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ON A LOCAL LIMIT THEOREM FOR LATTICE DISTRIBUTIONS

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Dedicated to the memory of A.B. Mukhin

Abstract. Sufficient conditions for a local limit theorem for sums of independent integer-valued random variables to be valid are discussed in terms of a characteristic of "smoothness" of distributions of sums of separate groups of summands.

The notion of "smoothness" of the probability distribution $\mathcal{P}_\xi$ of an integer-valued random variable ξ (see [3], [13]) was for the first time used in [4] to derive a local limit theorem (l.l.t.) for sums of independent integer-valued random variables. In [3] this notion is introduced as a variation norm of the signed measure $\mathcal{P}_\xi - \mathcal{P}_{\xi+1}$, i.e.,

$$\delta(\mathcal{P}_\xi) = \delta(\xi) = \sum_{m\in\mathbb{Z}} |P(\xi = m) - P(\xi = m-1)|.$$

The investigation by Mukhin in his remarkable paper [8] include the case treated in [4]. Later, in [5] an attempt was made at extending somehow the class of sequences from [4] for which conditions for the l.l.t. to be valid are given in terms of $\delta(\cdot)$. Below we try to get an insight into the real possibilities of $\delta(\cdot)$ in the l.l.t. and discuss the connection of the l.l.t. proved by us with some known results.

1. Let $\Xi = \{\xi_1, \xi_2, \dots\}$ be a sequence of independent integer-valued random variables with finite variances,

$$S_n = \xi_1 + \cdots + \xi_n, \quad B_n^2 = DS_n, \quad S_{n,m} = S_{n+m} - S_n, \quad n, m \in \mathbb{N},$$

and $\Phi(x)$ and $\varphi(x)$ be the standard normal distribution function and the corresponding density, respectively. We will deal with the conditions under

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which the central limit theorem (c.l.t.) for Ξ, i.e., the relation $P(B_n^{-1}(S_n - ES_n) < x) \to \Phi(x)$ as $n \to \infty$ for each $x \in \mathbb{R}$ implies the l.l.t., i.e., the convergence

$$\Delta_n = \sup_{m\in\mathbb{Z}} \Big| B_n P(S_n = m) - \varphi\Big(\frac{m - ES_n}{B_n}\Big)\Big| \to 0$$

as $n \to \infty$.

Before formulating the l.l.t. in terms of $\delta(\cdot)$, we would like to recall the following properties of the characteristic $\delta(\cdot)$, where ξ, η and $\eta_n, n \in \mathbb{N}$, are integer-valued random variables with maximal span 1 (cf. [3], [4]):

1°. If ξ and η are independent, then $\delta(\xi + \eta) \le \min(\delta(\xi), \delta(\eta))$.

2°. If $\mathcal{P}_\xi^{k*}$ denotes the k-th convolution power of $\mathcal{P}_\xi$, then $\delta(\mathcal{P}_\xi^{k*}) < Ck^{-1/2}$ with the constant C depending on $\mathcal{P}_\xi$.

3°. For the characteristic function (c.f.) $f(t,\xi) = E\exp(it\xi)$ of ξ we have the inequality

$$|f(t,\xi)| \le \delta(\xi)/\sqrt{2} \quad \text{if} \quad \pi/2 \le |t| \le \pi.$$

4°. If η_n converges to 0 in probability as $n \to \infty$, then $\delta(\eta_n) \to 2$.

Let now $\mathcal{I}_j$, $j \in \mathbb{N}$, be nonempty, finite and pairwise disjoint subsets of $\mathbb{N}$. Denote

$$S_{(j)} = \sum_{i\in\mathcal{I}_j} \xi_i, \quad r_n = \#\big\{j : \mathcal{I}_j \subset \{1,\dots,n\}\big\}, \ \ j,n \in \mathbb{N}.$$

Theorem. *If for the sequence* Ξ*:*

(a) *the c.l.t.holds,*

(b) *for any* $j \in \mathbb{N}$, $\mathcal{P}_{S_{(j)}}$ *has a maximal span* 1 *and there exists* $\ell \in \mathbb{N}$ *such that* $\delta(\mathcal{P}_{S_{(j)}}^{\ell*}) \le \lambda < \sqrt{2}$,

(c) *there exist* $c, 0 < c \le \pi$, *and a function* $g(t) \in L(\mathbb{R})$ *such that* $|f(B_n^{-1}t, S_n)| \le g(t)$ *for* $|t| \le cB_n$,
and if $c < \pi$, *the following extra condition holds*

(d) $B_n = o(e^{\mu c^2 r_n})$ *with* $\mu = (1 - \lambda^2/2)/(2\pi^2\ell)$,
then the l.l.t. also holds for the sequence Ξ.

