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ON A BOUNDARY VALUE PROBLEM OF LINEAR CONJUGATION FOR UNCLOSED ARCS OF THE CLASS R

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1. Denote by Γ some simple rectifiable line, and by Γ_{ab} an unclosed simple continuous arc with the ends a and b, directed from a to b. Sometimes the use will be made of the notation $\Gamma_{[a,b]}$ and $\Gamma_{(a,b)}$ in case, when the end points belong or do not belong to the arc.

We say that $\Gamma \in R$, if the singular integral

$$(S\varphi)(\tau) \equiv \frac{1}{\pi i} \int\limits_{\Gamma} \frac{\varphi(t)dt}{t-\tau}$$

forms a bounded in $L_p(\Gamma)$, $p>1$ operator, i.e.,

$$\|S\varphi\|_{L_p(\Gamma)} \leq M_p \|\varphi\|_{L_p(\Gamma)}, \quad p>1, \quad \forall \varphi \in L_p(\Gamma).$$

Such lines are called regular or Carleson lines.

We say that an analytic on the plane, cut along Γ, function $\phi(z)$ belongs to the class $\{K_{\Gamma,p}\}$ if it is representable by the Cauchy type integral with density from $L_p(\Gamma)$, i.e.,

$$\phi(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\varphi(t)}{t-z}\, dt \equiv (K_\Gamma \varphi)(z), \quad \varphi \in L_p(\Gamma), \quad z \overline{\in} \Gamma.$$

Similarly, $\phi(z) \in \{K_{\Gamma,p} + P_n\}$, if $\phi(z) = (K_\Gamma \varphi)(z) + P_n(z)$ where $\varphi \in L_p(\Gamma)$, and $P_n(z)$ is the polynomial of the n-th degree.

Under the boundary value problem of linear conjugation we mean the problem which is formulated as follows: Find an analytic on the plane, cut along Γ, function $\phi(z)$ of preassigned class whose boundary values satisfy the condition

$$\phi^+(t) = G(t)\phi^-(t) + g(t), \quad t \in \Gamma. \tag{1}$$

(For the exact definition of $\phi^\pm$ see [1] and [2]). The functions $G(t)$ and $g(t)$ in formula (1) are given a priori.

Problem (1) for piecewise smooth curves, when the given functions belong to the classes H, H_0, H^* and the unknown function is piecewise holomorphic, has been studied thoroughly in [1], in which the reader can also find the definitions of the above-mentioned classes. For Lyapunov curves, problem (1) in discontinuous statement has been investigated in [2], where besides basic results the work presents the definition of continuous and discontinuous statement of the problem. Problem (1) was being studied by many authors under different assumptions on the contour, as well as on the given and unknown functions.

Of special interest is the consideration of problem (1) in the discontinuous statement for lines of the class R, because this class of lines is the most common, and the singular integral operator, corresponding to that problem, may be Noetherian in L_p.

In [3] we have considered the discontinuous problem (1) for closed contours of the class R in the case if $G(t)$ are continuous, and for the contours of the class R_ρ in the

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case of piecewise continuous coefficients with some additional requirement at the points of discontinuity $G(t)$. The function $\rho(t)$ represents the power weight with the corresponding restriction to the exponents. This investigation was carried out when nobody yet knew David's theorem [4] and geometry of curves of the class R. At present, many authors consider power weights for $\Gamma \in R$ and it becomes clear that the requirement $\Gamma \in R_\rho$ in [3] is superfluous, because it is always fulfilled if $\Gamma \in R$.

After [4] it became obvious that the lines, considered by Salaev and his pupils for continuous problems under certain conditions for G and g, coincide with the lines of the class R. Although this work does not involve the question on the consideration of problem (1) for $g \in L_p$ or $\phi \in (K_{\Gamma,p} + P)$ and the question on the boundedness of the operator S in $L_p(\Gamma)$, but some properties of these lines and of the singular integral on these lines have been studied. We will use the following result of Seyfulaev (Salaev's pupil).

Theorem. (Seyfulaev [5]). If $\Gamma_{ab} \in R$, then

$$\varlimsup_{z\to c} \frac{\arg(z-c)}{|\ln|z-c||} = \overline{\Delta}_c, \quad \varliminf \frac{\arg(z-c)}{|\ln|z-c||} = \underline{\Delta}_c,$$

where c is any end of the arc, i.e., $c = a$ or $c = b$, $\overline{\Delta}_c$ and $\underline{\Delta}_c$ are finite numbers.

