Georgian Mathematical Journal
Volume 11 (2004), Number 4, 655–670

POLYTOPES AND K-THEORY

W. BRUNS AND J. GUBELADZE

To the memory of Professor G. Chogoshvili

Every so often you should try a damn-fool experiment —
from J. Littlewood's A Mathematician's Miscellany

Abstract. This is an overview of results from our experiment of merging two seemingly unrelated disciplines – higher algebraic K-theory of rings and the theory of lattice polytopes. The usual K-theory is the "theory of a unit simplex". A conjecture is proposed on the structure of higher polyhedral K-groups for certain class of polytopes for which the coincidence of Quillen's and Volodin's theories is known.

2000 Mathematics Subject Classification: 14M25, 19C09, 19D06, 19M05, 52B20.
Key words and phrases: Lattice polytopes, semigroup rings, automorphism groups, Milnor's group, Volodin's theory, + construction.

1. Introduction

We overview the results from our experiment of merging two seemingly unrelated disciplines – the higher algebraic K-theory of rings and the theory of lattice polytopes. The usual K-theory is the "theory of a unit simplex".

The text is based on the works [2, 6, 7].

At the end of the paper we propose a general conjecture on the structure of higher polyhedral K-groups for certain class of polytopes for which the coincidence of Quillen's and Volodin's theories is known.

All rings, considered below, are commutative and for a ring R its multiplicative group of units is denoted by R^*.

2. Motivation and Applications

To defuse the impression on the experiment to be too damn-fool, here we describe the motivation behind our polyhedral K-theory.

Demazure's paper [10] that initiated the theory of toric varieties in the early 1970s gave an exhaustive description of the automorphism group of a complete smooth toric variety. (Much later this was extended to arbitrary complete toric varieties by Cox [9] and Buehler [8].) Theorem 3.3 below gives an analogous result for the graded automorphism group of the affine cone over a projective toric variety, not necessarily smooth. As explained in Section 3.E, this approach leads to *polytopal generalizations* of the groups $\mathrm{GL}_n(k)$, k a field and the standard fact that $\mathrm{SL}_n(k) = \mathrm{E}_n(k)$. Our motivating question is: to what extent the *polytopal linear groups* and the associated higher K-groups resemble

ISSN 1072-947X / \$8.00 / © Heldermann Verlag *www.heldermann.de*

the ordinary K-groups? We work with the techniques of Quillen's + construction and Volodin's definition of higher K-groups. This seems the only possible framework in our essentially non-additive situation.

On the level of K_2, polyhedral K-theory can be thought of as complementary to the theory of *universal Chevalley groups* [11, 17, 18]. This is so because polytopal linear groups are semidirect products of unipotent groups and reductive groups of type A_l, see [6, Section 1].

For higher groups one is naturally led to the study of the integral homology of interesting examples of linear groups, see Section 8.

As an application to toric geometry, we have obtained results on *retractions* of toric varieties [3], automorphisms of *arrangements* of toric varieties [4], and *autoequivalences* of the category of toric varieties [5].

3. Polytopes, Their Algebras, and Their Linear Groups

3.A. **General polytopes.** By a *polytope* $P\subset\mathbb{R}^n$, $n\in\mathbb{N}$, we always mean a *finite convex* polytope, i.e., P is the convex hull of a finite subset $\{x_1,\dots,x_k\}\subset\mathbb{R}^n$:

$$P=\operatorname{conv}(x_1,\dots,x_k):=\{a_1x_1+\dots+a_kx_k\ :\ 0\le a_1,\dots,a_k\le 1,\ a_1+\dots+a_k=1\}.$$

Polytopes of dimension 1 are called *segments* and those of dimension 2 are called *polygons*.

The *affine hull* $\operatorname{aff}(X)$ of a subset $X\subset\mathbb{R}^n$ is the smallest affine subspace of $\mathbb{R}^n$ containing X. If $\dim\operatorname{aff}(X)=k-1$ for a subset $X=\{x_1,\dots,x_k\}$ of cardinality k, then $x_1,\dots,x_k$ are *affinely independent* and the polytope $P=\operatorname{conv}(x_1,\dots,x_k)$ is called a *simplex*.

For a halfspace $\mathcal{H}\subset\mathbb{R}^n$ containing P, the intersection $P\cap\partial\mathcal{H}$ of P with the affine hyperplane $\partial\mathcal{H}$ bounding $\mathcal{H}$ is called a *face* of P. The polytope itself is also considered as a face.

The faces of P are themselves polytopes. Faces of dimension 0 are *vertices* and those of codimension 1 (i.e., of dimension $\dim P-1$) are called *facets*. A polytope is the convex hull of the set $\operatorname{vert}(P)$ of its vertices. If $\dim P\subset\mathbb{R}^n$ has dimension n, then there is a unique halfspace $\mathcal{H}$ for each facet $F\subset P$ such that $P\subset\mathcal{H}$ and $\partial\mathcal{H}\cap P=F$.

3.B. **Lattice polytopes.** A polytope $P\subset\mathbb{R}^n$ is called a *lattice polytope* if the vertices of P belong to the integral lattice $\mathbb{Z}^n$. More generally, a lattice in $\mathbb{R}^n$ is a subset $\mathcal{G}=x_0+\mathcal{G}_0$ with $x_0\in\mathbb{R}^n$ and an additive subgroup $\mathcal{G}_0$ generated by n linearly independent vectors. A polytope P with $\operatorname{vert}(P)\subset\mathcal{G}$ is called a $\mathcal{G}$-polytope if the vertices of P belong to $\mathcal{G}$. However, since all the properties of $\mathcal{G}$-polytopes we are interested in remain invariant under an affine automorphism of $\mathbb{R}^n$ mapping $\mathcal{G}$ to $\mathbb{Z}^n$, we can always assume that our polytopes have vertices in $\mathbb{Z}^n$. More generally, lattice polytopes P and Q that are isomorphic under an integral-affine equivalence of $\operatorname{aff}(P)$ and $\operatorname{aff}(Q)$ are equivalent objects in our theory. We simply speak of *integral-affinely equivalent polytopes*.

Faces of a lattice polytope are again lattice polytopes.

