Georgian Mathematical Journal
Volume 11 (2004), Number 2, 229–242

ON THE UNIQUENESS OF SOLUTIONS OF SOME QUASI-VARIATIONAL INEQUALITIES FROM CONTROL THEORY

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Abstract. The existence and uniqueness problems for some quasi-variational inequalities are studied on the basis of the L^∞-estimates for solutions of the variational inequalities and their differences. An implicit obstacle problem is stated by analogy with one quasi-variational inequality studied by Benoussan and Lions (1982) and Vescan (1982) and its unique solvability is proved. Some conclusions are given concerning the uniqueness of solutions for an impulse control problem with bilateral restrictions and for a quasi-variational inequality appearing in dynamic programming.

2000 Mathematics Subject Classification: 35J85, 35B50, 58E35, 49J40, 49N25.
Key words and phrases: Variational and quasi-variational inequalities, impulse control, dynamic programming, implicit Signorini problem.

1. A Maximum Principle in Variational Inequalities

Let Ω be a bounded domain in $\mathbb{R}^n$, Γ be the boundary of $\Omega : \Gamma = \partial\Omega$, $\Gamma \in C^2, \nu$ be the outward unit normal vector to Ω. Let $H^1(\Omega)$ be the real Sobolev space and $\widetilde{H}^1(\Omega) := \{v \in H^1(\Omega) : \ v|_\Gamma = 0\}$. The norm in $H^1(\Omega)$ is denoted by $\|\cdot\|_1$.

Suppose that

$$V = \widetilde{H}^1(\Omega) \ \text{ or } \ V = H^1(\Omega).$$

Let us define the bilinear form on the space $H^1(\Omega) \times H^1(\Omega)$ as follows:

$$\begin{gathered} a(u,v) = \sum_{i,j=1}^n \int_\Omega a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} dx + \sum_{i=1}^n \int_\Omega a_i \frac{\partial u}{\partial x_i} v\, dx + \int_\Omega a_0 u\, v\, dx, \\ a_{ij}, a_i, a_0 \in L^\infty(\Omega), \quad \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geq \alpha_0|\xi|^2, \\ a_0(x) \geq a^0, \quad a^0 = \text{const} > 0. \end{gathered} \tag{1.1}$$

Suppose that the form $a(u,v)$ is coercive:

$$a(u,u) \geq \alpha \|u\|_1^2, \quad \alpha = \text{const} > 0, \ \forall u \in H^1(\Omega).$$

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Set the variational problem with the unilateral restrictions. Find a function $u \in K$ such that

$$a(u, v-u) \geq \int\limits_{\Omega} f\,(v-u)dx, \qquad \forall v \in K, \tag{1.2}$$

where

$$K = K_{\psi}\{v \in V : v \geq \psi, \ \text{ in } \Omega\} \ \text{ or} \tag{1.3}$$

$$K = K^{\varphi}\{v \in V : v \leq \varphi, \ \text{ in } \Omega\}, \tag{1.4}$$

where $f \in L_2(\Omega),\ h \in H^1(\Omega)$ are given functions and $a(u,v)$ is the form defined by (1.1). Let us now consider the same problem with double obstacles, i.e.,

$$a(u, v-u) \geq \int\limits_{\Omega} f\,(v-u)dx, \quad u \in K_{\psi}^{\varphi}, \quad \forall v \in K_{\psi}^{\varphi}, \tag{1.5}$$

where

$$K_{\psi}^{\varphi} = \{v \in V : \psi \leq v \leq \varphi\} \tag{1.6}$$

and $f \in L_2(\Omega),\ \varphi, \psi \in H^1(\Omega),\ \psi \leq \varphi$ are given functions.

Here and in what follows the inequalities between functions from H^1 or L_2 are understood almost everywhere, while "sup" stands for "ess sup".

The unique solvability of these problems follows from the coercivity and boundedness of the form $a(u,v)$ and from the closure and convexity of the sets K_{ψ}^{φ}, K_{ψ} and K^{φ} (see [2], [6]).

Below we will give the maximum principles for the above inequalities and estimate the differences of their solutions. Similar results can be found in [1], [2] and [6], but we will prove them in the form we need for further applications. First of all we give several definitions and lemmas.