Under $\arg(z-a)$ we mean a continuous branch on the plane, cut along $\Gamma_{[a,b]} \cup \Gamma_{(b,\infty)}$, $(\Gamma_{[a,b]} \cap \Gamma_{(b,\infty)} = \varnothing)$ and for $z \in \Gamma$ we mean boundary value of that branch from the left.

We will consider a particular case assuming that $\overline{\Delta}_c = \underline{\Delta}_c = \Delta_c$, i.e.,

$$\lim_{z\to c} \frac{\arg(z-c)}{|\ln|z-c||} = \Delta_c, \tag{2}$$

which is equivalent to

$$\arg(z-c) = \Delta_c\big|\ln|z-c|\big| + o\big(\ln|z-c|\big), \tag{2'}$$

in the neighborhood of the point c.

In some cases our requirement will be more strict, i.e.,

$$\arg(z-c) = \Delta_c\big|\ln|z-c|\big| + O(1), \tag{3}$$

in the neighborhood of the point c.

2. The goal of paper is to consider problem (1) in the case in which $\Gamma = \Gamma_{ab} \in R$, the function $G(t)$ is continuous on $\Gamma_{[a,b]}$, $g \in L_p(\Gamma_{ab})$, $\phi(z) \in \{K_{\Gamma,p}\}$ and (3) is satisfied.

3. Let $G(t)$ be the function, continuous on $\Gamma_{[a,b]}$ and different from zero. Under $\ln G(t)$ is meant the value of some continuous branch. We represent this function by analogy with [6] in the form

$$\ln G(t) = \omega_1(t) + \omega_2(t),$$

where

$$\begin{aligned} \omega_1(t) &= \ln G(t) - \ln G(a) - \frac{\ln G(b) - \ln G(a)}{b-a}\,(t-a), \\ \omega_2(t) &= \ln G(a) + \frac{\ln G(b) - \ln G(a)}{b-a}\,(t-a). \end{aligned} \tag{4}$$

Evidently, the both functions are continuous and $\omega_1(a) = \omega_1(b) = 0$. In [6], Chibrikova considers the continuous problem when the line is piecewise smooth and $G(t)$ the function of Hölder class. In our case we adopted from [6] only representation (4), and for the estimation of the functions $(\exp K\omega_2)(z)$ and $(\exp K\omega_1(z)$ our reasoning somewhat differs. To study the function $\exp K\omega_1(z)$ in our assumptions presented in Section 2, we have used the Banach theorem on the invertibility of operators and our theorem from [3] for continuous $G(t)$ and closed contours. We obtain

$$X_1(z) \equiv \big(\exp K\omega_1\big)(z) \in \bigcap_{p>1} \{K_{\Gamma,p}\} \tag{5}$$

$$\big(X_1^{p_1}(t)\big)^+ = \big(\exp(p_1 K\omega_1)\big)^+ \in W_p(\Gamma), \quad \forall p_1 \geq 1, \quad p > 1. \tag{6}$$

Here $W_p(\Gamma)$ is the set of measurable functions $\rho(t)$ satisfying the inequality

$$\left\|\rho S \rho^{-1} \varphi\right\|_{L_p} \leq M_{p,\rho} \left\|\varphi\right\|_{L_p}, \quad \forall \varphi \in L_p(\Gamma).$$

Further, we investigate the function $(\exp K\omega_2)(z)$. It is not difficult to calculate the integral $(K\omega_2)(z)$:

$$\begin{aligned}\big(K\omega_2\big)(z) = \frac{\ln G(b)}{2\pi i} \ln(z-b) - \frac{\ln G(a)}{2\pi i} \ln(z-a) + \\ + \frac{\ln G(b) - \ln G(a)}{2\pi i} \big((z-b)\ln(z-b) - (z-a)\ln(z-a)\big).\end{aligned} \tag{7}$$

Here under $\ln(z-a)$ and $\ln(z-b)$ we mean fully definite continuous branches on the plane with the cutting $\Gamma_{[a,b]} \cup \Gamma_{(b,\infty)}$ whose difference is the branch of the function $\ln \frac{z-b}{z-a}$, vanishing at infinity. The functions $\ln(z-a)$ and $\ln(z-b)$ are discontinuous when passing through the line $\Gamma_{(b,\infty)}$, but the branches are chosen such that $\ln \frac{z-b}{z-a}$ is continuous for $z \in \Gamma_{(b,\infty)}$. By means of Seyfulin's theorem, in the neighborhood of the point b from (7) we obtain

$$\big(\exp K\omega_2\big)(z) = \phi_b(z)|z-b|^{-\frac{\ln G(b)}{2\pi}\Delta_b + \varepsilon_1(z) + \frac{\arg G(b)}{2\pi}} \tag{8}$$

and in the neighborhood of the point a we have

$$\big(\exp K\omega_2\big)(z) = \phi_a(z)|z-a|^{\frac{\ln G(a)}{2\pi}\Delta_b + \varepsilon_2(z) - \frac{\arg G(a)}{2\pi}}\,. \tag{8$'$}$$

In (8) and (8^1) it is assumed that $|\phi_c(z)| < M$, $\lim\limits_{z\to c} \varepsilon_i(z) = 0$ where $c = a$ or $c = b$, and $i = 1, 2$.