For a lattice polytope $P \subset \mathbb{R}^n$ we put $\mathrm{L}_P = P \cap \mathbb{Z}^n$. A simplex Δ is called *unimodular* if $\sum_{z\in\mathrm{vert}(\Delta)} \mathbb{Z}(z - z_0)$ is a direct summand of $\mathbb{Z}^n$ for some (equivalently, every) vertex z_0 of Δ. All unimodular simplices of dimension n are integral-affinely equivalent. Such a simplex is denoted by Δ_n and called a *unit n-simplex*. Standard realizations of Δ_n are $\mathrm{conv}(O, e_1, \dots, e_n) \subset \mathbb{R}^n$ or $\mathrm{conv}(e_1, \dots, e_{n+1}) \subset \mathbb{R}^{n+1}$. ($e_i$ is the ith unit vector.)

There is no loss of generality in assuming that a given lattice polytope P is full dimensional (i. e. $\dim P = n$) and that $\mathbb{Z}^n$ is the smallest affine lattice containing L_P. In fact, we choose $\mathrm{aff}(P)$ as the space in which P is embedded and fix a point $x_0 \in \mathrm{L}_P$ as the origin. Then the lattice $x_0 + \sum_{x\in\mathrm{L}_P} \mathbb{Z}(x - x_0)$ can be identified with $\mathbb{Z}^r$, $r = \dim P$.

Under this assumption let F be a facet of P and choose a point $z_0 \in F$. Then the subgroup

$$F_{\mathbb{Z}} := \big(-z_0 + \mathrm{aff}(F)\big) \cap \mathbb{Z}^n \subset \mathbb{Z}^n$$

is isomorphic to $\mathbb{Z}^{n-1}$. Moreover, there is a unique group homomorphism $\langle F, -\rangle : \mathbb{Z}^n \to \mathbb{Z}$, written as $x \mapsto \langle F, x\rangle$, such that $\mathrm{Ker}(\langle F, -\rangle) = F_{\mathbb{Z}}$, $\mathrm{Coker}(\langle F, -\rangle) = 0$, and on the set L_P, $\langle F, -\rangle$ attains its minimum b_F at the lattice points of F.

The $\mathbb{Z}$-linear form $\langle F, -\rangle$ can be extended in a unique way to a linear function on $\mathbb{R}^n$. The description of P as an intersection of halfspaces yields that $x \in P$ if and only if $\langle F, x\rangle \geq b_F$ for all facets F of P.

All polytopes, considered below, are lattice polytopes.

3.C. **Column structures.** Let $P \subset \mathbb{R}^n$ be a polytope. A nonzero element $v \in \mathbb{Z}^n$ is called a *column vector* for P if there exists a facet $F \subset P$ such that $x + v \in P$ whenever $x \in \mathrm{L}_P \setminus F$. In this situation F is uniquely determined and called the *base facet* of v. We use the notation $F = P_v$. The set of column vectors of P is denoted by $\mathrm{Col}(P)$. A *column structure* is a pair of type (P, v), $v \in \mathrm{Col}(P)$. Figure 1 gives an example of a column structure. Familiar examples

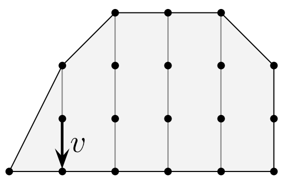

FIGURE 1. A column structure

of column structures are the unit simplices Δ_n with their edge vectors.

3.D. **Polytopal semigroups and their rings.** To a polytope $P \subset \mathbb{R}^n$ one associates the additive subsemigroup $S_P \subset \mathbb{Z}^{n+1}$, generated by $\{(z, 1) : z \in \mathrm{L}_P\} \subset \mathbb{Z}^{n+1}$. Let $C_P \subset \mathbb{R}^{n+1}$ be the cone $\{az \ : \ a \in \mathbb{R}_+,\ z \in P\}$. Then C_P is the convex hull of S_P. It is a *finite rational pointed* cone. In other words, C_P is the intersection of a finite system of halfspaces in $\mathbb{R}^{n+1}$ whose boundaries are rational hyperplanes containing the origin $O \in \mathbb{R}^{n+1}$, and there is no affine line contained in C_P.

As in Subsection 3.B, there is no loss of generality in assuming that $\mathbb{Z}^n$ is the lattice spanned affinely by L_P in $\mathbb{R}^n$. This is equivalent to $\mathrm{gp}(S_P) = \mathbb{Z}^{n+1}$.

While the points $x \in \mathrm{L}_P$ are identified with $(x, 1) \in \mathbb{Z}^{n+1}$, a column vector v is to be identified with $(v, 0) \in \mathbb{Z}^{n+1}$.

Let F be a facet of P. We use the function $\langle F, - \rangle$ to define the *height* of $x = (x', x'') \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ above the hyperplane $\mathcal{H}$ through the facet C_F of C_P by setting

$$\mathrm{ht}_F(x) = \langle F, x' \rangle - x'' b_F.$$

For lattice points x the function ht_F counts the number of hyperplanes between $\mathcal{H}$ and x (in the direction of P) that are parallel to, but different from $\mathcal{H}$ and pass through lattice points. If v is a column vector, then ht_v stands for ht_{P_v}. Moreover, we are justified in calling $\mathrm{ht}_F(v, 0) = \langle F, v \rangle$ the *height of v with respect to F*, since v is identified with $(v, 0)$.

Although the semigroup S_P may miss some integral points in the cone C_P this cannot happen on the segments parallel to a column vector v. More precisely, the following holds:

$$z + v \in S_P \quad \text{for all} \quad z \in S_P \setminus C_{P_v}. \tag{1}$$

($C_{P_v} \subset C_P$ is the face subcone, corresponding to P_v.)

Let R be a ring and $P \subset \mathbb{R}^n$ a lattice polytope. The semigroup ring $R[P] := R[S_P]$ – the *polytopal R-algebra of P* – carries a graded structure $R[P] = R \oplus R_1 \oplus \cdots$ in which $\deg(x) = 1$ for all $x \in \mathrm{L}_P$. By definition of S_P it follows that R_1 generates $R[P]$ over R.

We are interested in the group $\mathrm{gr.aut}_R(P)$ of graded R-algebra automorphisms of $R[P]$. For a field $R = k$ the group $\mathrm{gr.aut}_k(P)$ is naturally a k-linear group. In fact, it is a closed subgroup of $\mathrm{GL}_m(k)$, $m = \# \mathrm{L}_P$. We call $\mathrm{gr.aut}_k(P)$ the *polytopal k-linear group* of P. Its structure will be given in Theorem 3.3.

In the special case when P is a unimodular simplex, the ring $R[P]$ is isomorphic to a polynomial algebra $R[X_1, \dots, X_m]$, $m = \# \mathrm{L}_P$. Therefore, the category $\mathrm{Pol}(R)$ of polytopal R-algebras and graded homomorphisms between them contains a full subcategory that is equivalent to the category of free R-modules.