Definition. For $a_0(x)$ from (1.1) and any $f \in L_2(\Omega)$ define the constants

$$M_f^+ := \sup_{\Omega} \frac{f}{a_0}, \quad M_f^- := \inf_{\Omega} \frac{f}{a_0} \quad \text{when} \quad V = H^1(\Omega),$$

$$M_f^+ := \max\left(\sup_{\Omega} \frac{f}{a_0}, 0\right), \quad M_f^- := \min\left(\inf_{\Omega} \frac{f}{a_0}, 0\right) \quad \text{when} \quad V = \widetilde{H}^1(\Omega).$$

Lemma 1.1 (cf. [5], [6]). *Let $u \in V$. Then*

(i) $\displaystyle \frac{\partial u}{\partial x_i}\bigg|_D = 0, \quad 1 \leq i \leq n,$ *where $D = \{x \in \Omega,\ u(x) = 0\}$.*

(ii) *If*

$$\max(u,0) = \begin{cases} u, & \{x \in \Omega, \ \ u(x) > 0\}, \\ 0, & \{x \in \Omega, \ \ u(x) \leq 0\}, \end{cases}$$

then $\max(u,0) \in V$ *and*

$$[\max(u,0)]_{x_i} = \begin{cases} u_{x_i}, & \{x \in \Omega, \ \ u(x) > 0\}, \\ 0, & \{x \in \Omega, \ \ u(x) \leq 0\}. \end{cases}$$

Note that sometimes we will use the notation $u^+ = \max(u,0)$ and $u^- = \min(u,0)$.

Lemma 1.2 ([5]). *Let u be the solution of problem* (1.2), (1.3) *and u_f be the solution of the problem*

$$u_f \in V, \quad a(u_f, v) = \int\limits_\Omega f\, v dx, \qquad \forall v \in V, \tag{1.7}$$

which is the Neumann or the Dirichlet problem with the homogeneous boundary condition, depending on whether $V = H^1(\Omega)$ or $V = \widetilde{H}^1(\Omega)$. Then

$$u_f \le u \le z, \tag{1.8}$$

where z satisfies the conditions

$$z \in V,\ z \ge \psi, \quad a(z, v) \ge \int\limits_\Omega f\, v\, dx, \quad \forall v \in V, \ \ v \ge 0. \tag{1.9}$$

Proof. We will prove the right-hand side of (1.8). Let z satisfy conditions (1.9). Take

$$w := \min(z-u, 0) = (z-u)^-.$$

We will show that $w = 0$. Due to (ii) of Lemma 1.1 we have the following properties of the function w:

$$\begin{gathered} w \in V, \quad \frac{\partial w}{\partial x_i}\frac{\partial (z-u)}{\partial x_j} = \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j}, \quad \frac{\partial (z-u)}{\partial x_i} w = \frac{\partial w}{\partial x_i} w, \\ (z-u)w = w^2. \end{gathered} \tag{1.10}$$

Substituting the function w into the form (1.1) and taking into account (1.10), we obtain

$$a(z-u, w) = a(w,w). \tag{1.11}$$

It is easy to see that $\psi \le \min(z,u) = w + u$; so we can put $v = \min(z,u)$ in inequality (1.2) to obtain

$$a(u,w) \ge \int\limits_\Omega fw\, dx. \tag{1.12}$$

At the same time we have

$$a(z,w) \le \int\limits_\Omega fw\, dx. \tag{1.13}$$

After subtracting (1.12) from (1.13), by virtue of (1.11) we obtain $a(w,w) \le 0$, which implies $w = 0$ due to property (1.2).

Let us prove the left-hand side of (1.8). Take $v = u - (u-u_f)^-$. Obviously, $v \in K_\psi$ and if we substitute it into problem (1.2), (1.3), we then obtain

$$a(u, (u-u_f)^-) \le \int\limits_\Omega f(u-u_f)^-\, dx.$$

Further, taking into account the fact that due to (1.7), $a(u_f,(u-u_f)^-) = \int_\Omega f(u-u_f)^-\,dx$, by a reasoning analogous to that we have used for (1.10) and (1.11) we derive

$$0 \geq a(u-u_f,(u-u_f)^-) = a((u-u_f)^-,(u-u_f)^-),$$

from which it follows that $(u-u_f)^- = 0$ and $u \geq u_f$. □

Let us prove the principles of maximum and minimum for the above-formulated variational inequalities and problem (1.7).

Theorem 1.3. *Let $f \in L_2(\Omega)$, $\varphi,\psi \in H^1(\Omega)$, $\psi \leq \varphi$.*

(i) *For a solution u_f of problem* (1.7) *there holds*

$$M_f^- \leq u_f \leq M_f^+. \tag{1.14}$$

(ii) *If u is a solution of the unilateral problem* (1.2), (1.3) ((1.2), (1.4)), *then*

$$u \leq \max(M_f^+, \sup\psi) \quad (u \geq \max(M_f^-, \inf\varphi)). \tag{1.15}$$

(iii) *If u is a solution of problem* (1.5), (1.6), *then*

$$\min(M_f^-, \inf\varphi) \leq u \leq \max(M_f^+, \sup\psi). \tag{1.16}$$

Proof. (i) Take $\psi \leq M_f^+$ in problem (1.2), (1.3). Then it is clear that $z = M_f^+$ satisfies (1.9). So we take $z = M_f^+$ in (1.8).

