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ON SPECTRAL FACTORIZATION OF MATRIX FUNCTIONS

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Abstract. In this note it is reviewed some results of the authors in matrix spectral factorization theory.

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Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit circle in the complex plane and $\mathbb{T} := \partial\mathbb{D}$. $L_p = L_p(\mathbb{T})$, $p > 0$, is the Lebesgue space of p-integrable complex functions and $H_p = H_p(\mathbb{D})$ is the Hardy space of analytic functions in $\mathbb{D}$. H_p^O denotes the set of outer analytic functions from the Hardy space H_p. Let $L_p^+ = L_p^+(\mathbb{T})$ be the space of boundary values of functions from H_p. For $f(z) \in H_p$, we write $f(z)|_{z=t} = f(t) \in L_p^+$ and thus these two classes $H_p(\mathbb{D})$ and $L_p^+(\mathbb{T})$ will be identified in usual way. It is well known that for $p \geq 1$, $L_p^+ = \{f \in L_p : f(t) \sim \sum_{n=0}^{\infty} c_n(f)t^n\}$ where $\sim$ stands for the Fourier expansion sign. We also deal with $L_p^- = \{f : \overline{f} \in L_p^+(\mathbb{T})\}$. Let $\mathcal{P}$ be the set of trigonometric polynomials, $\mathcal{P}^\pm := \mathcal{P} \cap \mathcal{L}_\infty^\pm$, $\mathcal{P}_n^+ = \{f : f(t) = \sum_{k=0}^{n} c_k t^k\}$ and $\mathcal{P}_n^- = \{f : f(t) = \sum_{k=0}^{n} c_k t^{-k}\}$.

For a class of functions $\mathcal{K}$ and $r \geq 1$, let $\mathcal{K}(r \times r)$ be the set of matrices with entries from $\mathcal{K}$. $H_2^O(r \times r)$ denotes the set of outer analytic matrix functions, i.e. the set of those $S \in H_2(r \times r)$ for which $\det S \in H_{2/r}^O$ (see [2]).

We are now ready to formulate Wiener's matrix spectral factorization theorem [10] which represents integrable matrix function on $\mathbb{T}$ as a product of boundary values of two invertible analytic matrix functions defined, respectively, on $\mathbb{D}_+ = \mathbb{D}$ and $\mathbb{D}_- = \mathbb{C} \cup \{\infty\} \backslash \overline{\mathbb{D}}$.

Theorem 1. *Let $S(t) \in L_1(r \times r)$ be a positive definite matrix function with integrable logarithm of determinant*

$$\log \det S(t) \in L_1(\mathbb{T}). \tag{1}$$

Then it admits a (left) spectral factorization

$$S(t) = S^+(t)S^-(t), \tag{2}$$

where $S^+(z) \in H_2^O(r \times r)$ and $S^-(z) = \big(S^+(1/\overline{z})\big)^*$.

The right factorization can be obtained by the left factorization of S^T. This theorem is the special positive definite case of a general matrix factorization theorem where the diagonal term in the middle of the product in (2) might appear which determines the partial indices of the matrix function S. It should be noted that the partial indices are equal to 0 in the positive definite case.

The condition (1) is also a necessary one for the factorization (2) to exist. A spectral factor $S^+(z)$ is unique up to a constant right unitary multiplier (see, e.g., [4] for a simple proof), and the unique spectral factor with an additional requirement that $S^+(0)$ be positive definite is called *canonical*. We always assume that the spectral factor in (2) is canonical. In the scalar case, $r = 1$, the canonical spectral factor S^+ can be explicitly written by the formula

$$S^+(z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\log S(e^{i\theta})\,d\theta\right). \tag{3}$$

However, there is no analog of this formula in the matrix case because, generally speaking, $e^{A+B} \neq e^A e^B$ for non-commutative matrices A and B. This is the main reason why the matrix spectral factorization is more demanding than the scalar spectral factorization.

In the present note we would like to review some results obtained by the authors in a matrix spectral factorization theory. These results appeared in print recently, and we would like to emphasize the relationships between them as well.