3.E. **Polytopal linear groups.** Assume R is a ring and P a polytope. Let (P, v) be a column structure and $\lambda \in R$. As pointed out above, we identify the vector v with the degree 0 element $(v, 0) \in \mathbb{Z}^{n+1}$, and further with the corresponding monomial in $R[\mathbb{Z}^{n+1}]$. Then we define a mapping from S_P to $R[\mathbb{Z}^{n+1}]$ by the assignment

$$x \mapsto (1 + \lambda v)^{\mathrm{ht}_v x} x.$$

Since ht_v is a group homomorphism $\mathbb{Z}^{n+1} \to \mathbb{Z}$, our mapping is a homomorphism from S_P to the multiplicative monoid of $R[\mathbb{Z}^{n+1}]$. Now it is immediate from (1) in Subsection 3.D that the (isomorphic) image of S_P lies actually in $R[P]$. Hence this mapping gives rise to a graded R-algebra endomorphism e_v^λ of $R[P]$ preserving the degree of an element. But then e_v^λ is actually a graded automorphism of $R[P]$ because $e_v^{-\lambda}$ is its inverse.

It is clear that e_v^λ is just an elementary matrix in the special case when $P = \Delta_n$, after the identification $\operatorname{gr.aut}_R(P) = \mathrm{GL}_{n+1}(R)$. Accordingly, the automorphisms of type e_v^λ are called *elementary*, and the group they generate in $\operatorname{gr.aut}_R(P)$ is denoted by $\mathbb{E}_R(P)$.

Remark 3.1. Above we have generalized the basic building blocks of higher K-theory of rings to the polytopal setting: general linear groups and their elementary subgroups. As mentioned in Section 2, the real motivation for us to pursue the analogy has been the main result of [2] (Theorem 3.3 below). It is the polytopal version of the fact that an invertible matrix over a field can be diagonalized by elementary transformations on rows (or columns) – or, putting it in different words, the group SK_1 is trivial for fields.

Proposition 3.2. *Let R be a ring, P a polytope, and $v_1, \dots, v_s$ pairwise different column vectors for P with the same base facet $F = P_{v_i}$, $i = 1, \dots, s$. Then the mapping*

$$\varphi : (R, +)^s \to \operatorname{gr.aut}_R(P), \qquad (\lambda_1, \dots, \lambda_s) \mapsto e_{v_1}^{\lambda_1} \circ \cdots \circ e_{v_s}^{\lambda_s},$$

is an embedding of groups. In particular, $e_{v_i}^{\lambda_i}$ and $e_{v_j}^{\lambda_j}$ commute for all $i, j \in \{1, \dots, s\}$, and the inverse of $e_{v_i}^{\lambda_i}$ is $e_{v_i}^{-\lambda_i}$.

In the special case, when R is a field the homomorphism φ is an injective homomorphisms of algebraic groups.

For the rest of this subsection we assume that k is a field, $n = \dim P$, and $\mathbb{A}(F)$ is the image of the map φ in Proposition 3.2

After $\mathbb{A}(F)$ we introduce some further subgroups of $\operatorname{gr.aut}_k(P)$. First, the $(n+1)$-torus $\mathbb{T}_{n+1} = (k^*)^{n+1}$ acts naturally on $k[P]$ by restriction of its action on $k[\mathbb{Z}^{n+1}]$ that is given by

$$(\xi_1, \dots, \xi_{n+1})(e_i) = \xi_i e_i, \qquad i \in [1, n+1];$$

here e_i is the i-th standard basis vector of $\mathbb{Z}^{n+1}$. This gives rise to an algebraic embedding $\mathbb{T}_{n+1} \subset \operatorname{gr.aut}_k(P)$, and we will identify $\mathbb{T}_{n+1}$ with its image. It consists precisely of those automorphisms of $k[P]$ that multiply each monomial by a scalar from k^*.

Second, the automorphism group $\Sigma(P)$ of the semigroup S_P is in a natural way a finite subgroup of $\operatorname{gr.aut}_k(P)$. It is the group of integral affine transformations mapping P onto itself.

Third, we have to consider a subgroup of $\Sigma(P)$ defined as follows. Assume v and $-v$ are both column vectors. Then for every point $x \in P \cap \mathbb{Z}^n$ there is a unique $y \in P \cap \mathbb{Z}^n$ such that $\mathrm{ht}_v(x, 1) = \mathrm{ht}_{-v}(y, 1)$ and $x - y$ is parallel to v. The mapping $x \mapsto y$ gives rise to a semigroup automorphism of S_P: it 'inverts columns' that are parallel to v. It is easy to see that these automorphisms generate a normal subgroup of $\Sigma(P)$, which we denote by $\Sigma(P)_{\mathrm{inv}}$.

Finally, $\mathrm{Col}(P)$ is the set of column structures on P. Now the main result of [2] is:

Theorem 3.3. *Let P be an n-dimensional polytope and k a field.*

(a) *Every element* $\gamma \in \operatorname{gr.aut}_k(P)$ *has a* (*not uniquely determined*) *presentation*

$$\gamma = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r \circ \tau \circ \sigma,$$

where $\sigma \in \Sigma(P)$, $\tau \in \mathbb{T}_{n+1}$, *and* $\alpha_i \in \mathbb{A}(F_i)$ *such that the facets* F_i *are pairwise different and* $\#(F_i \cap \mathbb{Z}^n) \leq \#(F_{i+1} \cap \mathbb{Z}^n)$, $i \in [1, r-1]$;

(b) *For an infinite field k the connected component of unity* $\operatorname{gr.aut}_k(P)^0 \subset \operatorname{gr.aut}_k(P)$ *is generated by the subgroups* $\mathbb{A}(F_i)$ *and* $\mathbb{T}_{n+1}$. *It consists precisely of those graded automorphisms of* $k[P]$ *which induce the identity map on the divisor class group of the normalization of* $k[P]$;

(c) $\dim \operatorname{gr.aut}_k(P) = \#\operatorname{Col}(P) + n + 1$;

(d) *One has* $\operatorname{gr.aut}_k(P)^0 \cap \Sigma(P) = \Sigma(P)_{\mathrm{inv}}$ *and*

$$\operatorname{gr.aut}_k(P)/\operatorname{gr.aut}_k(P)^0 \approx \Sigma(P)/\Sigma(P)_{\mathrm{inv}}.$$

Furthermore, if k is infinite, then $\mathbb{T}_{n+1}$ *is a maximal torus of* $\operatorname{gr.aut}_k(P)$.