To prove the left-hand side inequality in (1.14), note that

$$-u_f = u_{-f} \leq M_{-f}^+ = -M_f^-.$$

(ii) Let u be a solution of problem (1.2), (1.3). Then to obtain (1.15) we can put $z = \max(M_f^+, \sup\psi)$ in (1.8). If u is a solution of problem (1.2), (1.4), then $-u$ is a solution of problem (1.2), (1.3) with $-f$ and $\psi = -\varphi$. Thus from $M_{-f}^+ = -M_f^-$ follows the second inequality of (1.15).

(iii) Let us prove the right-hand side inequality in (1.16). Let $u_\psi \in V$, $u_\psi \geq \psi$ be a solution of the variational inequality

$$a(u_\psi, v-u_\psi) \geq \int_\Omega f\,(v-u_\psi)dx, \qquad \forall v \in V, \ \ v \geq \psi. \tag{1.17}$$

Take $v = \min(u,u_\psi)$. Obviously, $v \in K_\psi^\varphi$ and we can put it in (1.5) to obtain

$$a(u,(u_\psi-u)^-) \geq \int_\Omega f\,(u_\psi-u)^- dx. \tag{1.18}$$

At the same time, due to (1.17) we can write

$$a(u_\psi,(u_\psi-u)^-) \leq \int_\Omega f\,(u_\psi-u)^- dx. \tag{1.19}$$

After subtracting (1.18) from (1.19), by the same reasoning as in Lemma 1.2, we derive

$$0 \geq a(u_\psi - u, (u_\psi - u)^-) = a((u_\psi - u)^-, (u_\psi - u)^-),$$

which implies $(u_\psi - u)^- = 0$ and $u_\psi \geq u$. Now, by estimate (1.15) for a solution of a unilateral variational inequality we have $u \leq u_\psi \leq \max(M_f^+, \sup \psi)$.

Note that $-u$ is a solution of problem (1.5), (1.6) with $-f$, and $K_{-\varphi}^{-\psi} = \{v \in V : -\varphi \leq v \leq -\psi\}$. Thus the above-proved result implies the first inequality of (1.16). □

Next, we will estimate the differences of solutions corresponding to different data of inequalities with unilateral and bilateral restrictions.

Theorem 1.4. *Let u_1 and u_2 be solutions of inequalities*

$$u_1 \in K_1, \quad a(u_1, v - u_1) \geq \int_\Omega f_1 (v - u_1) dx, \quad \forall v \in K_1, \tag{1.20}$$

$$u_2 \in K_2, \quad a(u_2, v - u_2) \geq \int_\Omega f_2 (v - u_2) dx, \quad \forall v \in K_2, \tag{1.21}$$

with

$$K_i = \{v \in V : v \leq \varphi_i\}, \quad \varphi_i \in V, \quad i = 1, 2.$$

Then

$$\min(M_{f_1 - f_2}^-, \inf(\varphi_1 - \varphi_2)) \leq u_1 - u_2 \leq \max(M_{f_1 - f_2}^+, \sup(\varphi_1 - \varphi_2)). \tag{1.22}$$

Proof. Let $k = \max(M_f^+, \sup(\varphi_1 - \varphi_2))$. It can be verified that

$$(u_1 - u_2 - k)^+ \leq \varphi_2 - u_2.$$

So, if we put $v_1 = u_1 - (u_1 - u_2 - k)^+$ and $v_2 = u_2 + (u_1 - u_2 - k)^+$ in (1.20) and (1.21), respectively, and sum them, we arrive at

$$a(u_1 - u_2, (u_1 - u_2 - k)^+) \leq \int_\Omega (f_1 - f_2) (u_1 - u_2 - k)^+ \, dx.$$

Hence, by the same reasoning as we have used for equality (1.11), we derive

$$a((u_1 - u_2 - k)^+, (u_1 - u_2 - k)^+) + \int_\Omega k a_0 (u_1 - u_2 - k)^+ \, dx$$
$$\leq \int_\Omega (f_1 - f_2) (u_1 - u_2 - k)^+ \, dx,$$

which implies $(u_1 - u_2 - k)^+ = 0$ and thus the right-hand side of estimate (1.22) is proved.

To prove the left-hand side of (1.22) we take $k = \min(M_f^-, \inf(\varphi_1 - \varphi_2))$. Further, we check that $(u_1 - u_2 - k)^- \geq u_1 - \varphi_1$ and substitute $v_1 = u_1 - (u_1 - u_2 - k)^-$ and $v_2 = u_2 + (u_1 - u_2 - k)^-$ into inequalities (1.20) and (1.21) respectively. Then, in the same way as above we derive $(u_1 - u_2 - k)^- = 0$. □

2. Quasi-Variational Inequalities

Let us consider a quasi-variational inequality, i.e., an inequality with an obstacle depending on the solution:

$$\begin{gathered} a(u, v-u) \geq \int_\Omega f\,(v-u)dx, \quad \forall v \in K(u), \\ K(u) = \{v \in H^1(\Omega) : v \geq h - a(u,\phi)\} \end{gathered} \tag{2.1}$$

with

$$f \in L_2(\Omega), \quad h,\, \phi \in H^1(\Omega), \quad \phi \geq 0. \tag{2.2}$$

The problem with an obstacle of this kind appearing on the boundary is considered as an implicit Signorini problem in the monograph [1] and in some other references. The uniqueness of a solution of this problem is proved in [4]. Using one result of Section 1, we will show that problem (2.1) is solvable uniquely.

