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On a Relationship Between the Integrabilities of Various Ergodic Maximal Functions

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Let $(X,\mathbb{S},\mu)$ be a σ-finite measure space and let $(T_t)_{t\in\mathbb{R}}$ be an ergodic group of measure-preserving transformations on X. We consider various types of ergodic maximal operators defined by

$$M_j(f)=\sup_{\Delta\in\mathcal{I}_j}\frac{1}{|\Delta|}\bigg|\int\limits_{\Delta} f(T_t x)\,dt\bigg|,\quad f\in L(X),$$

where $\mathcal{I}_j$, $j=0,1,2,3$, are the following subclasses of the class of intervals on the real line

$$\begin{aligned}\mathcal{I}_0&=\{(a,b):a<0<b\},\\ \mathcal{I}_1&=\{(a,b):b=0\},\\ \mathcal{I}_2&=\{(a,b):a=0\},\\ \mathcal{I}_3&=\{(a,b):a=-b\}.\end{aligned}$$

Obviously, $M_j(f)\leq M_0(f)\leq M_1(f)+M_2(f)$, $j=1,2,3$.

Let

$$\mathcal{V}_j=\{f\in L(X):M_j(f)\in L(X)\},\quad j=0,1,2,3. \tag{1}$$

(When we write $\mathcal{V}$, it is assumed that it equals $\mathcal{V}_j$ for some j, and it does not matter to which one.)

It is well-known that, for finite measure spaces, $\mu(X)<\infty$, if $f\in L\log L$, then $f\in\mathcal{V}$. The converse is true for positive functions: $f\geq 0$, $f\in\mathcal{V}$ *imply* $f\in L\log L$ (see [3]). On the other hand, there exist integrable functions outside $L\log L$ which belong to $\mathcal{V}$. It was proved in [1] that for each $f\in L(X)$ there exists a measurable function s with absolute value one, $s=\pm1$, such that $sf\in\mathcal{V}$.

For infinite measure spaces, $\mu(X)=\infty$, if $f\geq 0$ and $f\not\equiv 0$, then $f\notin\mathcal{V}$. If we do not require f to be positive, then it may happen that $f\in\mathcal{V}$ for a function not almost everywhere equal to zero. This cannot happen for the group of translations on the real line, $T_t(x)=x+t$, $x\in\mathbb{R}$, and, furthermore, there exist examples of conservative ergodic groups of measure-preserving transformations on infinite Lebesgue measure spaces such that $f\in\mathcal{V}$ imply $f\equiv 0$ (see [1], Th. 2).

In general, the characterization of classes (1) in any explicit integral form is impossible

We claim the validity of the following

Theorem. *For every ergodic group of measure-preserving transformations on any σ-finite measure space we have*

$$\mathcal{V}_0=\mathcal{V}_1=\mathcal{V}_2=\mathcal{V}_3$$

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For the group of rotations on the unit circle $T_t(e^{ix}) = e^{i(x+t)}$, this theorem was proved in [2]. In the same paper the theorem was proved in the discrete case.

The equality of sets $\mathcal{V}_1 = \mathcal{V}_2$ means that the distribution functions of variable $\lambda \in \mathbb{R}^+$

$$\lambda \mapsto \mu(M_1(f) > \lambda) \quad \text{and} \quad \lambda \mapsto \mu(M_2(f) > \lambda)$$

can be integrable only simultaneously. In spite of this, it may happen that for some integrable f

$$\int_0^\infty \big|\mu(M_1(f) > \lambda) - \mu(M_2(f) > \lambda)\big| \, d\lambda = \infty.$$

References

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