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On the uniqueness of the one-sided maximal functions of Borel measures

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Abstract. We prove that if ν and μ are arbitrary (signed) Borel measures (on the unit circle) such that $M_+\nu(x) = M_+\mu(x)$ for each x, where M_+ is the one-sided maximal operator (without modulus in the definition), then $\nu = \mu$. The proof is constructive and it shows how ν can be recovered from $M_+\nu$ in the unique way.

1. Introduction.

The Hardy-Littlewood maximal function

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|} \int_I |f(t)|\, dt$$

plays an important role in the field of Harmonic Analysis and Real Analysis. Namely, the maximal functions are useful to estimate the various norms of different integral operators. The maximal operator is not linear, so we cannot use Banach space techniques in its study. In particular, the surjectivity and injectivity properties of the operator, first considered in [**9**], are difficult to handle. However, the first partial result in the study of the uniqueness has been obtained in [**1**]. Namely, it was proved that one-sided maximal operator (without modulus sign in the definition) on the unit circle is one-to-one. It should be mentioned that the continuous character of the system of intervals with respect to which the supremum is taken in the definition of the maximal function plays a crucial role in the validity of this theorem. Otherwise, if we define the maximal function with respect to some discrete system of partitions, like in the case of the dyadic maximal function, the uniqueness property fails to hold in general (see [**9**]). Our arguments in the present paper are rather elementary, however we need careful observations

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on the behavior of the maximal functions. In some sense, we propose a method of reconstructing f from Mf. Although our method would not work in the higher dimensional case, we think this should be a concern of our next study.

For a real locally integrable function $f \in L_{\mathrm{loc}}(\boldsymbol{R})$, let M_+f denote the one-sided maximal function,

$$M_+f(x) = \sup_{y>x} \frac{1}{y-x} \int_x^y f(t)dt, \quad x \in \boldsymbol{R}. \tag{1}$$

As it was mentioned, unlike the classical definition of the Hardy-Littlewood maximal function, we do not take the modulus of f in the right-hand side of (1) since we deal with the uniqueness problem of the operator M_+.

Let $L(\boldsymbol{T}) \subset L_{\mathrm{loc}}(\boldsymbol{R})$ be the class of 2π-periodic integrable (on $\boldsymbol{T} = [0, 2\pi)$) functions. The following uniqueness theorem has been proved in [**1**].

THEOREM A. *Let $f, g \in L(\boldsymbol{T})$ and*

$$M_+f(x) = M_+g(x) \quad \textit{for each } x \in \boldsymbol{R}. \tag{2}$$

Then

$$f(x) = g(x) \quad \textit{for a.a. } x \in \boldsymbol{R}.$$

We consider periodic functions in Theorem A, which is equivalent to taking the functions on the unit circle $\partial\boldsymbol{D}$ in the complex plane ($\partial\boldsymbol{D}$ is naturally identified with $\boldsymbol{T}$), because the theorem is correct only for finite measure spaces. Indeed, if we consider just integrable functions on $\boldsymbol{R}$, then $M_+f \equiv 0$ for every negative $f \in L(\boldsymbol{R})$.

Theorem A has been generalized for the ergodic maximal functions. Let $(T_t)_{t\in\Gamma}$ be an ergodic group of measure-preserving transformations on a finite measure space $(X, \boldsymbol{S}, P)$, where Γ is either $\boldsymbol{Z}$ (the discrete case) or $\boldsymbol{R}$ (the continuous case), and let

$$f^*(x) = \begin{cases} \sup_{b>0} \frac{1}{b} \int_0^b f(T_t x)dt \text{ in the continuous case,} \\ \sup_{n>0} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \text{ in the discrete case.} \end{cases}$$

The following theorem has been proved for the continuous case in [**2**] and for the discrete case in [**3**]. In the latter case another simple proof of the theorem has been also proposed in [**7**].

THEOREM B. *Let $f, g \in L(X)$ and*

$$f^*(x) = g^*(x) \textit{ for a.a. } x \in X \textit{ (with respect to the measure } P). \tag{3}$$

Then

$$f(x) = g(x) \quad \textit{for a.a. } x \in X.$$

One can find further motivations for proving Theorems A and B in the introduction of [**3**].