This theorem agrees with the general picture of the l.l.t. given in [8]: only the conditions imposed on the arithmetic structure of summands (one can assume (b) to be such a condition) do not guarantee the l.l.t.

2. We begin proving the theorem by noting that the modified version of the classical Cramér's inequality ([1], Ch. IV, p. 27) gives for the c.f. $f(t)$ the estimate

$$|f(t)| \le 1 - (1-\gamma^2)t^2/(8\tau^2), \quad |t| < \tau, \quad \tau > 0,$$

if it is known that $|f(t)| \le \gamma < 1$ for $\tau \le |t| < 2\tau$ (the right-hand side of Cramér's inequality being greater than γ as $\tau \le |t| < 2\tau$). Hence by property 3° of the characteristic $\delta(\cdot)$ and the condition (b) we find that the following inequality holds:

$$|f(t, S_{(j)})| \leq \exp(-\mu t^2), \quad |t| \leq \pi, \quad j \in \mathbb{N}.$$

Using the standard way of proving the l.l.t., by the inversion formulas for the c.f., we arrive at an estimate

$$2\pi\Delta_n \leq \int\limits_{|t|\leq A} |f(t, B_n^{-1}(S_n - ES_n)) - e^{-t^2/2}|dt + \int\limits_{|t|>A} e^{-t^2/2}dt +$$
$$+ \int\limits_{A<|t|\leq cB_n} |f(t, B_n^{-1}S_n)|dt + B_n \int\limits_{c<|t|\leq\pi} |f(t, S_n)|dt =$$
$$= I_1 + I_2 + I_{31} + I_{32},$$

where $A > 0$ is a fixed number to be chosen next. It can be taken large enough for I_2 and, by the condition (c), I_{31} must be less than a given positive ε. By the condition (a), if A is fixed, then $I_1 \to 0$ as $n \to \infty$. Further, I_{32} does not exceed $2(\pi - c)B_n e^{-\mu c^2 r_n}$ which, by the condition (d), tends to 0 as $n \to \infty$. The theorem is proved.

3. If $B_n^2 = O(n^\alpha), 0 < \alpha \leq 1$, i.e., when the variance of the sum S_n grows not faster than linearly with respect to n, and if in the condition (b) one takes $S_{(j)} = \xi_j, j \in \mathbb{N}$, then for all t in $[-\pi, \pi]$ we obtain that

$$|f(t, B_n^{-1}S_n)| \leq e^{-\frac{n}{B_n^2}\mu t^2} \leq e^{-C\mu n^{1-\alpha}t^2}$$

with a positive constant C. Thus for $\alpha < 1$ and fixed A the sum $I_{31} + I_{32}$ tends to 0 as $n \to \infty$; for $\alpha = 1$ it can be done sufficiently small by chosing A large enough (cf. [4]). Property 2° of the characteristic $\delta(\cdot)$ then immediately gives the sufficiency part of the classical Gnedenko theorem for identically distributed summands [6].

In the case of random variables uniformly bounded by a constant K, Liapunoff's inequality (see, e.g., [1], Ch. VII, p. 75) implies, that

$$|f(t, B_n^{-1}S_n)| \leq e^{-t^2/3}, \quad |t| \leq (4K)^{-1}B_n \tag{1}$$

(the same follows from Doob's inequality for c.f.; see [2], p. 47). Even when $B_n \to \infty$, i.e., both conditions (a) and (c) are fulfilled, this case needs extra information about the arithmetic structure of summands in order to be covered by our theorem (cf. [11]). In [5] it is stated that if $B_n^2 = O(n^\alpha)$, $\alpha > 1$, the c.l.t. holds and the condition (b)is fulfilled in a weaker sense than in [4], being imposed on $S_{j,m}$ for a fixed m and each $j \in \mathbb{N}$, then the l.l.t. holds too; in the proof (1) is misused. Had this l.l.t. been true, the well-known l.l.t. from [7] would be its consequence for $\Xi = \{\xi_j = j\eta_j$, $j \in \mathbb{N}\}$, where $P(\eta_j = -1) = P(\eta_j = 1) = 1/4$, $P(\eta_j = 0) = 1/2$ for each $j \in \mathbb{N}$. Indeed, here $B_n = O(n)$ and (b) is easy to check for $m = 2$ (see [5]).