Let u_λ be a solution of the following variational inequality:

$$\begin{gathered} a(u_\lambda, v-u_\lambda) \geq \int_\Omega f\,(v-u_\lambda)dx, \quad \forall v \in H^1(\Omega), \quad v \geq h-\lambda, \\ u_\lambda \in H^1(\Omega), \quad u_\lambda \geq h - \lambda, \end{gathered} \tag{2.3}$$

where f and h are the same as in problem (2.1) and $\lambda \in \mathbb{R}$.

Let $S = \big\{y \in \mathbb{R} : \ \ y \geq \int_\Omega f\,\phi\,dx\big\}$ for f and ϕ from (2.2). Define the mapping $F:\ \mathbb{R} \to S$ for problem (2.1) as follows:

$$F(\lambda) = a(u_\lambda, \phi), \quad \lambda \in \mathbb{R}, \tag{2.4}$$

with ϕ from (2.1), (2.2). Clearly, if $F(\lambda) = \lambda$, then the corresponding u_λ is a solution of problem (2.1). Conversely, if u is a solution of problem (2.1), then for $\lambda = a(u,\phi)$ we have $u_\lambda = u$; therefore $F(\lambda) = \lambda$, i.e., the problem of solvability and the number of solutions of the problem (2.1) are reduced to defining the number of fixed points of mapping (2.4). The continuity and lack of increase of the function $F(\lambda)$ are sufficient to have a unique fixed point. First we will show that mapping (2.4) is continuous.

Let $\lambda_1,\ \lambda_2 \in \mathbb{R}$. Put $v = u_{\lambda_1} + \lambda_1 - \lambda_2$ and $v = u_{\lambda_2} + \lambda_2 - \lambda_1$ in inequality (2.3) for $\lambda = \lambda_2$ and $\lambda = \lambda_1$, respectively, to obtain

$$a(u_{\lambda_2}, u_{\lambda_1} - u_{\lambda_2}) \geq \int_\Omega f\,(u_{\lambda_1} - u_{\lambda_2} + \lambda_1 - \lambda_2)dx + (\lambda_2 - \lambda_1)\int_\Omega a_0 u_{\lambda_2}\,dx,$$

$$a(u_{\lambda_1}, u_{\lambda_2} - u_{\lambda_1}) \geq \int_\Omega f\,(u_{\lambda_2} - u_{\lambda_1} + \lambda_2 - \lambda_1)dx + (\lambda_1 - \lambda_2)\int_\Omega a_0 u_{\lambda_1}\,dx.$$

Sum these inequalities and recall that the form $a(u,v)$ is coercive:

$$\alpha\|u_{\lambda_1}-u_{\lambda_2}\|_1^2 \le a(u_{\lambda_1}-u_{\lambda_2},u_{\lambda_1}-u_{\lambda_2}) \le (\lambda_2-\lambda_1)\int\limits_\Omega (u_{\lambda_1}-u_{\lambda_2})\,dx$$

$$\le |\lambda_1-\lambda_2|\,\|u_{\lambda_1}-u_{\lambda_2}\|_{L_2}.$$

Thus

$$\|u_{\lambda_1}-u_{\lambda_2}\|_1 \le |\lambda_1-\lambda_2|.$$

Now we need to use the boundedness of form (1.1). Assume that $c = \max\limits_{1\le i,j\le n}(\|a_{ij}\|_{L^\infty}, \|a_i\|_{L^\infty}, \|a_0\|_{L^\infty})$ and write

$$F(\lambda_1)-F(\lambda_2) = a(u_{\lambda_1}-u_{\lambda_2},\phi) \le c\|u_{\lambda_1}-u_{\lambda_2}\|_1\,\|\phi\|_1 \le |\lambda_1-\lambda_2|\,\|\phi\|_1,$$

which implies that the function $F(\lambda)$ is continuous.

Now we show that $F(\lambda)$ is nonincreasing, i.e.,

$$a(u_{\lambda_1}-u_{\lambda_2},\phi)\ge 0, \quad \lambda_1\ge\lambda_2.$$

To prove this under conditions (2.2) it is equivalent to prove

$$a(u_0-u_\lambda,\phi)\ge 0, \quad \forall\phi\ge 0, \quad \forall\lambda\ge 0. \tag{2.5}$$

where u_0 is a solution of problem (2.3) for $\lambda = 0$.

