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ON THE BOUNDARY VALUE PROBLEM OF LINEAR CONJUGATION OF AN UNCLOSED CARLESON ARC

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The aim of the present paper is to improve our results obtained in [1], i.e. to solve the boundary value problem not imposing the condition $(2')$ from [1] on the contour.

Just as in [1], we denote by Γ_{ab} the arc with the ends a and b directed from a to b. By $\{\mathcal{K}_p(\Gamma_{ab})\}$ we denote a class of functions representable by the Cauchy type integral

$$\{\mathcal{K}_p(\Gamma_{ab})\} = \left\{\phi_0 : \phi_0(z) = \frac{1}{2\pi i}\int\limits_{\Gamma_{ab}} \frac{\varphi(t)dt}{t-z} = (\mathcal{K}_p\varphi)(z),\ z \notin \Gamma, \varphi \in L_p(\Gamma_{ab})\right\},$$

and by $\{\mathcal{K}_p(\Gamma_{ab}) + P_n\}$ we denote a class $\{\mathcal{K}_p(\Gamma_{ab}) + P_n\} = \{\phi : \phi(z) = \phi_0(z) + P_n(z),\ P_n(z)$ is the nth degree polynomial$\}$,

The boundary value problem of linear conjugation for the arc Γ_{ab} is called the following problem: Find the function $\phi(z) \in \{\mathcal{K}_p(\Gamma_{ab}) + P_n\}$ which satisfies the condition

$$\phi^+(t) = G(t)\phi^-(t) + g(t), \quad t \in \Gamma, \tag{1}$$

where $G(t)$ and $g(t)$ are the given functions. In our case, $G(t)$ is continuous on Γ_{ab}, and $g(t) \in L_p(\Gamma_{ab})$, $p > 1$. Moreover, we will seek, as commonly, for a solution vanishing at infinity, i.e. $\phi(z) \in \{\mathcal{K}_p(\Gamma_{ab})\}$. The arc Γ_{ab} is assumed to satisfy David's condition ([2]); in this case we write $\Gamma_{ab} \in R$, and the arc is called regular, or Carleson's arc.

Short description of the results for the problem (1) can be found in [1].

In what follows, the use will be made of the results obtained by Seiffulaev [3]. If $\Gamma_{ab} \in R$ and $c = a$, or b, then

$$\overline{\lim} \frac{\arg(z-c)}{|\ln|z-c||} = \overline{\Delta}_c, \quad \underline{\lim} \frac{\arg(z-c)}{|\ln|z-c||} = \underline{\Delta}_c, \tag{2}$$

where $\overline{\Delta}_c$ and $\underline{\Delta}_c$ are finite numbers.

To solve the problem (1), we represent the function $G(t)$ in the form

$$\ln G(t) = \omega_1(t) + \omega_2(t),$$

where

$$\omega_1(t) = \ln G(t) - \ln G(a) - \frac{\ln G(b) - \ln G(a)}{b-a}(t-a),$$
$$\omega_2(t) = \ln G(a) + \frac{\ln G(b) - \ln G(a)}{b-a}(t-a).$$

As is shown in [1], if $\Gamma_{ab} \in R$, $\overline{\Delta}_c = \underline{\Delta}_c$ then

$$X_1(z) \equiv \exp(K\omega_1)(z) \in \prod_{p>1} \{\mathcal{K}_p(\Gamma_{ab}) + P_0\}$$

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and moreover, $X_1^+ \in \prod\limits_{p>1}\{W_p(\Gamma_{ab})\}$, where $\{W_p(\Gamma_{ab})\}$ denotes a class of the functions $\{W_p(\Gamma_{ab})\} = \{\rho : \|\rho S_\rho^{-1}\varphi\|_p \le M_p\|\varphi\|_p, \forall p \in L_p\}$; $S_{\Gamma_{ab}}$ is the singular integral

$$(S_{\Gamma_{ab}}\psi)(\tau) = \frac{1}{\pi i}\int\limits_{\Gamma_{ab}} \frac{\varphi(t)dt}{t-\tau}.$$

Regarding $(\exp\mathcal{K}\omega_2)(z)$, it is proved in [1] that if the integers $\varkappa_a$ and $\varkappa_b$ are chosen such that

$$-\frac{\ln|G(b)|}{2\pi}\Delta_b + \frac{\arg G(b)}{2\pi} = \varkappa_b + \alpha_b \tag{3}$$

$$\frac{\ln G(a)}{2\pi}\Delta_a - \frac{\arg G(a)}{2\pi} = \varkappa_a + \alpha_a \tag{4}$$

and

$$-\frac{1}{p} < \alpha_a < \frac{1}{q}, \ \ ; \frac{1}{p} < \alpha_b < \frac{1}{q}, \tag{5}$$

then

$$X_2(z) = (z-a)^{-\varkappa_a}(z-b)^{-\varkappa_b}(\exp\mathcal{K}\omega_2)(z) \in \{\mathcal{K}_p(\gamma_{ab}) + P\}. \tag{6}$$

As a result, we obtain

$$X(z) \equiv X_1(z)X_2(z) = (z-a)^{-\varkappa_a}(z-b)^{-\varkappa_b}\exp(\mathcal{K}\ln G) \in \{\mathcal{K}_p(\Gamma_{ab}) + P\}.$$

In the sequel, we have to show that $X^+ \in W_p(\Gamma_{ab})$. In [1] we have done this with supplementary restriction to the contour. Here we will perform this without any restriction. Towards this end, we have to show that $X_2^+(t) \in W_p(\Gamma_{ab})$.