4. Stable Groups of Elementary Automorphisms and Polyhedral K_2

4.A. **Product of column vectors.** The product of two column vectors $u, v \in \operatorname{Col}(P)$ is defined as follows: we say that the product uv exists if $u + v \neq 0$ and for every point $x \in \mathrm{L}_P \setminus P_u$ the condition $x + u \notin P_v$ holds. In this case, we define the product as $uv = u + v$. It is easily seen that $uv \in \operatorname{Col}(P)$ and $P_{uv} = P_u$.

Figure 2 shows a polytope with all its column vectors and the two existing products $w = uv$ and $u = w(-v)$.

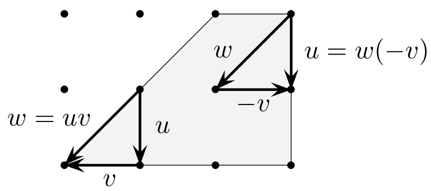

Figure 2. The product of two column vectors

In the case of a unimodular simplex the product of two oriented edges, viewed as column vectors, exists if and only if they are not opposite to each other and the end point of the first edge is the initial point of the second edge.

4.B. **Balanced polytopes.** A polytope P is called *balanced* if $\langle P_u, v\rangle \leq 1$ for all $u, v \in \operatorname{Col}(P)$. One easily observes that P is balanced if and only if $|\langle P_u, v\rangle| \leq 1$ for all $u, v \in \operatorname{Col}(P)$.

The reason we introduce balanced polytopes is that the main results of [6, 7] are only proved for this class of polytopes. However, it is not yet excluded that everything generalizes to arbitrary polytopes.

We give the classification result in dimension 2. It uses the notion of projective equivalence: n-dimensional polytopes $P, Q \subset \mathbb{R}^n$ are called *projectively equivalent* if and only if P and Q have the same dimension, the same combinatorial type, and the faces of P are parallel translates of the corresponding ones of Q. An alternative definition in terms of *normal fans* is given in Subsection 6.D.

Recall the notation $\Delta_n = \operatorname{conv}(O, (1, \dots, 0), \dots, (0, \dots, 1))$ for the unit n-simplex.

Theorem 4.1. *For a balanced polygon P there are exactly the following possibilities* (*up to integral-affine equivalence*)*:*

(a) *P is a multiple of the unimodular triangle $P_a = \Delta_2$. Hence* $\operatorname{Col}(P) = \{\pm u, \pm v, \pm w\}$ *and the column vectors are subject to the obvious relations;*
(b) *P is projectively equivalent to the trapezoid* $P_b = \operatorname{conv}\big((0,0),(2,0),(1,1),(0,1)\big)$*, hence* $\operatorname{Col}(P) = \{u, \pm v, w\}$ *and the relations in* $\operatorname{Col}(P)$ *are $uv = w$ and $w(-v) = u$;*
(c) $\operatorname{Col}(P) = \{u, v, w\}$ *and $uv = w$ is the only relation;*
(d) $\operatorname{Col}(P)$ *has any prescribed number of column vectors, they all have the same base edge* (*clearly, there are no relations between them*)*;*
(e) *P is projectively equivalent to the unit lattice square P_e, hence* $\operatorname{Col}(P) = \{\pm u, \pm v\}$ *with no relations between the column vectors;*
(f) $\operatorname{Col}(P) = \{u, v\}$ *so that $P_u \neq P_v$ with no relations in* $\operatorname{Col}(P)$.

It turns out that polyhedral K-groups are invariants of the projective equivalence classes of polytopes (in arbitrary dimension); see Proposition 6.4 below.

4.C. **Doubling along a facet.** Let $P \subset \mathbb{R}^n$ be a polytope and $F \subset P$ be a facet. For simplicity we assume that $0 \in F$, a condition that can be satisfied by a parallel translation of P. Denote by $H \subset \mathbb{R}^{n+1}$ the n-dimensional linear subspace that contains F and whose normal vector is perpendicular to that of $\mathbb{R}^n = \mathbb{R}^n \oplus 0 \subset \mathbb{R}^{n+1}$ (with respect to the standard scalar product on $\mathbb{R}^{n+1}$). Then the upper half space $H \cap \big(\mathbb{R}^n \times \mathbb{R}_+\big)$ contains a congruent copy of P which differs from P by a 90° rotation. Denote the copy by $P^{|_F}$, or just by $P^{|}$ if there is no danger of confusion.

Note that $P^{|}$ is not always a lattice polytope with respect to the standard lattice $\mathbb{Z}^{n+1}$. However, it is so with respect to the sublattice $(\mathbb{Z}^n)^{|_F}$ which is the image of $\mathbb{Z}^n$ under the 90° rotation.

The operator of doubling along a facet is then defined by

$$P^{\lrcorner_F} = \operatorname{conv}(P, P^{|}) \subset \mathbb{R}^{n+1}.$$

The doubled polytope is a lattice polytope with respect to the subgroup $(\mathbb{Z}^n)^{\lrcorner_F} = \mathbb{Z}^n + (\mathbb{Z}^n)^{|_F} \subset \mathbb{R}^{n+1}$. After a change of basis in $\mathbb{R}^{n+1}$ that does not affect $\mathbb{R}^n$ we can replace $(\mathbb{Z}^n)^{\lrcorner_F}$ by $\mathbb{Z}^{n+1}$, and consider $P^{\lrcorner_F}$ as an ordinary lattice polytope in $\mathbb{R}^{n+1}$. In what follows, whenever we double a lattice polytope $P \subset \mathbb{R}^n$ along a facet F, the lattice of reference in $\mathbb{R}^{n+1}$ is always $\mathbb{Z}^n + (\mathbb{Z}^n)^{|_F}$. For simplicity of notation this lattice will be denoted by $\mathbb{Z}^{n+1}$.

In case $F = P_v$ for some $v \in \operatorname{Col}(P)$ we will use the notation $P^{\lrcorner_F} = P^{\lrcorner_v}$.

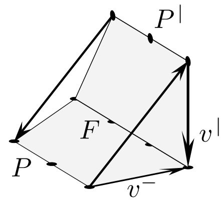

FIGURE 3. Doubling along the facet F

4.D. **The stable group of elementary automorphisms.** An ascending infinite chain of lattice polytopes $\mathfrak{P} = (P = P_0 \subset P_1 \subset \dots)$ is called a *doubling spectrum* if the following conditions hold:

(i) for every $i \in \mathbb{Z}_+$ there exists a column vector $v \subset \operatorname{Col}(P_i)$ such that $P_{i+1} = P_i^{\lrcorner v}$;
(ii) for every $i \in \mathbb{Z}_+$ and any $v \in \operatorname{Col}(P_i)$ there is an index $j \geq i$ such that $P_{j+1} = P_j^{\lrcorner v}$.