Since the Wiener's existence theorem was proved, there were numerous efforts for constructing an algorithm for approximate computation of matrix coefficients of S^+ for a given spectral density S (see the survey papers [7], [9]). In [6], a new algorithm for matrix spectral factorization is proposed. The decisive role in this algorithm plays the constructive proof of the following

Theorem 2. *Let $N \geq 1$. For any matrix function $F(t) \in \mathcal{P}(m\times m)$ of the form*

$$F(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \zeta_1^-(t) & \zeta_2^-(t) & \zeta_3^-(t) & \cdots & \zeta_{m-1}^-(t) & f^+(t) \end{pmatrix}, \tag{4}$$

where

$$\zeta_j^-(t) \in \mathcal{P}_n^-,\ j = 1,2,\dots,m-1,\ \text{and } f^+(t) \in \mathcal{P}_n^+,\ f^+(0) \neq 0, \tag{5}$$

there exists a unitary matrix function $U(t)$, $U(t)U^*(t) = I$, of the form

$$U(t) = \begin{pmatrix} u_{11}^+(t) & u_{12}^+(t) & \cdots & u_{1,m-1}^+(t) & u_{1m}^+(t) \\ u_{21}^+(t) & u_{22}^+(t) & \cdots & u_{2,m-1}^+(t) & u_{2m}^+(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1}^+(t) & u_{m-1,2}^+(t) & \cdots & u_{m-1,m-1}^+(t) & u_{m-1,m}^+(t) \\ \overline{u_{m1}^+(t)} & \overline{u_{m2}^+(t)} & \cdots & \overline{u_{m,m-1}^+(t)} & \overline{u_{mm}^+(t)} \end{pmatrix}, \tag{6}$$

where $u_{ij}^+(t) \in \mathcal{P}_n^+$, $i,j = 1,2,\dots,m$, with determinant 1, $\det U(t) = 1$, such that

$$F(t)U(t) \in \mathcal{P}^+. \tag{7}$$

It turned out that unitary matrix functions of this type are closely related with *compact wavelet matrices* ([8] p. 41). Thus Theorem 2 can be used to completely parameterize them [1].

In order to prove Theorem 2, we write a simple system of linear conditions, which is actually equivalent to (6), (7), and can be solved explicitly.

Lemma 1. *For given functions $\zeta_j^-(t)$, $j=1,2,\dots,m-1$, and $f^+(t)$ satisfying* (5)*, the columns of matrix function* (6) *satisfy the system of conditions* (*when $x_i=u_{ij}$, $i=1,2,\dots,m$*)

$$\begin{cases}\zeta_1^-(t)x_m^+(t)-f^+(t)\overline{x_1^+(t)}\in\mathcal{P}^+,\\ \zeta_2^-(t)x_m^+(t)-f^+(t)\overline{x_2^+(t)}\in\mathcal{P}^+,\\ \cdot\qquad\cdot\qquad\cdot\\ \zeta_{m-1}^-(t)x_m^+(t)-f^+(t)\overline{x_{m-1}^+(t)}\in\mathcal{P}^+,\\ \zeta_1^-(t)x_1^+(t)+\zeta_2^-(t)x_2^+(t)+\dots+\zeta_{m-1}^-(t)x_{m-1}^+(t)+f^+(t)\overline{x_m^+(t)}\in\mathcal{P}^+,\end{cases}$$

Theorem 2 is further extended in [5].

Theorem 3. *For any matrix function $F(t)\in L_2(m\times m)$ of the form* (4)*, where*

$$\zeta_j(t)\in L_2^-(\mathbb{T}),\ j=1,2,\dots,m-1,\ \text{and}\ f(t)\in H_2^O\subset L_2^+(\mathbb{T}),$$

there exists a unitary matrix function $U(t)$ of the form (6) *where $u_{ij}^+(t)\in L_\infty^+$, $i,j=1,2,\dots,m$, with determinant 1, such that*

$$F(t)U(t)\in L_2^+.$$

The Wiener's existence theorem is used in [5] to prove Theorem 3, but we would like to emphasize that this theorem can be proved directly applying the ideas developed in [6], [5] (this is important since Theorem 3 is also used in an analytic proof of the existence theorem itself which is described in [3]). The proof can be carried out applying the following theorem to the L_2-approximation $\zeta_n(t)=\sum_{k=0}^n c_k(\zeta)t^{-k}$ of $\zeta(t)$ (see [5], Theorem 3)