A general l.l.t. for sums of identically distributed integer-valued random variables weighted with integers is announced in [14].

If Ξ is a k-sequence, i.e., there is only a finite number k of different distributions among $\mathcal{P}_{\xi_j}, j \in \mathbb{N}$, using the existence of a positive number c_j for each non-singular ξ_j such that

$$|f(t, \xi_j)| \le e^{-(1/4)\sigma_j^2 t^2}, \quad |t| \le c_j, \quad j \in \mathbb{N},$$

where σ_j^2 is the variance of ξ_j (see, e.g., [10], Ch. I, p. 11), then we obtain, except for the trivial case with Ξ consisting only of constants, that for $c = \min_j c_j$ the following inequality holds:

$$|f(t, B_n^{-1} S_n)| \le e^{-(1/4)t^2}, \quad |t| \le cB_n.$$

By Petrov's theorem [9] (see also [10], Ch. VII, p. 189), for l.l.t. to be valid in the case of k-sequence of integer-valued random variables with finite variances, the relation g.c.d.$(h_1, \dots, h_\ell) = 1$ is necessary and sufficient, where $h_1, \dots, h_\ell$, $1 \le \ell \le k$, are the maximal spans of the distributions $\mathcal{P}_1, \dots, \mathcal{P}_\ell$, respectively, which occur infinitely often. If now $\mathcal{I}_1$ is composed of the numbers of the first terms of Ξ with distributions $\mathcal{P}_1, \dots, \mathcal{P}_\ell$, respectively, $\mathcal{I}_2$ consists of the numbers of the second terms with the same distributions and so on, then $S_{(1)}, S_{(2)}, \dots$ are identically distributed with maximal span 1. If we restrict ourselves to k-sequences for which $ne^{-\lambda c^2 r_n} \to 0$ as $n \to \infty$, where $r_n = \min_{1 \le i \le \ell} s_i(n)$ and $s_i(n)$ is the number of summands with distribution $\mathcal{P}_i$ among the first n terms of Ξ, $i = 1, \dots, \ell$, then all the conditions of our theorem will be fulfilled and therefore in this particular case this theorem will imply the sufficiency part of Petrov's theorem.

In [12], besides a certain arithmetic condition which is fulfilled in [10] and [11], Rozanov introduced the condition which implies our condition (c)(and which is fulfilled in [10] and [11] as well), namely,

$$E(\xi_j - E\xi_j)^2 I([|\xi_j - E\xi_j| < N]) \to \sigma_j^2 \quad (N \to \infty) \tag{2}$$

uniformly in $j \in \mathbb{N}$, where $I(U)$ denotes the indicator of an event U. If (2) takes place, then so do Lindeberg condition and hence the c.l.t., but the Lindeberg condition is not sufficient for (c) to be fulfilled. Thus we can conclude that if the conditions (a) and (c) are replaced by the condition (2), our theorem remains true.

4. We give an example helpful for understanding the role of $\delta(\cdot)$ in the l.l.t. for integer-valued random variables (as is known, $\delta(S_n) \to 0$ when $n \to \infty$ if the l.l.t. holds for Ξ [3]).

Let $0 < \alpha < 1$, $P(\xi_j = -1) = P(\xi_j = 1) = p_j = [j^\alpha - (j-1)^\alpha]/2$, $P(\xi_j = 0) = 1 - 2p_j$, $j \in \mathbb{N}$, $p_j \to 0$ $(j \to \infty)$. Since ξ_j tends to 0 in probability as $j \to \infty$, we have, as noted above, $\delta(\mathcal{P}_{\xi_j}) \to 2$ as $j \to \infty$ and, by the same reason, even $\delta(\mathcal{P}_{\xi_j}^{\ell *}) \to 2$ for fixed $\ell \in \mathbb{N}$. Here $B_n^2 = n^\alpha$, the condition (b) is not fulfilled for individual summands and (for the given partition of S_n into groups) Ξ is not covered by the Theorem. However by Prokhorov's theorem [11] which states that the l.l.t. holds for a sequence of uniformly bounded random variables with zero modes and B_n tending to ∞ if and only if

$$\text{g.c.d.}\left\{m : \sum_{j=1}^{\infty} P(\xi_j = m) = \infty\right\} = 1,$$

our sequence obeys the l.l.t.

Let us note finally that, like the theorems by Prokhorov, Rozanov and Petrov, our theorem is a l.l.t. in strengthened form, that is, it remains true when any finite number of terms of Ξ is replaced by random variables having other distributions on $\mathbb{Z}$ with finite variances. This notion was introduced in [11].

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