Here we use the natural inclusion $\operatorname{Col}(P_i) \subset \operatorname{Col}(P_{i+1})$.

One says that $v \in \operatorname{Col}(P_i)$ is *decomposed* at the jth step in $\mathfrak{P}$ for some $j \geq i$ if $P_{j+1} = P_j^{\lrcorner v}$.

Associated to a doubling spectrum $\mathfrak{P}$ is the 'infinite polytopal' algebra

$$R[\mathfrak{P}] = \lim_{i\to\infty} R[P_i]$$

and the filtered union

$$\operatorname{Col}(\mathfrak{P}) = \lim_{i\to\infty} \operatorname{Col}(P_i).$$

The product of two vectors from $\operatorname{Col}(\mathfrak{P})$ is defined in the obvious way, using the definition for a single polytope. Also, we can speak of systems of elements of $\operatorname{Col}(\mathfrak{P})$ having the same base facets, etc.

Elements $v \in \operatorname{Col}(\mathfrak{P})$ and $\lambda \in R$ give rise to a graded automorphism of $R[\mathfrak{P}]$ as follows: we choose an index i big enough so that $v \in \operatorname{Col}(P_i)$. Then the elementary automorphisms $e_v^\lambda \in \mathbb{E}_R(P_j)$, $j \geq i$ constitute a compatible system and, therefore, define a graded automorphism of $R[\mathfrak{P}]$. This automorphism will also be called 'elementary' and it will be denoted by e_v^λ.

The group $\mathbb{E}(R, \mathfrak{P})$ is by definition the subgroup of $\operatorname{gr.aut}_R(R[\mathfrak{P}])$, generated by all elementary automorphisms.

Remark 4.2. Unlike the classical situation of unimodular simplices, the group $\mathbb{E}(R, \mathfrak{P})$ can *not* be represented as a direct limit of the 'unstable' groups $\mathbb{E}_R(P_i)$, $i \in \mathbb{Z}_+$.

Theorem 4.3. *Let R be a ring and P be a polytope* (*not necessarily balanced*) *admitting a column structure. Assume $\mathfrak{P} = (P \subset P_1 \subset P_2 \subset \dots)$ is a doubling spectrum. Then:*

(a) $\mathbb{E}(R, \mathfrak{P})$ *is naturally isomorphic to* $\mathbb{E}(R, \mathfrak{Q})$ *for any other doubling spectrum* $\mathfrak{Q} = (P \subset Q_1 \subset Q_2 \subset \dots)$;
(b) $\mathbb{E}(R, \mathfrak{P})$ *is perfect;*
(c) *The center of* $\mathbb{E}(R, \mathfrak{P})$ *is trivial;*

(d) $e_u^\lambda \circ e_u^\mu = e_u^{\lambda+\mu}$ *for every* $u \in \mathrm{Col}(\mathfrak{P})$ *and* $\lambda, \mu \in R$*;*
(e) *If* P *is balanced,* $u, v \in \mathrm{Col}(\mathfrak{P})$*,* $u + v \neq 0$ *and* $\lambda, \mu \in R$ *then*

$$[e_u^\lambda, e_v^\mu] = \begin{cases} e_{uv}^{-\lambda\mu} & \textit{if} \quad uv \quad \textit{exists}, \\ 1 & \textit{if} \quad u + v \notin \mathrm{Col}(\mathfrak{P}). \end{cases}$$

The difficult parts of this theorem are the claims (c) and (e), which in the special case $P = \Delta_n$ are just standard facts.

Thanks to Theorem 4.3(a) we can use the notation $\mathbb{E}(R, P)$ for $\mathbb{E}(R, \mathfrak{P})$.

Remark 4.4. Theorem 4.3(e) is the generalization of Steinberg's relations between elementary matrices to balanced polytopes.

4.E. **The Schur multiplier.** Let P be a balanced polytope and $\mathfrak{P} = (P \subset P_1 \subset P_2 \subset \dots)$ be a doubling spectrum. Then for a ring R we define the *stable polytopal Steinberg group* $\mathbb{S}\mathrm{t}(R, P)$ as the group generated by symbols x_v^λ, $v \in \mathrm{Col}(\mathfrak{P})$, $\lambda \in R$, which are subject to the relations

$$x_v^\lambda x_v^\mu = x_v^{\lambda+\mu}$$

and

$$[x_u^\lambda, x_v^\mu] = \begin{cases} x_{uv}^{-\lambda\mu} & \text{if} \quad uv \quad \text{exists}, \\ 1 & \text{if} \quad u + v \notin \mathrm{Col}(\mathfrak{P}) \cup \{0\}. \end{cases}$$

The use of the notation $\mathbb{S}\mathrm{t}(R, P)$ is justified by the fact that, like in Theorem 4.3(a), the stable Steinberg groups are determined by the underlying doubling spectra (with the same initial polytope) up to canonical isomorphism.

The central result of [6] is the following

Theorem 4.5. *For a ring* R *and a balanced polytope* P *the natural surjective group homomorphism* $\mathbb{S}\mathrm{t}(R, P) \to \mathbb{E}(R, P)$ *is a universal central extension whose kernel coincides with the center of* $\mathbb{S}\mathrm{t}(R, P)$*.*

The group $\mathrm{Ker}\big(\mathbb{S}\mathrm{t}(R, P) \to \mathbb{E}(R, P)\big)$ is called the *polyhedral Milnor group.* We denote it by $K_2(R, P)$. Clearly, when P is a unimodular simplex $K_2(R, P)$ is the usual Milnor group $K_2(R)$ [13].

5. Rigid Systems of Column Vectors

We can speak of the product $\prod_{i=1}^m v_i$ of elements $v_i \in \mathrm{Col}(P)$ whenever the following two conditions are satisfied:

(i) the products $v_i v_{i+1}$ exist for all $i \in [1, m-1]$;
(ii) $\sum_{i=r}^s v_i \neq 0$ for all $1 \leq r < s \leq m$.

In this case every bracketing of the sequence $v_1 v_2 \dots v_m$ yields pairs of column vectors whose products exist.