Theorem 4. *Let $F^{\{n\}}(t)$, $n=0,1,2,\dots$, be a sequence of matrix functions of the form* (4)*, where the last row is replaced by a row $\left(\zeta_1^{\{n\}}(t),\zeta_2^{\{n\}}(t),\cdots,\zeta_{m-1}^{\{n\}}(t),f^{\{n\}}(t)\right)$ with $\zeta_j^{\{n\}}(t)\in\mathcal{P}_n^-$, $j=1,2,\dots,m-1$, $f^{\{n\}}(t)\in\mathcal{P}_n^+$, $f^{\{n\}}(0)\neq 0$, and let $U_{F^{\{n\}}}(t)$, $n=0,1,2,\dots$, be a sequence of the corresponding unitary matrix functions determined according to Theorem 2. If*

$$\lim_{n\to\infty}\|F^{\{n\}}(t)-F(t)\|_{L_2}=0,$$

then $U_{F^{\{n\}}}(t)$ is convergent in measure and FU_F is a spectral factor of FF^*, where U_F is the limit of $U_{F^{\{n\}}}$.

This theorem can be proved exactly in the same manner as Theorem 2 in [5] using Lemma 1 instead of Lemma 5 of [5].

In our method of spectral factorization first we perform the lower-upper triangular factorization of S with outer analytic entries on the diagonal, and then step-by-step make analytic $m \times m$ left-upper submatrices of the factor, $m = 2, 3, \dots, r$, using the unitary matrices of Theorem 3.

The main result of [5] is the stability criteria of spectral factorization:

Theorem 5. *Let $S_n(t)$, $n = 1, 2, \dots$, be a sequence of positive definite $r \times r$ matrix functions with integrable entries such that*

$$\log\det S_n(t) \in L_1(\mathbb{T}), \quad n = 1, 2, \dots, \tag{8}$$

and let $(S_n)^+(t)$, $n = 1, 2, \dots$, be the sequence of corresponding spectral factors. If

$$\|S_n(t) - S(t)\|_{L_1} \to 0 \tag{9}$$

and

$$\int_0^{2\pi} \log\det S_n(e^{i\theta})\, d\theta \to \int_0^{2\pi} \log\det S(e^{i\theta})\, d\theta, \tag{10}$$

then

$$\|(S_n)^+ - S^+\|_{H_2} \to 0. \tag{11}$$

It is well-known that, in general, (9) alone does not imply (11) even in the scalar case. On the other hand, if (11) holds, then $S_n^+(0) \to S^+(0) \Longrightarrow \det S_n^+(0) \to \det S^+(0)$, and since Wiener's matrix spectral factorization theorem provides the scalar spectral factorization of the determinant $\big(\det S\big)(t) = \big(\det S\big)^+(t)\big(\det S\big)^-(t) = \det S^+(t) \det S^-(t)$ and $\big(\det S_n\big)^+(0) = \exp\Big(\frac{1}{4\pi}\int_0^{2\pi} \log\det S_n(e^{i\theta})\, d\theta\Big)$, $n = 1, 2, \dots$ (see (3)), we have that (10) is valid. Thus one can easily see that the condition (10) is necessary for the convergence (11) to hold.

One more important practical consequence of Theorem 5 is the following fact: as is well known (see, e.g., [4]), if

$$S(t) = \sum\nolimits_{k=-N}^{N} \sigma_k t^k \tag{12}$$

is a trigonometric polynomial matrix function, then

$$S^+(t) = \sum\nolimits_{k=0}^{N} \rho_k t^k \tag{13}$$

is a polynomial (of the same order N) matrix function. If a sequence of the considered spectral densities are matrix polynomials of fixed order N, then (9) always implies (10) and consequently (11) follows directly from (9). Thus it should be expected that a "small" perturbation of the coefficients σ_k in (12) will not much affect the coefficients ρ_k in (13) even in the case where the determinant of $S(t)$ has the zeros on the boundary. This fact was empirically observed during computer simulations of different numerical polynomial spectral factorization algorithms.

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