Obviously, one can always choose integers $\varkappa_a$ and $\varkappa_b$ such that

$$-\frac{\ln|G(b)|}{2\pi}\Delta_b + \frac{\arg G(b)}{2\pi} = \varkappa_b + \alpha_b, \tag{9}$$

$$\frac{\ln|G(b)|}{2\pi}\Delta_a - \frac{\arg G(b)}{2\pi} = \varkappa_a + \alpha_a, \tag{9$'$}$$

$$-\frac{1}{p} \leq \alpha_a < \frac{1}{q}, \quad -\frac{1}{p} \leq \alpha_b < \frac{1}{q}, \tag{10}$$

where $q = p(p-1)^{-1}$.

We choose p such that the strict inequality

$$-\frac{1}{p} < \alpha_a < \frac{1}{q}, \quad -\frac{1}{p} < \alpha_b < \frac{1}{q}, \tag{11}$$

is fulfilled. Note that if for a particular p inequality (10) is fulfilled, but not fulfilled inequality (11), then its fulfilment will depend on sufficiently close to p numbers $p_1 \in (p-\varepsilon, p+\varepsilon)$.

Taking now in (8) and ($8'$) the neighborhoods of end points sufficiently small, the expressions (9), ($9'$) and (11) will be true (if p in the latter expression is taken appropriately).

Thus we can show that

$$X_2(z) \equiv (z-a)^{-\varkappa_a}(z-b)^{-\varkappa_b}\big(\exp K\omega_2\big)(z) \in \big\{K_{\Gamma,p} + P\big\}. \tag{12}$$

From the above-said we can easily prove the following

Theorem 1. *If* $\Gamma_{ab} \in R$, $\underline{\Delta}_c = \overline{\Delta}_c$ *where* c *are end points* a *and* b, $G(t)$ *is continuous on* $\Gamma_{[a,b]}$, *then*

$$\begin{aligned}X(z) = (z-a)^{-\varkappa_a}(z-b)^{-\varkappa_b}\big(\exp\ln G\big)(z) \in \big\{K_{\Gamma,\rho} + P\big\} \\ X^{-1}(z) = (z-a)^{\varkappa_a}(z-b)^{\varkappa_b}\big(\exp\ln G^{-1}\big)(z) \in \big\{K_{\Gamma,q} + P\big\}.\end{aligned} \tag{13}$$

To use $X(z)$ for the solution of problem (1) in $\{K_{\Gamma,p}+P\}$, it is necessary that $X^{+}(t)\in W_p(\Gamma_{ab})$, and towards this end we have to show that X_2^{+} is the weight function. This can be easily achieved if we additionally require (2^1). But this condition is seemingly not necessary, but here we restrict ourselves just to this case because we get easily visible result.

Denote $\varkappa=\varkappa_a+\varkappa_b$.

Theorem 2. *If $\Gamma_{ab}\in R$, condition $(2')$ is fulfilled, $G(t)$ is continuous on $\Gamma_{[a,b]}$, and $g\in L_p$, then the solution (if any) of problem (1) in $\{K_{\Gamma,p}\}$ has the form*

$$\Phi(z)=X(z)\int\limits_{\Gamma_{ab}}\frac{g(t)dt}{X^{+}(t)(t-z)}+P_{\varkappa-1}(z)X(z),$$

where $P_n(z)$ is the polynomial of the n-th degree for $n\geq 0$ and $P_n(z)\equiv 0$ if $n<0$.

If $\varkappa>0$, then the problem has $\varkappa$ linearly independent solutions; if $\varkappa=0$, then the solution is unique; if $\varkappa<0$, then for the solvability of the problem it is necessary and sufficient that the conditions

$$\int\limits_{\Gamma_{ab}}\frac{t^k g(t)}{X^{+}(t)}=0,\quad k=0,1,\dots,\varkappa-1$$

be fulfilled.

Thus the best possible result differs from the classical one only by the expression for the index $\varkappa$ which is given by formulas (9), $(9')$ and by the function $X(z)$ which depends on the number Δ_c characterizing rotation of the line Γ_{ab} at the end points.

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