It is useful to have another, a weaker notion of product. We say that $\prod_{i=1}^m v_i$ exists *weakly* if there is a bracketing of the sequence

$$v_1 v_2 \cdots v_m$$

such that all the recursively defined products of pairs of column vectors exist. Since $v_1 \cdots v_n = v_1 + \cdots + v_n$ in the case of weak existence, the value of the product does not depend on the bracketing.

By $\langle V \rangle$ we denote the hull of V in $\mathrm{Col}(P)$ under products (of two column vectors). One has $v \in \langle V \rangle$ if and only if there exist $v_1, \dots, v_m \in V$ such that $v = v_1 \cdots v_m$ is their weak product.

For simplicity we introduce the following convention: $v_1 \cdots v_m \in [V]$ means that the product of $v_1, \dots, v_m$ exists (in the strong sense), whereas $v_1 \cdots v_m \in \langle V \rangle$ means that the product of $v_1, \dots, v_m$ exists in the weak sense.

We will represent certain partial product structures on sets of column vectors by equivalence classes of directed paths in graphs. The *graphs* considered here are finite directed graphs $\mathbf{G}$ satisfying the following conditions:

(i) $\mathbf{G}$ has no isolated vertices;
(ii) $\mathbf{G}$ has no multiple edges and no edges from a vertex to itself;
(iii) if vertices a and b are connected by an edge, then there is no other directed path connecting a and b.

Condition (iii) implies that there are no directed cycles in $\mathbf{G}$ (but the existence of non-directed cycles is not excluded). A *path* is always assumed to be oriented.

The set of nonempty paths in a graph $\mathbf{F}$ carries a natural partial product structure – ll' exists if the end point of the path l is the initial point for l'. The set of all paths in $\mathbf{F}$ is denoted by $\mathrm{path}\,\mathbf{F}$. There is an equivalence relation on $\mathrm{path}\,\mathbf{F}$: two paths are considered to be equivalent if they have the same initial and the same end point. We let $\overline{\mathrm{path}\,\mathbf{F}}$ denote the corresponding quotient set.

Definition 5.1. A system of column vectors $V \subset \mathrm{Col}(P)$ is called *rigid* if the following conditions are satisfied:

(a) $[V]$ does not contain a subset of type $\{v, -v\}$, $v \in \mathrm{Col}(P)$;
(b) $[V] = \langle V \rangle$;
(c) there exist a graph $\mathbf{F}$ and an isomorphism $[V] \approx \overline{\mathrm{path}\,\mathbf{F}}$ of partial product structures.

6. Higher Polyhedral K-Groups

In this section we assume that R is a ring and P is a balanced polytope admitting a column structure.

6.A. **Triangular subgroups in $\mathbb{E}(R, P)$ and $\mathbb{St}(R, P)$.** We fix a doubling spectrum $\mathfrak{P} = (P \subset P_1 \subset \cdots)$. Thanks to Theorem 4.3(a) (and its straightforward analogue for polyhedral Steinberg groups) all the objects defined below are independent of the fixed spectrum.

We say that $V \subset \mathrm{Col}(\mathfrak{P})$ is a rigid system if there exists an index $j \in \mathbb{N}$ such that V is a subset of $\mathrm{Col}(P_j)$ and is rigid.

Definition 6.1.

(a) A subgroup $G \subset \mathbb{E}(R, P)$ is called *triangular* if there exists a rigid system $V \subset \mathrm{Col}(\mathfrak{P})$ such that G is generated by the elementary automorphisms

e_v^λ, where λ runs through R and v through V. The triangular subgroup corresponding to a rigid system V is denoted by $G(R,V)$, and $\mathbf{T}(R,P)$ is the family of all triangular subgroups of $\mathbb{E}(R,P)$;

(b) The triangular subgroups of $\mathbb{S}\mathrm{t}(R,P)$ are defined similarly.

6.B. **Volodin's theory.**

Definition 6.2.

(a) The d-simplices of the *Volodin simplicial set* $\mathbb{V}(\mathbb{E}(R,P))$ are those sequences $(\varepsilon_0,\dots,\varepsilon_d) \in (\mathbb{E}(R,P))^{d+1}$ for which there exists a triangular group $G \in \mathbf{T}(R,P)$ such that $\varepsilon_k\varepsilon_l^{-1} \in G$, $k,l \in [0,d]$. The ith face (resp. degeneracy) of $\mathbb{V}(\mathbb{E}(R,P))$ is obtained by omitting (resp. repeating) ε_i;

(b) The simplicial set $\mathbb{V}(\mathbb{S}\mathrm{t}(R,P))$ is defined analogously;

(c) The higher Volodin polyhedral K-groups of R are defined by

$$K_i^{\mathrm{V}}(R,P) = \pi_{i-1}\big(|\mathbb{V}(\mathbb{E}(R,P))|, (\mathbf{Id})\big), \quad i \geq 2.$$

where $|-|$ refers to the geometric realization of a simplicial set.

The definition of the Volodin simplicial set is independent of the choice of $\mathfrak{P}$ and one has

$$K_i^{\mathrm{V}}(R,P) = \pi_{i-1}\big(\mathbb{V}(\mathbb{S}\mathrm{t}(R,P))\big), \quad i \geq 3.$$

When P is a unimodular simplex of arbitrary dimension Definition 6.2 gives the usual Volodin theory [20].

6.C. **Quillen's theory.** We define *Quillen's higher polyhedral K-groups* by

$$K_i^{\mathrm{Q}}(R,P) = \pi_i\big(\mathrm{B}\,\mathbb{E}(R,P)^+\big), \qquad i \geq 2,$$

where $\mathrm{B}\,\mathbb{E}(R,P)^+$ refers to Quillen's $+$ construction applied to $\mathrm{B}\,\mathbb{E}(R,P)$ with respect to the whole group $\mathbb{E}(R,P) = [\mathbb{E}(R,P),\mathbb{E}(R,P)]$ (Theorem 4.3(b)).

We have the equalities

$$K_i^{\mathrm{Q}}(R,P) = \pi_i\big(\mathrm{B}\,\mathbb{S}\mathrm{t}(R,P)^+\big), \qquad i \geq 3,$$

where the $+$ construction is considered with respect to the whole group $\mathbb{S}\mathrm{t}(R,P)$.

Proposition 6.3. $K_2^{\mathrm{Q}}(R,P) = K_2(R,P) = K_2^{\mathrm{V}}(R,P)$.

For a unimodular simplex $P = \Delta_n$ we recover Quillen's theory [15].