In the discrete case Theorem B has been generalized for the two sided maximal operator $f \mapsto \sup_{n,m \geq 0} \frac{1}{n+m+1} \sum_{k=-n}^{m} f(T^k x)$ in [**4**], while in the continuous case, the proof of the uniqueness theorem has not yet been found even for the Hardy-Littlewood maximal operator $Mf(x) = \sup_{a<x<b} \frac{1}{b-a} \int_a^b f(t)dt$ on the real line.

It is natural to require in the ergodic case that the equality (3) holds almost everywhere. One can slightly strengthen Theorem A in this direction since $M_+ f = M_+ g$ almost everywhere implies (2). This follows from the equality

$$M_+ f(x) = \lim_{\delta \to 0+} \operatorname*{ess\,inf}_{y \in (x, x+\delta)} M_+ f(y),$$

which was proved in [**2**] (see Lemma 3 therein).

THEOREM C. *Let $f, g \in L(\boldsymbol{T})$ and*

$$M_+ f(x) = M_+ g(x) \quad \textit{for a.a. } x \in \boldsymbol{R}.$$

Then

$$f(x) = g(x) \quad \textit{for a.a. } x \in \boldsymbol{R}.$$

In the present paper we consider the problem of generalization of Theorems A and C for Borel measures. Let $\mathscr{M}(\boldsymbol{T})$ denote the set of (signed) Borel measures ν on $\partial \boldsymbol{D}$. For notational convenience we assume that the one-sided maximal functions $M_+ \nu$ of such measures are 2π-periodic functions on the real line defined by

$$M_+ \nu(x) = \sup_{y \in (x, x+2\pi]} \frac{\nu[x, y)}{y - x}, \tag{4}$$

where it is always naturally assumed that $\nu(B) = \nu\{e^{i\theta} : \theta \in B\}$, whenever a Borel measurable set $B \subset [x, x + 2\pi)$ for some $x \in \boldsymbol{R}$. At the same time, without causing any confusion, we assume that $\nu(B) = \nu(B \cap \boldsymbol{T})$, whenever B is a 2π-periodic subset of $\boldsymbol{R}$.

Remark 1. Since the weak (1,1) type inequality

$$m(\boldsymbol{T} \cap \{M_+\nu > \lambda\}) \leq \frac{C}{\lambda}|\nu|(\boldsymbol{T}), \quad \lambda > 0, \tag{5}$$

holds for operator M_+ (see [**8**]), where m stands for the Lebesgue measure on the line, we have

$$M_+\nu(x) < \infty \text{ for a.a. } x \in \boldsymbol{R}. \tag{6}$$

As the following counter example shows, there exist $\nu, \mu \in \mathscr{M}(\boldsymbol{T})$ such that $M_+\mu = M_+\nu$ almost everywhere (with respect to m) while $\mu \neq \nu$.

Example. Let $\delta_{\{x\}} \in \mathscr{M}(\boldsymbol{T})$ be the Dirac measure concentrated at e^{ix} and $\mu, \nu \in \mathscr{M}(\boldsymbol{T})$ be defined by the equalities: $\nu = \delta_{\{0\}} = \delta_{\{2\pi\}}$ and $\mu = g dm - \delta_{\{\pi\}} + \delta_{\{2\pi\}}$, where $g(e^{ix}) = \pi/(2\pi - x)^2$ for $x \in (0, \pi)$ and $g(e^{ix}) = 0$ for $x \in [\pi, 2\pi]$. Then, for each $x \in (0, 2\pi]$, we have

$$M_+\nu(x) = \frac{1}{2\pi - x} \quad \text{and} \quad M_+\mu(x) = \begin{cases} \frac{1}{2\pi - x} \text{ for } x \neq \pi, \\ 1/4\pi \text{ for } x = \pi. \end{cases}$$

Thus Theorem C cannot be generalized for Borel measures. Nevertheless, we claim that the following generalization of Theorem A is valid.

Theorem 1. *Let $\nu, \mu \in \mathscr{M}(\boldsymbol{T})$ and*

$$M_+\nu(x) = M_+\mu(x) \quad \textit{for every } x \in \boldsymbol{R}. \tag{7}$$

Then

$$\mu = \nu.$$

A general idea of proving Theorem 1 is similar to the one used for the proof of Theorem A, but the details are much more involved.