Set

$$h_0 = h - u_\lambda.$$

Clearly, $h_0\le\lambda$. Consider the variational problem

$$\begin{gathered} a(w,v-w)\ge 0, \quad \forall v\in H^1(\Omega),\ \ v\ge h_0,\\ w\in H^1(\Omega), \quad w\ge h_0. \end{gathered} \tag{2.6}$$

It has a unique solution w. Since $h_0\le\lambda$, $\lambda\ge 0$, due to Theorem 1.3 (see (1.15)) we obtain

$$w\le\lambda \quad \text{in } \Omega. \tag{2.7}$$

Further, let

$$z = u_\lambda + w.$$

Let us show that z is a solution of problem (2.3) for $\lambda = 0$, i.e., $z = u_0$. Indeed, we have $z\ge h$. Further, by (2.6) and (2.7), we conclude that $v-u_\lambda\ge h-u_\lambda$ and $v-w\ge h-\lambda$ hold for any $v\in H^1(\Omega)$, $v\ge h$. Applying the latter inequalities we obtain

$$a(z,v-z) = a(u_\lambda+w,v-u_\lambda-w) \ge a(u_\lambda, v-w-u_\lambda)$$

$$\ge \int\limits_\Omega f\,(v-w-u_\lambda)dx = \int\limits_\Omega f\,(v-z)dx, \quad v\in H^1(\Omega), \quad v\ge h.$$

Thus z is a solution of (2.3) for $\lambda=0$ and therefore $z=u_0$. So $u_0-u_\lambda = w$ and (2.6) implies (2.5), from which it follows that the function F is nonincreasing.

Thus $F: S \to S$ is a continuous and nonincreasing function, having a unique fixed point. Hence, we have proved

Theorem 2.1. *The quasi-variational inequality* (2.1) *has a unique solution under conditions* (2.2)*.*

Let us study the question of uniqueness for an impulse control problem with double implicit obstacles.

Take

$$\begin{gathered} f \in L^\infty(\Omega),\ k_1, k_2 > 0, \quad k_1 \geq \frac{\|f^-\|_{L^\infty}}{a^0}, \quad k_2 \geq \frac{\|f^+\|_{L^\infty}}{a^0}, \\ c_1, c_2 : (\mathbb{R}^n)^+ \to \mathbb{R}^+, \quad c_1(0) = c_2(0) = 0, \\ M_1\varphi(x) = \inf_{\substack{\xi \geq 0, \\ x+\xi \in \overline{\Omega}}} [\varphi(x+\xi) + c_1(x+\xi)], \\ M_2\varphi(x) = \sup_{\substack{\xi \geq 0, \\ x+\xi \in \overline{\Omega}}} [\varphi(x+\xi) - c_2(x+\xi)]. \end{gathered} \tag{2.8}$$

Note that the inequalities between vectors from $\mathbb{R}^n$ are understood componentwise.

Consider the quasi-variational inequality

$$\begin{gathered} a(u, v-u) \geq \int_\Omega f\,(v-u)dx, \\ u \in V, \quad -k_2 + M_2 u \leq u \leq k_1 + M_1 u, \\ \forall v \in V, \quad -k_2 + M_2 u \leq v \leq k_1 + M_1 u. \end{gathered} \tag{2.9}$$

This inequality was considered in [1] for (1.1) for $c_1 = c_2 = 0$. It was proved there that the problem is solvablle, but the question of uniqueness remained open. Problem (2.9) was also considered with unilateral restrictions in [1]:

$$\begin{gathered} a(w, v-w) \geq \int_\Omega f\,(v-w)dx, \quad f \in L^\infty(\Omega), \\ w \in V, \quad u \leq k + M_1 w, \quad k > 0, \\ \forall v \in V, \quad v \leq k + M_1 w, \end{gathered} \tag{2.10}$$

In the coercive case the unique solvability was proved when $f \geq 0$, a_{ij} are sufficiently smooth and c_1 satisfies some other conditions. Since the restrictions on a_{ij} and c_1 are needed for the regularization purpose and not for the existence and uniqueness, we can ignore them. Also, the condition $f \geq 0$ is not necessary for the unique solvability of (2.10) either. Indeed, let f be not nonnegative; then consider (2.10) with $\overline{f} = f - \inf f$, which has a unique solution $\overline{w}$. Now we need only to make sure that there is a one-to-one correspondence between the solutions of the problems with f and with $\overline{f}$: $\overline{w} = w - \inf f$.