6.D. **Functorial properties.** Let Q be another balanced polytope. If there exists a mapping $\mu : \mathrm{Col}(P) \to \mathrm{Col}(Q)$, such that the conditions

(i) $\langle P_w, v\rangle = \langle Q_{\mu(w)}, \mu(v)\rangle$ and (ii) $\mu(vw) = \mu(v)\mu(w)$ if vw exists,

hold for all $v,w \in \mathrm{Col}(P)$, then the assignment $x_v^\lambda \mapsto x_{\mu(v)}^\lambda$ induces a homomorphism

$$\mathbb{S}\mathrm{t}(R,\mu) : \mathbb{S}\mathrm{t}(R,P) \to \mathbb{S}\mathrm{t}(R,Q).$$

Moreover, if μ is bijective, then

$$\mathbb{S}\mathrm{t}(R,P) \approx \mathbb{S}\mathrm{t}(R,Q), \quad \mathbb{E}(R,P) \approx \mathbb{E}(R,Q), \quad K_2(R,P) \approx K_2(R,Q).$$

This observation allows one to study polyhedral K-theory as a functor also in the polytopal argument. The map μ is called a *K-theoretic morphism* from P to Q. Though we cannot prove K_2-functoriality for *all* maps μ, it is useful to note the $\mathbb{S}\mathrm{t}$-functoriality, since it implies bifunctoriality of the higher polyhedral K-groups with covariant arguments:

$$K_i^{\mathrm{Q}}(-,-), K_i^{\mathrm{V}}(-,-) : \underline{\textit{Commutative Rings}} \times \underline{\textit{Balanced Polytopes}} \to \\ \to \underline{\textit{Abelian Groups}}, \quad i \geq 3.$$

The *normal fan* $\mathcal{N}(P)$ of a finite convex (not necessarily lattice) polytope $P \subset \mathbb{R}^n$ is defined as the complete fan in the dual space $(\mathbb{R}^n)^* = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R})$ given by the system of cones

$$\bigl(\{\varphi \in (\mathbb{R}^n)^* \mid \max_P(\varphi) = F\},\ F \text{ a face of } P\bigr).$$

Two polytopes $P, Q \subset \mathbb{R}^n$ are projectively equivalent (see Section 4.B) if and only if $\mathcal{N}(P) = \mathcal{N}(Q)$.

Proposition 6.4. *If P and Q are projectively equivalent balanced polytopes, then $K_i^{\mathrm{Q}}(R,P) \approx K_i^{\mathrm{Q}}(R,Q)$ and $K_i^{\mathrm{V}}(R,P) \approx K_i^{\mathrm{V}}(R,Q)$ for $i \geq 2$.*

7. On the Coincidence of Quillen's and Volodin's Theories

All polytopes are assumed to be balanced and to admit a column vector, unless specified otherwise.

Definition 7.1. A (balanced) polytope P is Col-*divisible* if its column vectors satisfy the following condition:

(CD_1) if ac and bc exist and $a \neq b$, then $a = db$ or $b = da$ for some d;

(CD_2) if $ab = cd$ and $a \neq c$, then there exists t such that $at = c$, $td = b$, or $ct = a$, $tb = d$.

(See Figure 4.)

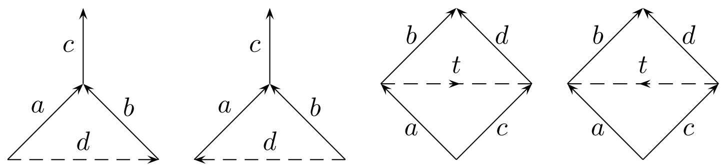

Figure 4. Col-divisibility

The main result of [7] is the following

Theorem 7.2. *Suppose P is a* Col-*divisible polytope. Then*

$$K_i^{\mathrm{Q}}(R,P) = K_i^{\mathrm{V}}(R,P), \qquad i \geq 2.$$

The proof is a 'polytopal extension' of Suslin's proof [19] of the coincidence of the usual theories.

However, we expect that Quillen's and Volodin's theories diverge for general balanced polytopes, see Remark 8.4.

8. Computations

8.A. **The case of polygons.** The class of Col-divisible polytopes may at first glance seem rather restricted. However, it follows immediately from Theorem 4.1 that *all* balanced polytopes of dimension 2 are Col-divisible.

Let R be a ring. In Theorem 4.1 we have grouped all balanced polygons in six infinite series which give rise to the following isomorphism classes of stable elementary automorphism groups:

$$\mathbb{E}_a = \mathrm{E}(R), \tag{a}$$

$$\mathbb{E}_b = \begin{pmatrix} \mathrm{E}(R) & \mathrm{End}_R(\oplus_{\mathbb{N}} R) \\ 0 & \mathrm{E}(R) \end{pmatrix}, \tag{b}$$

$$\mathbb{E}_c = \begin{pmatrix} \mathrm{E}(R) & \mathrm{End}_R(\oplus_{\mathbb{N}} R) & \mathrm{Hom}_R(\oplus_{\mathbb{N}} R, R) \\ 0 & \mathrm{E}(R) & \mathrm{Hom}_R(\oplus_{\mathbb{N}} R, R) \\ 0 & 0 & 1 \end{pmatrix}, \tag{c}$$

$$\mathbb{E}_{d,t} = \begin{pmatrix} \mathrm{E}(R) & \mathrm{Hom}_R(\oplus_{\mathbb{N}} R, R^t) \\ 0 & \mathbf{Id}_t \end{pmatrix}, \quad t \in \mathbb{N}, \tag{d}$$

$$\mathbb{E}_e = \mathrm{E}(R) \times \mathrm{E}(R), \tag{e}$$

$$\mathbb{E}_f = \begin{pmatrix} \mathrm{E}(R) & \mathrm{Hom}_R(\oplus_{\mathbb{N}} R, R) \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} \mathrm{E}(R)\, \mathrm{Hom}_R(\oplus_{\mathbb{N}} R, R) & \\ 0 & 1 \end{pmatrix}. \tag{f}$$

Definition 8.1. A ring R is an $S(n)$*-ring* if there are $r_1, \dots, r_n \in R^*$ such that the sum of each nonempty subfamily is a unit. If R is an $S(n)$-ring for all $n \in \mathbb{N}$, then R has *many units*.

The class of rings with many units includes local rings with infinite residue fields and algebras over rings with many units.