2. Some propositions concerning the operator M_+.

PROPOSITION 1. *Let $\nu \in \mathscr{M}(\boldsymbol{T})$ and $x \in \boldsymbol{R}$. We have the following relations*:

$$\text{(i) } M_+\nu(x) \le \liminf_{y\to x-} M_+\nu(y);$$

$$\text{(ii) } \nu\{x\} \le 0 \Rightarrow M_+\nu(x) \le \liminf_{y\to x+} M_+\nu(y);$$

$$\text{(iii) } \nu\{x\} < 0 \Rightarrow M_+\nu(x) < \liminf_{y\to x+} M_+\nu(y).$$

PROOF. Take arbitrary $\lambda < M_+\nu(x)$ and find $z > x$ such that $\nu[x,z)/(z-x) > \lambda$. There exists $\delta_1 > 0$ such that $\nu[y,z)/(z-y) > \lambda$ for each $y \in (x-\delta_1, x]$. Thus, $M_+\nu(y) > \lambda$ for each $y \in (x-\delta_1, x]$ and (i) follows. At the same time, if $\nu\{x\} \le 0$, then there exists $\delta_2 > 0$ such that $\nu[y,z)/(z-y) > \lambda$ for each $y \in [x, x+\delta_2)$. Consequently $M_+\nu(y) > \lambda$ for each $y \in [x, x+\delta_2)$ and (ii) follows.

For each $z \in (x, x+2\pi]$, we have

$$\frac{\nu[x,z)}{z-x} \le \frac{\nu(x,z)}{z-x} - \frac{|\nu\{x\}|}{2\pi} = \lim_{y\to x+} \frac{\nu[y,z)}{z-y} - \frac{|\nu\{x\}|}{2\pi} \le \liminf_{y\to x+} M_+\nu(y) - \frac{|\nu\{x\}|}{2\pi}.$$

Thus (iii) holds. □

For $\lambda \in \boldsymbol{R}$, let

$$G_\lambda = \{x \in \boldsymbol{R} : M_+\nu(x) > \lambda\}. \tag{8}$$

Obviously G_λ is a 2π-periodic set. Note that in general G_λ may not be open, but Proposition 1(i) implies that it is always open from the left, i.e. for each $x \in G_\lambda$ there exists $\delta > 0$ such that $(x-\delta, x] \subset G_\lambda$. Hence each connected component of G_λ has a form $(a,b\rangle$, where the angle "$\rangle$" here and always in similar situations indicates that $(a,b\rangle$ is either $(a,b]$ or (a,b) (this will be clear from the context). It follows from Proposition 1(ii) that

$$b \in (a,b\rangle \Longrightarrow \nu\{b\} > 0. \tag{9}$$

Obviously

$$\nu\{x\} > 0 \Longrightarrow M_+\nu(x) = +\infty. \tag{10}$$

The representation of G_λ as a union of disjoint connected components has the form

$$G_\lambda = \cup_{n=1}^{\infty}(a_n, b_n\rangle, \tag{11}$$

and if $\nu \in \mathscr{M}(\boldsymbol{T})$ and $G_\lambda \neq \boldsymbol{R}$, then $0 < b_n - a_n \leq 2\pi$ for each n. At the same time, if $(a, b]$ is a connected component of G_λ, then $b - a < 2\pi$.

In the sequel we use some facts about the connected components of (11) which were obtained in [**5**]. Namely, for arbitrary connected component $(a, b\rangle$ of G_λ which is finite ($b - a \leq 2\pi$), we have

$$\nu(a, b\rangle \geq \lambda(b - a), \tag{12}$$

$$\nu[x, b\rangle \geq \lambda(b - x) \text{ for each } x \in (a, b) \tag{13}$$

and

$$\nu\langle b, y) \leq \lambda(y - b) \text{ for each } y \in (b, b + 2\pi), \tag{14}$$

where $\langle b, y) = (a, y) \setminus (a, b\rangle$, (see, respectively, (11), (12) and (17), (18) in [**5**]).