We take advantage of the fact that it makes no difficulty to calculate $(\mathcal{K}\omega_2)(z)$. Indeed,

$$(\mathcal{K}\omega_2(z)) = \frac{G(b)}{2\pi i}\ln(z-b) - \frac{\ln G(a)}{2\pi i}\ln(z-a) +$$
$$+\frac{\ln G(b) - \ln G(a)}{2\pi i(b-a)}((z-b)\ln(z-b) - (z-a)\ln(z-a) + (b-a)) \tag{7}$$

In [1] we made mechanical error. (in (6) we missed one summand $(b-a)$, but the results remains unchanged. Under $\ln(z-a)$ and $\ln(z-b)$ are meant branches, analytic on the plane, cut along $\Gamma_{ab}\cap\Gamma_{b,\infty}$ and chosen such that the function $\ln\frac{z-b}{z-a}$ is continuous when $z \in \Gamma_{(b,\infty)}$.

According to (2), it is clear that in the neighborhood of the points c,

$$\arg(z-c) = \Delta_c|\ln|z-c|| + \varepsilon_c(z),$$

where $\lim_{z\to c}\varepsilon_c(z) = 0$.

Consider now the expression

$$N_c(z) \equiv \exp\frac{\ln G(c)}{2\pi i}\ln(z-c).$$

In the neighborhood of the points c we have

$$N_c(z) = \exp\frac{\ln|G(c)| + i\arg G(c)}{2\pi i}(\ln|z-c| + i\arg(z-c)) =$$
$$= M_c(z)\exp\frac{\ln|G(c)|}{2\pi}\arg(z-c) + \frac{\arg G(c)}{2\pi}\ln|z-c|, \tag{7'}$$

where $|M_c(z)|$ are bounded in the neighborhood of $c=a$ and $c=b$ and not equal to zero. Taking into account ($7'$), we obtain

$$N_c(z)=M_c(z)\exp\frac{\ln|G(c)|}{2\pi}\frac{\arg(z-c)}{\ln|z-c|}\ln|z-c|+$$
$$+\frac{\arg G(c)}{2\pi}\ln|z-c|=M_c(z)\exp\frac{\ln|G(c)|}{2\pi}(-\Delta_c+\varepsilon_c(z)\ln|z-c|)+$$
$$+\frac{\arg G(c)}{2\pi}\ln|z-c|=M_c(z)|z-c|^{-\frac{\ln G(c)}{2\pi}}\Delta_c+\frac{\ln|G(c)|}{2\pi}\varepsilon_c(z)+\frac{\arg G(c)}{2\pi}.$$

By virtue of (6), in the neighborhood of b we have

$$\exp(\mathcal{K}\omega_2)(z)=A_1(z)M_b(z)|z-b|^{\frac{-\ln|G(b)|}{2\pi}\Delta_b+\frac{\ln|G(b)|}{2\pi}\varepsilon_p(z)+\arg\frac{G(b)}{2\pi}}, \tag{8}$$

and in the neighborhood of the point a,

$$\exp(\mathcal{K}\omega_2)(z)=A_2(z)M_a(z)|z-a|^{\frac{\ln|G(a)|}{2\pi}\Delta_a-\frac{\ln|G(a)|}{2\pi}\varepsilon_p(z)-\arg\frac{G(a)}{2\pi}} \tag{9}$$

$A_1(z)$ and $A_2(z)$ are the bounded functions, different from zero.

We now take $\varkappa_b$ and $\varkappa_a$ the same as in (3) and (4). Choose $\delta>0$ such that $|\varepsilon_c(z)|$ are sufficiently small in order that if

$$\alpha_c'\equiv\frac{\ln|G(c)|}{2\pi}\operatorname{Re}\varepsilon_c(z)+\alpha_c,\quad\text{then}\quad-\frac{1}{p}<\alpha_0'<\frac{1}{q}$$

whence it follows that the function

$$X_2(z)=(z-a)^{-\varkappa_a}(z-b)^{-\varkappa_b}\exp(\mathcal{K}\omega_2)(z)=B(z)|z-a|^{\alpha_a}|z-b|^{\alpha_b}, \tag{10}$$

where $B(z)$ is the function, bounded and not equal to zero in the neighborhood of the points a and b. Let Γ_{aa_1} and Γ_{b_1b} be just these neighborhoods, and moreover, the points d and e belong to them, $\Gamma_{aa_1}\subset\Gamma_{ab}$, $d\in\Gamma_{aa_1}$, $\Gamma_{b_1b\subset\Gamma_{ab}}$, $e\in\Gamma_{b_1b}$. It is seen from (10) that $X_2^+(t)\in W_p(\Gamma_{aa_1})\cap W_p(\Gamma_{b_1b})$. Obviously, $X_2^+(t)\in W_p(\Gamma_{de})$. It now follows from the results of [4] that $X_2^+\in W_p(\Gamma_{ab})$. It is not difficult to find that $X^+=X_1^+X_2^+\in W_p(\Gamma_{ab})$. Thus we can state that Theorem 2 of [1] is valid without the condition $2'$. Indeed, if we denote $\varkappa=\varkappa_a+\varkappa_b$, then the solution, vanishing at infinity, of the problem of conjugation has index $\varkappa$, and for $\varkappa<0$, both the solution and the condition of solvability have classical form. Here we imposed on the line of class R only one restriction $\overline{\Delta}_c=\Delta_c$ which is not necessary, but it provides us with the formula for the index and also a sufficiently simple proof. In the sequel, we will try to eliminate this restriction and to consider the problem in the class $L^{p(\cdot)}(\Gamma_{ab})$, like in [5]. Moreover, we will consider piecewise continuous coefficients.

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