It can be shown that the following unilateral problem is also uniquely solvable:

$$\begin{gathered} a(z, v-z) \geq \int_\Omega f\,(v-z)dx, \quad f \in L^\infty(\Omega), \\ z \in V, \quad u \geq -k + M_2 z, \quad k > 0, \\ \forall v \in V, \quad v \geq -k + M_2 z. \end{gathered} \tag{2.10$'$}$$

Let us estimate the solutions of problems (2.9), (2.10) and (2.10)′. Definitions (2.8) immediately imply

$$\sup M_2\varphi \leq \sup \varphi, \quad \inf M_1\varphi \geq \inf \varphi. \tag{2.11}$$

Due to Theorem 1.3 (estimates (1.15), (1.16)) and to (2.11), we obtain

$$M_f^- \leq u \leq M_f^+, \tag{2.12}$$

$$w \geq M_f^-, \tag{2.13}$$

$$z \leq M_f^+. \tag{2.13$'$}$$

Note that (2.8) implies $k_2 \geq M_f^+$ and $k_1 \geq -M_f^-$,

Theorem 2.2. *If $k_2 \geq M_f^+ - M_f^-$ ($k_1 \geq M_f^+ - M_f^-$), then problem* (2.9) *has a unique solution u, which is a solution of the quasi-variational inequality* (2.10) ((2.10)′) *with the same f and with $k = k_1$, ($k = k_2$).*

Proof. Let u_0 be one of solutions of the quasi-variational inequality (2.9) and u_1 be a solution of the variational inequality

$$\begin{gathered} a(u_1, v-u_1) \geq \int_\Omega f\,(v-u_1)dx, \\ u_1 \in V, \quad u_1 \leq k_1 + M_1 u_0, \\ \forall v \in V, \quad v \geq k_1 + M_1 u_0, \end{gathered}$$

Since for u_0 we have (2.12), by (2.11) we obtain $M_2 u_0 \leq M_f^+$. Applying the latter estimate and (2.13), we derive

$$u_1 \geq M_f^- \geq M_f^+ - k_2 \geq M_2 u_0 - k_2$$

Thus u_1 is a solution of the variational inequality

$$\begin{gathered} a(u_1, v-u_1) \geq \int_\Omega f\,(v-u_1)dx, \\ u_1 \in V, \quad -k_2 + M_2 u_0 \leq u_1 \leq k_1 + M_1 u_0, \\ \forall v \in V, \quad -k_2 + M_2 u_0 \leq v \leq k_1 + M_1 u_0. \end{gathered}$$

It implies that $u_1 = u_0$. Hence u_0 is a solution of the quasi-variational inequality (2.10) with $k = k_1$. As we have noted, this problem is solvable uniquely.

Analogously, we prove the unique solvability for $k_1 \geq M_f^+ - M_f^-$ by considering problem (2.10)′ and applying (2.13)′. □

Let us now study one more quasi-variational inequality considered in [1], [3] and [7]. Set

$$M(v) = h + \int_\Omega g\, v\, dx, \qquad \forall v \in H^1(\Omega), \tag{2.14}$$

where

$$h \in H^1(\Omega),\ h \geq k,\ k = \text{const} > 0,\ g \in L_2(\Omega). \tag{2.15}$$

Consider the problem

$$\begin{gathered} a(u, v-u) \geq \int_\Omega f\,(v-u)dx, \quad f \in L_2(\Omega), \\ u \in H^1(\Omega), \quad u \leq M(u), \\ \forall v \in H^1(\Omega), \quad v \leq M(u). \end{gathered} \tag{2.16}$$

In references [1], [3] problem (2.14)–(2.16) is considered for the form

$$a(u,v) = \sum_{i,j=1}^n \int_\Omega \frac{\partial u}{\partial x_i}\, \frac{\partial v}{\partial x_j}\, dx + \int_\Omega u\, v\, dx.$$

We consider problem (2.14)–(2.16) for the form (1.1) because the latter form does not cause any difficulty.

Define $F(\lambda)$ as

$$F(\lambda) = \int_\Omega g\, u_\lambda\, dx, \tag{2.17}$$

where u_λ is a solution of the problem

$$\begin{gathered} a(u_\lambda, v-u_\lambda) \geq \int_\Omega f\,(v-u_\lambda)dx, \quad u_\lambda \in H^1(\Omega), \\ \forall v \in H^1(\Omega), \quad v \leq h+\lambda, \quad u_\lambda \leq h+\lambda, \end{gathered} \tag{2.18}$$

with the same f and h as in (2.15) and (2.16).

As above, the number of solutions of problem (2.14)–(2.16) coincides with the the number of fixed points of $F(\lambda)$.

In [1] and [7] it is shown that $F(\lambda)$ is continuous and

$$\frac{F(\lambda)}{\lambda} \to \int_\Omega g\, dx, \ \ \lambda \to -\infty, \quad F(\lambda) \to \int_\Omega g\, u_f dx, \ \ \lambda \to +\infty, \tag{2.19}$$

where

$$u_f \in H^1(\Omega), \quad a(u_f, v) = \int_\Omega f\, v dx, \qquad \forall v \in H^1(\Omega). \tag{2.20}$$

Hence, in view of the continuity of $F(\lambda)$, (2.19) implies that if

$$\int_\Omega g\, dx < 1, \tag{2.21}$$

then problem (2.14)–(2.16) has a solution.