PROPOSITION 2. *Let $\nu \in \mathscr{M}(\boldsymbol{T})$. We have*

$$\inf_{x \in \boldsymbol{T}}\{M_+\nu(x)\} = \frac{\nu(\boldsymbol{T})}{2\pi}. \tag{15}$$

PROOF. Let

$$\lambda_{\min} = \inf_{x \in \boldsymbol{T}}\{M_+\nu(x)\}. \tag{16}$$

If we take $y = x + 2\pi$ in the definition (4), it is clear that $\lambda_{\min} \geq \nu(\boldsymbol{T})/2\pi$. In order to prove (15), we need to obtain the reverse inequality

$$\lambda_{\min} \leq \frac{\nu(\boldsymbol{T})}{2\pi}. \tag{17}$$

Consider two cases:

i) $M_+\nu(x) > \lambda_{\min}$ for each $x \in \boldsymbol{R}$. Then

$$\cup_{n=1}^{\infty} G_{\lambda_n} = G_{\lambda_{\min}} = \boldsymbol{R} \tag{18}$$

for each sequence $\{\lambda_n\}_{n=1}^{\infty} \searrow \lambda_{\min}$. For each $\lambda > \lambda_{\min}$, $G_\lambda \neq \boldsymbol{R}$ because of (16). So that G_λ consists of the union of finite connected components $G_\lambda = \cup_{n=1}^{\infty}(a_n, b_n\rangle$. Since $\nu(a_n, b_n\rangle \geq \lambda(b_n - a_n)$ for each n (see (12)), we have $\nu(G_\lambda) \geq \lambda m(G_\lambda)$. Thus, by virtue of (18), we have $\nu(\boldsymbol{T}) = \lim_{n\to\infty} \nu(G_{\lambda_n}) \geq \lim_{n\to\infty} \lambda_n m(G_{\lambda_n}) = \lambda_{\min} \cdot 2\pi$ and (17) is proved.

ii) There exists $x_0 \in \boldsymbol{T}$ such that

$$M_+\nu(x_0) = \lambda_{\min}. \tag{19}$$

Then, for each $y \in (x_0 + 2\pi, x_0 + 4\pi]$, we have

$$\lambda_{\min} = M_+\nu(x_0) = M_+\nu(x_0 + 2\pi) \geq \frac{\nu[x_0 + 2\pi, y)}{y - x_0 - 2\pi}. \tag{20}$$

Take any

$$\lambda < \lambda_{\min} \tag{21}$$

and let us show that

$$\nu[x_0, x_0 + 2\pi) \geq \lambda \cdot 2\pi. \tag{22}$$

This establishes (17).

Let us first show that, for each $x \in (x_0, x_0 + 2\pi)$,

$$\nu[x, y) > \lambda(y - x) \text{ for some } y \in (x, x_0 + 2\pi]. \tag{23}$$

Indeed, if

$$\sup_{y\in(x,x_0+2\pi]} \frac{\nu[x, y)}{y - x} \leq \lambda, \tag{24}$$

then

$$M_+\nu(x) = \sup_{y\in(x_0+2\pi,x+2\pi]} \frac{\nu[x, y)}{y - x}, \tag{25}$$

since we know that $M_+\nu(x) \geq \lambda_{\min} > \lambda$ and the relations (4) and (24) hold. But when $y \in (x_0 + 2\pi, x + 2\pi]$, we have

$$\frac{\nu[x,y)}{y-x} = \frac{\nu[x,x_0+2\pi)+\nu[x_0+2\pi,y)}{x_0+2\pi-x+y-x_0-2\pi} = \frac{\alpha+\beta(y)}{\gamma+\delta(y)} \le \frac{\alpha+\lambda_{\min}\delta(y)}{\gamma+\delta(y)} =: \varphi(y), \tag{26}$$

where $\alpha = \nu[x, x_0+2\pi)$, $\beta(y) = \nu[x_0+2\pi, y)$, $\gamma = x_0+2\pi-x$ and $\delta(y) = y-x_0-2\pi$ and the inequality in (26) holds because of (20).

Taking $y = x_0 + 2\pi$ in (24), we get $\alpha \le \lambda\gamma$. Consequently $\alpha + \lambda_{\min}\delta(y) < \lambda_{\min}(\gamma + \delta(y))$, by (21). Thus $\varphi(y) < \lambda_{\min}$ for each $y \in [x_0 + 2\pi, x + 2\pi]$ and

$$\sup_{y\in[x_0+2\pi,x+2\pi]} \varphi(y) < \lambda_{\min} \tag{27}$$

since φ is a continuous function. By virtue of (25), (26) and (27), we get $M_+\nu(x) < \lambda_{\min}$ which is a contradiction, by (16). Hence (23) holds.