Theorem 8.2. *For every ring R and every index $i \geq 2$ we have:*
(a) $\pi_i(\mathrm{B}\,\mathbb{E}_a^+) = K_i(R)$;
(b) $\pi_i(\mathrm{B}\,\mathbb{E}_b^+) = K_i(R) \oplus K_i(R)$;
(c) $\pi_i(\mathrm{B}\,\mathbb{E}_c^+) = K_i(R) \oplus K_i(R)$ *if R has many units;*
(d) $\pi_i(\mathrm{B}\,\mathbb{E}_{d,t}^+) = K_i(R)$ *if R has many units;*
(e) $\pi_i(\mathrm{B}\,\mathbb{E}_e^+) = K_i(R) \oplus K_i(R)$;
(f) $\pi_i(\mathrm{B}\,\mathbb{E}_f^+) = K_i(R) \oplus K_i(R)$ *if R has many units.*

The proof is based on homological computations for the corresponding matrix groups due to Nesterenko-Suslin [14] and Quillen [16].

8.B. **Higher dimensional polytopes.** It seems that a similar 'almost triangular' matrix group interpretation is possible for the group of elementary automorphisms for *all* Col-divisible polytopes. Then, based on the techniques of Berrick and Keating [1, 12], the corresponding K-groups should be computable in terms of the usual K-groups of the underlying ring. This remark leads us to the following

Conjecture 8.3. *For a commutative ring R and a* Col*-divisible polytope P of arbitrary dimension we have*

$$K_i(R,P) = \underbrace{K_i(R) \oplus \cdots \oplus K_i(R)}_{\mathfrak{c}(P)}, \qquad i \geq 2,$$

where $\mathfrak{c}(P) \leq \dim P$ is a natural number explicitly computable in terms of the partial product table of $\operatorname{Col}(P)$*.*

Remark 8.4. For balanced but not Col-divisible polytopes we may expect that Quillen's and Volodin's theories diverge and we get really new K-groups. The simplest candidate for such a deviation from the usual theory is the pyramid over the unit square shown below – its column vectors are the four oriented edges of the square and four oriented edges emerging from the top vertex. This polytope has shown up several times in our papers as a counterexample to several natural conditions.

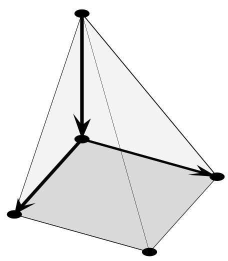

FIGURE 5. The pyramid over the unit square

ACKNOWLEDGEMENT

Both authors were supported by Mathematical Sciences Research Institute.

REFERENCES

1. A. J. BERRICK and M. E. KEATING, The K-theory of triangular matrix rings. *Applications of algebraic K-theory to algebraic geometry and number theory,* Part I, II (*Boulder, Colo.,* 1983), 69–74, *Contemp. Math.,* 55, *Amer. Math. Soc., Providence, RI,* 1986.
2. W. BRUNS and J. GUBELADZE, Polytopal linear groups. *J. Algebra* **218**(1999), No. 2, 715–737.
3. W. BRUNS and J. GUBELADZE, Polyhedral algebras, arrangements of toric varieties, and their groups. *Computational commutative algebra and combinatorics* (*Osaka,* 1999), 1–51, *Adv. Stud. Pure Math.,* 33, *Math. Soc. Japan, Tokyo,* 2002.
4. W. BRUNS and J. GUBELADZE, Polytopal linear retractions. *Trans. Amer. Math. Soc.* **354**(2002), No. 1, 179–203 (electronic).
5. W. BRUNS and J. GUBELADZE, Polytopal linear algebra. *Beiträge Algebra Geom.* **43**(2002), No. 2, 479–500.
6. W. BRUNS and J. GUBELADZE, Polyhedral K_2. *Manuscripta Math.* **109**(2002), No. 3, 367–404.

7. W. Bruns and J. Gubeladze, Higher polyhedral K-groups. *J. Pure Appl. Algebra* **184**(2003), No. 2-3, 175–228.
8. D. Buehler, Homogener Koordinatenring und Automorphismengruppe vollständiger torischer Varietäten. *Diplomarbeit, Basel,* 1996.
9. D. A. Cox, The homogeneous coordinate ring of a toric variety. *J. Algebraic Geom.* **4**(1995), No. 1, 17–50.
10. M. Demazure, Sous-groupes algbriques de rang maximum du groupe de Cremona. *Ann. Sci. École Norm. Sup. (4)* **3**(1970), 507–588.
11. W. van der Kallen and M. Stein, On the Schur multipliers of Steinberg and Chevalley groups over commutative rings. *Math. Z.* **155**(1977), No. 1, 83–94.
12. M. Keating, The K-theory of triangular matrix rings. II. *Proc. Amer. Math. Soc.* **100**(1987), No. 2, 235–236.
13. J. Milnor, Introduction to algebraic K-theory. *Annals of Mathematics Studies,* No. 72. *Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo,* 1971.
14. Yu. P. Nesterenko and A. A. Suslin, Homology of the general linear group over a local ring, and Milnor's K-theory. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **53**(1989), No. 1, 121–146; English transl.: *Math. USSR-Izv.* **34**(1990), No. 1, 121–145.
15. D. Quillen, On the cohomology and K-theory of the general linear groups over a finite field. *Ann. of Math. (2)* **96**(1972), 552–586.
16. D. Quillen, Characteristic classes of representations. *Algebraic K-theory* (*Proc. Conf., Northwestern Univ., Evanston,* Ill., 1976), 189–216. *Lecture Notes in Math.,* 551, *Springer, Berlin,* 1976.
17. R. Steinberg, Générateurs, relations et revêtements de groupes algébriques. 1962 *Colloq. Théorie des Groupes Algébriques* (*Bruxelles,* 1962), 113–127. *Librairie Universitaire, Louvain; Gauthier-Villars, Paris,* 1962.
18. M. Stein, Generators, relations and coverings of Chevalley groups over commutative rings. *Amer. J. Math.* **93**(1971), 965–1004.
19. A. A. Suslin, On the equivalence of K-theories. *Comm. Algebra* **9**(1981), No. 15, 1559–1566.
20. L. A. Volodin, Algebraic K-theory as an extraordinary homology theory on the category of associative rings with a unit. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **35**(1971), 844–873.

(Received 18.05.2004)

Authors' addresses:

Winfried Bruns
Universität Osnabrück
FB Mathematik/Informatik, 49069 Osnabrück
Germany
E-mail: Winfried.Bruns@mathematik.uni-osnabrueck.de

Joseph Gubeladze
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1 Aleksidze St., Tbilisi 0193
Georgia

Department of Mathematics
San Francisco State University
San Francisco, CA 94132, USA
E-mail: soso@math.sfsu.edu