Let us show that if condition (2.21) fails to be fulfilled, then problem (2.14)–(2.16) may have no solution at all. Indeed, examine the data

$$f \in L_2(\Omega), \quad f \geq \text{const} > 0, \quad \frac{f}{a_0} \neq \text{const}, \quad h = u_f, \quad \kappa = \operatorname{mes}\Omega,$$
$$g = \kappa^{-1} + \beta\Big(u_f - \kappa^{-1}\int\limits_\Omega u_f\,dx\Big), \quad \beta < \frac{\int\limits_\Omega u_f\,dx}{(\int\limits_\Omega u_f\,dx)^2 - \kappa\int\limits_\Omega u_f^2\,dx}. \tag{2.22}$$

Check that $(\int\limits_\Omega u_f\,dx)^2 \neq \kappa\int\limits_\Omega u_f^2\,dx$. First show that $u_f \neq \text{const}$. Indeed, if $u_f = \text{const}$, then

$$a(u_f, v) = u_f\int\limits_\Omega a_0\,v dx = \int\limits_\Omega f\,v dx, \quad \forall v \in H^1(\Omega).$$

Thus $u_f\,a_0 = f$, which contradicts $\frac{f}{a_0} \neq \text{const}$ in (2.22). Hence $u_f \neq \text{const}$ and Schwarz inequality implies

$$\Big(\int\limits_\Omega u_f\,dx\Big)^2 < \kappa\int\limits_\Omega u_f^2\,dx. \tag{2.23}$$

Now, observe that due to (2.22) and Theorem 1.3, $h = u_f \geq M_f^- > 0$.

Thus, data (2.22) are correct and satisfy conditions (2.15).

Lemma 2.3. *Problem* (2.14)–(2.16) *with conditions* (2.22) *has no solution.*

Proof. First of all observe that

$$u_\lambda = u_f, \quad \lambda \geq 0.$$

Theorem 1.4 implies that

$$0 \leq u_{\lambda_1} - u_{\lambda_2} \leq \lambda_1 - \lambda_2, \quad \lambda_1 \geq \lambda_2. \tag{2.24}$$

Since $u_0 = u_f = h$, then from (2.24) we derive

$$u_\lambda = u_0 + \lambda, \quad \lambda \leq 0,$$
$$F(\lambda) = \int\limits_\Omega g\,u_f\,dx + \lambda\int\limits_\Omega g\,dx = \int\limits_\Omega g\,u_f\,dx + \lambda, \quad \lambda \leq 0.$$

We have only to prove that $\int\limits_\Omega g\,u_f\,dx < 0$. Indeed, due to (2.22) and (2.23)

$$\int\limits_\Omega g\,u_f\,dx = \kappa^{-1}\int\limits_\Omega u_f\,dx + \beta\Big(\int\limits_\Omega u_f^2\,dx - \kappa^{-1}\Big(\int\limits_\Omega u_f\,dx\Big)^2\Big) < 0.$$

The lemma is proved. □

Let us consider the question of uniqueness of the solution of problem (2.14)–(2.16) with condition (2.21).

Theorem 2.4. *If* $\int\limits_\Omega g^+\,dx < 1$, *then problem* (2.14)–(2.16) *has a unique solution, while, if*

$$\frac{f}{a_0} \neq \text{const},\;\; f \geq \text{const} > 0,\;\; h = M_f^-(1-\alpha),\;\; g = \alpha\,\overline{g},$$

$$\overline{g} = \kappa^{-1} + \beta\Big(w - \kappa^{-1}\int\limits_\Omega w\,dx\Big),\;\; \beta < \frac{\int\limits_\Omega w\,dx}{(\int\limits_\Omega w\,dx)^2 - \kappa\int\limits_\Omega w^2\,dx},$$

$$w = \frac{\overline{u}_\mu - M_f^-}{\mu},\;\; \mu\in\mathbb{R}^+,\;\; \alpha = \frac{1}{\int\limits_\Omega \overline{g}\,w\,dx} \tag{2.25}$$

$$a(\overline{u}_\mu, v - \overline{u}_\mu) \geq \int\limits_\Omega f\,(v - \overline{u}_\mu)dx, \quad \forall v \leq M_f^- + \mu,\; v\in H^1(\Omega),$$

$$\overline{u}_\mu \in H^1(\Omega), \quad u_\mu \leq M_f^- + \mu,$$

with κ *from* (2.22), *then* f, g, h *satisfy* (2.15) *and* (2.21) *and problem* (2.14)–(2.16) *has at least two solutions.*

Proof. Let us prove the first claim. Suppose that problem (2.14) – (2.16) has two solutions u_1 and u_2 with $\int\limits_\Omega g\,u_1\,dx > \int\limits_\Omega g\,u_2\,dx$; then, due to Theorem 1.4,

$$0 \leq u_1 - u_2 \leq \int\limits_\Omega g\,(u_1-u_2)\,dx \leq \int\limits_\Omega g^+\,(u_1-u_2)\,dx < \sup(u_1-u_2),$$

which is a contradiction since $\int\limits_\Omega g^+\,dx < 1$.