Define

$$x = \sup\{y \in (x_0, x_0 + 2\pi] : \nu[x_0, y) > \lambda(y - x_0)\} \tag{28}$$

(because of (19) and (21), the set in (28) is not empty). The limiting argument shows that

$$\nu[x_0, x) \ge \lambda(x - x_0). \tag{29}$$

At the same time, if $x < x_0 + 2\pi$, then

$$\nu[x_0, y) > \lambda(y - x_0) \text{ for some } y \in (x, x_0 + 2\pi] \tag{30}$$

since we can sum up (29) and (23). The relation (30) contradicts the definition (28) of x being a supremum. Thus $x = x_0 + 2\pi$ and, by virtue of (29), the inequality (22) holds. □

3. Some auxiliary lemmas.

Although the following two lemmas look very similar to the Riesz "rising sun" lemma (see [**6**]) which was generalized for arbitrary Borel measures in [**5**], they have not been obtained therein. Therefore we prove them in the present paper.

LEMMA 1. *Let* $\nu \in \mathscr{M}(\boldsymbol{T})$ *and let* $(a, b\rangle \subset \boldsymbol{R}$ *be a finite* $(b - a \le 2\pi)$ *connected component of* G_λ *for some* $\lambda \in \boldsymbol{R}$ *(see (8)). If*

$$M_+\nu(a) = \lambda, \tag{31}$$

then

$$\nu[a,b\rangle = \lambda(b-a). \tag{32}$$

Proof. Because of (31), $\nu[a,y) \le \lambda(y-a)$ for each $y \in (a, a+2\pi]$, and passing to the limit, if necessary, we get $\nu[a,b\rangle \le \lambda(b-a)$. Thus to obtain (32) we need to show that

$$\nu[a,b\rangle \ge \lambda(b-a). \tag{33}$$

Recall that the inequalities (12)–(14) hold in the current situation.

Passing to the limit from the right, the inequalities (13) and (14) imply, respectively,

$$\nu(x,b\rangle \ge \lambda(b-x) \text{ for each } x \in (a,b) \tag{34}$$

and

$$\nu\langle b,y] \le \lambda(y-b) \text{ for each } y \in (b, b+2\pi). \tag{35}$$

If $\nu\{a\} \ge 0$, then (33) holds, because of (12).

Now consider the case where $\nu\{a\} < 0$. Since $\lim_{y\to a+} \nu[a,y)/(y-a) = -\infty$ in this case and (31) holds, we have

$$\lambda = M_+\nu(a) = \sup_{y\in(a+\delta,a+2\pi]} \frac{\nu[a,y)}{y-a}$$

for some $\delta > 0$. Hence there exist $y_0 \in (a+\delta, a+2\pi]$ and a sequence $\{y_n\}_{n=1}^{\infty}$ convergent to y_0 such that $\lambda = M_+\nu(a) = \lim_{n\to\infty} \nu[a,y_n)/(y_n-a)$. Since a convergent from the left or from the right subsequence can be extracted from this sequence, we have

$$\text{either } \nu[a,y_0) = \lambda(y_0-a) \text{ or } \nu[a,y_0] = \lambda(y_0-a). \tag{36}$$

If $y_0 < b$, then $\nu[y_0,b\rangle \ge \lambda(b-y_0)$ and $\nu(y_0,b\rangle \ge \lambda(b-y_0)$ by (13) and (34), respectively, and summing up one of these inequalities with the corresponding correct equality in (36), we get (33).

If $y_0 \in (b, b+2\pi)$, then $\nu\langle b,y_0) \le \lambda(y_0-b)$ and $\nu\langle b,y_0] \le \lambda(y_0-b)$ by (14) and (35), respectively, and subtracting one of these inequalities from the corresponding correct equality in (36), we get (33).

