substitution of $\mathrm{GL}(R)$ for $E(R)$ in the groups $\mathbb{E}_a, \mathbb{E}_b, \ldots, \mathbb{E}_f$. Then we have the equations $\mathbb{E}_a = [\mathbb{G}_a, \mathbb{G}_a]$, $\mathbb{E}_b = [\mathbb{G}_b, \mathbb{G}_b], \ldots, \mathbb{E}_f = [\mathbb{G}_f, \mathbb{G}_f]$. We have $\pi_i(\mathrm{B}\mathbb{G}_a^+) = \pi_i(\mathbb{E}_a^+)$, $\pi_i(\mathrm{B}\mathbb{G}_b^+) = \pi_i(\mathbb{E}_b^+), \ldots, \pi_i(\mathrm{B}\mathbb{G}_f^+) = \pi_i(\mathbb{E}_f^+)$, where the $+$ constructions $\mathrm{B}\mathbb{G}_a^+$, etc. are considered relative to the normal subgroups $\mathbb{E}_a \subset \mathbb{G}_a = \pi_1(\mathrm{B}\mathbb{G}_a)$, etc. which are also perfect. (See, for instance, [13, Theorem 5.2.7].)

Since we have $\mathrm{B}\mathbb{G}_f^+ \approx \mathrm{B}\mathbb{G}_{d,1}^+ \times \mathrm{B}\mathbb{G}_{d,1}^+$ (essentially due to the uniqueness of the $+$ construction) it is enough to show that

$$\mathrm{B}\mathrm{GL}(R)^+ \times \mathrm{B}\mathrm{GL}(R)^+ \approx \mathrm{B}\mathbb{G}_b^+ \approx \mathrm{B}\mathbb{G}_c^+.$$

and

$$\mathrm{B}\mathrm{GL}(R)^+ \approx \mathrm{B}\mathbb{G}_{d,t}^+, \quad t \in \mathbb{N}.$$

Because of the equations

$$H_1(\mathbb{G}_b, \mathbb{Z}) = (\mathbb{G}_b)_{\mathrm{ab}} = \pi_1(\mathbb{G}_b^+),$$

$$H_1(\mathbb{G}_c, \mathbb{Z}) = (\mathbb{G}_c)_{\mathrm{ab}} = \pi_1(\mathbb{G}_c^+),$$

$$H_1(\mathbb{G}_{d,1}, \mathbb{Z}) = (\mathbb{G}_{d,1})_{\mathrm{ab}} = \pi_1(\mathbb{G}_{d,1}^+)$$

in conjunction with Whitehead's theorem it is sufficient to establish that

(i) $H_i(\mathbb{G}_b, \mathbb{Z}) = H_i(\mathrm{GL}(R) \times \mathrm{GL}(R), \mathbb{Z}), \quad i \in \mathbb{N},$

(ii) $H_i(\mathbb{G}_c, \mathbb{Z}) = H_i(\mathrm{GL}(R) \times \mathrm{GL}(R), \mathbb{Z}), \quad i \in \mathbb{N},$

(iii) $H_i(\mathbb{G}_{d,t}, \mathbb{Z}) = H_i(\mathrm{GL}(R), \mathbb{Z}), \quad i, t \in \mathbb{N}.$

Now, (i) is proved in [12] (for not necessarily commutative rings), and the stronger unstable version of (iii) (for not necessarily commutative) rings with many units is proved in [10]. It only remains to notice that the validity of (i) and (iii) for not necessarily commutative rings implies (ii) as follows. Put

$$T = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}.$$

Then $\mathrm{GL}(T) \approx \mathbb{G}_b$ and the following natural *split* epimorphisms give the result

$$\begin{pmatrix} \mathrm{GL}(T) & \mathrm{Hom}(\oplus_{\mathbb{N}}T, T) \\ 0 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} \mathrm{GL}(R) & \mathrm{Hom}(\oplus_{\mathbb{N}}R, \oplus_{\mathbb{N}}R) & \mathrm{Hom}(\oplus_{\mathbb{N}}R, R) & \mathrm{Hom}(\oplus_{\mathbb{N}}R, R) \\ 0 & \mathrm{GL}(R) & 0 & \mathrm{Hom}(\oplus_{\mathbb{N}}R, R) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \mathbb{G}_c \to$$

$$\to \mathrm{GL}(R) \times \mathrm{GL}(R)$$

where the units refer to the unit elements respectively in T and in R, the second homomorphism is obtained by erasing the third column and third row, and the third homomorphism is obtained by picking the first two diagonal entries. (The condition of the existence of many units is inherited by T.) $\quad\square$

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