Now we, will prove the second claim. First let us verify that g and h from (2.25) satisfy (2.15) and (2.21). It is sufficient to show that $\alpha < 1$, but for our future purpose we will show that $\alpha < 0$. To this end, let us first verify that $w \neq$ const. Indeed, if $w =$ const, then $\overline{u}_\mu =$ const and for every $\phi\in H^1(\Omega)$, $\phi \leq 0$, there holds

$$a(\overline{u}_\mu, \phi) = \overline{u}_\mu \int\limits_\Omega a_0\,\phi\,dx \geq \int\limits_\Omega f\,\phi\,dx. \tag{2.26}$$

Hence it follows that $\overline{u}_\mu\,a_0 \leq f$ and $\overline{u}_\mu \leq M_f^-$. It can be easily verified that in this case (2.29) holds for every $\phi\in H^1(\Omega)$; therefore $\overline{u}_\mu = u_f =$ const, where u_f is defined from (2.20). As we have shown in the proof of Lemma 2.3, $\overline{u}_\mu =$ const contradicts $\dfrac{f}{a_0} \neq$ const. Thus $w \neq$ const. Now, as in the above-mentioned proof, we will show that the fulfilment of $\int\limits_\Omega \overline{g}\,w\,dx < 0$ implies $\alpha < 0$.

First, we will give one general assertion. Consider problem (2.14), (2.16) with data $\overline{h}\in H^1(\Omega)$, $\overline{g}\in L_2(\Omega)$, $f\in L_2(\Omega)$, with $\overline{u}_\lambda$ and $\overline{F}(\lambda)$ defined by (2.17) and (2.18) by the mentioned data and take

$$g = \alpha_1 \overline{g}, \quad h = \overline{h} + \alpha_2, \tag{2.27}$$

where α_1 and α_2 are arbitrary numbers. If we take an arbitrary $\theta \in \mathbb{R}$ with

$$\overline{F}(\theta) \neq \overline{F}(0) \tag{2.28}$$

and define

$$\alpha_1 = \frac{\theta}{\overline{F}(\theta) - \overline{F}(0)}, \quad \alpha_2 = -\alpha_1 \overline{F}(0), \tag{2.29}$$

then $\alpha_1 \overline{F}(0)$ and $\alpha_1 \overline{F}(\theta)$ will be the fixed points of $F(\lambda)$, which is defined from (2.17) and (2.18) using the data f, g, h.

It is a simple consequence of the equalities

$$u_\lambda = \overline{u}_{\lambda+\alpha_2}, \quad F(\lambda) = \alpha_1 \overline{F}(\lambda + \alpha_2),$$

where u_λ is defined from (2.18) by f, g, h.

Let us return to conditions (2.25). Take

$$\overline{g} = g_0, \quad \overline{h} = M_f^-, \quad \theta = \mu. \tag{2.30}$$

To complete the proof, we must show that (2.27) and (2.28) hold for g and h from conditions (2.25) and for $\overline{g}$, $\overline{h}$, α_1 and α_2 from conditions (2.29) and (2.30). To this end, we verify that

$$\int\limits_\Omega \overline{g}\, dx = 1, \quad \overline{u}_0 = M_f^-,$$

where $\overline{u}_0$ is defined from (2.18) with $\lambda = 0$, $h = \overline{h}$.

The first equality is clear. To prove the second one, note that due to the corresponding estimate (1.15) of Theorem 1.3, we have $\overline{u}_0 \geq M_f^-$. These equalities imply that $\alpha_1 = \alpha = \frac{\mu}{\overline{F}(\mu) - \overline{F}(0)}$ and $\alpha_2 = -\alpha M_f^-$, which proves (2.27) and (2.28) since we have shown that $\alpha < 0$.

Thus $F(\lambda)$, defined in (2.17) for f, g, h from conditions (2.25) has at least two fixed points $\alpha_1 \overline{F}(0) = \alpha M_f^-$ and $\alpha_1 \overline{F}(\mu) = \alpha \int_\Omega g_0\, u_\mu\, dx$. Hence we conclude that problem (2.14), (2.16) under conditions (2.25) has at least two solutions. □

Acknowledgement

The author wishes to express his gratitude to the referee for his valuable suggestions which led to the improvement of the paper.

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(Received 4.01.2003; revised 12.03.2004)

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