Consider the remaining case $y_0 = b$. If $b \notin (a, b\rangle$, i.e. $M_+\nu(b) \le \lambda$, then $\nu\{b\} \le 0$ (otherwise $M_+\nu(b) = +\infty$) and (33) follows from (36). If $b \in (a, b\rangle$ ($b - a < 2\pi$ in this case) and the first equality in (36) holds, i.e. $\nu[a, b) = \lambda(b-a)$, then $\nu[a, b] \ge \lambda(b - a)$ holds, because of (9), and (33) follows. □

LEMMA 2. *Under the hypothesis of Lemma* 1*, let* $x \in (a, b)$ *and*

$$\liminf_{y \to x+} M_+\nu(y) = \lambda. \tag{37}$$

Then

$$\nu(x, b\rangle = \lambda(b - x). \tag{38}$$

PROOF. We have $\nu(x, b\rangle = \lim_{y\to x+} \nu[y, b\rangle$ and, by virtue of (37), for each $\varepsilon > 0$, there exists a sequence $\{y_n\}_{n=1}^{\infty}$ convergent to x from the right such that

$$\lambda + \varepsilon > \liminf_{n\to\infty} M_+\nu(y_n) \ge \liminf_{n\to\infty} \frac{\nu[y_n, b\rangle}{b - y_n} = \frac{\nu(x, b\rangle}{b - x}.$$

Thus $\nu(x, b\rangle \le \lambda(b - x)$ and since (34) holds as well, we get (38). □

The following lemma has a general measure theoretical character.

LEMMA 3. *Let* $B \subset \boldsymbol{T}$ *be a Borel measurable set and* $h : B \to \boldsymbol{R}$ *be an integrable function, and let for each* $y \in B$ *there exists* $\delta_y > 0$ *such that*

$$\frac{\nu[y, z]}{z - y} \le h(y) \text{ for each } z \in (y, y + \delta_y). \tag{39}$$

Then, for each $S \subset B$,

$$\nu(S) \le \int_S h\, dm. \tag{40}$$

PROOF. First consider the case where $h \equiv \lambda$ is a constant function. Since we can consider the measure $\nu + \tau m$ and the function $h + \tau$, $\tau > 0$, if necessary, we can assume without loss of generality that $\lambda > 0$.

Take any Borel set $S \subset B$. Since the measure $|\nu| + m$ is regular, for each $\varepsilon > 0$, there exists an open set $\mathscr{O} \supset S$ such that

$$(|\nu| + m)(\mathscr{O} \backslash S) < \varepsilon. \tag{41}$$

Let us cover S with the system of closed intervals $\mathscr{I} = \{[y,z] : y \in S, z \in (y, y+\delta_y), [y,z] \subset \mathscr{O}\}$. We can extract disjoint subsystem $\mathscr{J} = \{[y_n, z_n] : n = 1, 2, \ldots\} \subset \mathscr{I}$ such that

$$(|\nu| + m)\big(S \setminus \cup_{n=1}^{\infty} [y_n, z_n]\big) = 0 \tag{42}$$

(see, e.g., Lemma 2 in [**5**]). Because of (39), we have

$$\nu[y_n, z_n] \le \lambda(z_n - y_n), \quad n = 1, 2, \ldots. \tag{43}$$

Obviously $(\cup_{n=1}^{\infty}[y_n, z_n]\setminus S) \subset (\mathscr{O}\setminus S)$, so that, by virtue of (41),

$$(|\nu| + m)(\cup_{n=1}^{\infty}[y_n, z_n]\setminus S) \le (|\nu| + m)(\mathscr{O}\setminus S) < \varepsilon. \tag{44}$$

It follows from the relations $S = (S\setminus A) \cup (S \cap A)$, $A = (A\setminus S) \cup (S \cap A)$, where $A = \cup_{n=1}^{\infty}[y_n, z_n]$, (42) and (44) that

$$\big|\nu(S) - \nu(\cup_{n=1}^{\infty}[y_n, z_n])\big| = \big|\nu\big(S\setminus(\cup_{n=1}^{\infty}[y_n, z_n])\big) - \nu(\cup_{n=1}^{\infty}[y_n, z_n]\setminus S)\big| \le \varepsilon \tag{45}$$

and

$$\big|m(S) - m(\cup_{n=1}^{\infty}[y_n, z_n])\big| = \big|m\big(S\setminus(\cup_{n=1}^{\infty}[y_n, z_n])\big) - m(\cup_{n=1}^{\infty}[y_n, z_n]\setminus S)\big| \le \varepsilon. \tag{46}$$

By virtue of (45), (43) and (46), we have

$$\begin{aligned}\nu(S) &\le \nu(\cup_{n=1}^{\infty}[y_n, z_n]) + \varepsilon \le \lambda m(\cup_{n=1}^{\infty}[y_n, z_n]) + \varepsilon \\ &\le \lambda(m(S) + \varepsilon) + \varepsilon \le \lambda m(S) + (1+\lambda)\varepsilon,\end{aligned} \tag{47}$$

and since ε is arbitrary, the inequality (40) holds for $h = \lambda$.

If now $h = \sum_{k=1}^{\infty} \lambda_k \mathbf{1}_{B_k}$, $B_i \cap B_j = \emptyset$, $\cup_{k=1}^{\infty} B_k = B$, is a simple integrable function, then

$$\nu(S) = \sum_{k=1}^{\infty} \nu(S \cap B_k) \le \sum_{k=1}^{\infty} \int_{S\cap B_k} \lambda_k \, dm = \int_S h \, dm. \tag{48}$$

For arbitrary h, we can approximate it by the simple functions $h_n = \sum_{k=-\infty}^{\infty} \frac{k}{n} \mathbf{1}_{\{\frac{k-1}{n} < h \le \frac{k}{n}\}}$ and pass to the limit

$$\nu(S) \leq \int_S h_n \, dm \longrightarrow \int_S h \, dm, \tag{49}$$

so that (40) holds. □

In an absolutely similar manner we can prove

Lemma 4. *Let $B \subset \boldsymbol{T}$ be a Borel measurable set and $h : B \to \boldsymbol{R}$ be an integrable function, and let for each $y \in B$, there exists $\delta_y > 0$ such that*

$$\frac{\nu[y,z]}{z-y} \geq h(y) \text{ for each } z \in (y, y+\delta_y)$$

Then, for each $S \subset B$,

$$\nu(S) \geq \int_S h \, dm.$$

Proof. We need to follow exactly the proof of the Lemma 3, just changing the signs of inequalities in (43), (48) and (49) (approximating by the simple functions $h_n = \sum_{k=-\infty}^{\infty} \frac{k-1}{n} \mathbf{1}_{\{\frac{k-1}{n} < h \leq \frac{k}{n}\}}$ from below), and the relations in (47) as follows

$$\begin{aligned} \nu(S) &\geq \nu(\cup_{n=1}^{\infty}[y_n, z_n]) - \varepsilon \geq \lambda m(\cup_{n=1}^{\infty}[y_n, z_n]) - \varepsilon \\ &\geq \lambda(m(S) - \varepsilon) - \varepsilon \leq \lambda m(S) - (1+\lambda)\varepsilon. \end{aligned}$$

□

Obviously, Lemmas 3 and 4 remain correct if, for each $y \in B$, we consider intervals $[z, y]$, $z < y$, instead of $[y, z]$, $z > y$. Thus, we have the following

Corollary 1. *Let $B \subset \boldsymbol{T}$ be a Borel measurable set and $h : B \to \boldsymbol{R}$ be a bounded measurable function, and let, for each $y \in B$, there exists $\delta_y > 0$ such that $\frac{\nu[y,z]}{z-y} \leq h(y)$ for each $z \in (y, y+\delta_y)$ and $\frac{\nu[z,y]}{y-z} \geq h(y)$ for each $z \in (y-\delta_y, y)$. Then*

$$\nu(S) = \int_S h \, dm \text{ for each } S \subset B.$$

4. A main lemma.

The following lemma plays a decisive role in the proof of Theorem 1.

Lemma 5. *Let $\nu, \mu \in \mathscr{M}(\boldsymbol{T})$ be such that (7) holds, and let $\lambda_0 \geq \lambda_{\min}$ be* *such that* $G_{\lambda_0} \neq \boldsymbol{R}$ *(see* (8)*). If* $(a,b\rangle \subset \boldsymbol{R}$ *is a connected component of* G_{λ_0}*, then, for each* $x \in (a,b)$*,*

$$\nu[x,b\rangle = \mu[x,b\rangle. \tag{50}$$

Let us first show how Theorem 1 follows from Lemma 5.

We say that $\mu = \nu$ on B, for a Borel measurable set $B \subset \boldsymbol{R}$, if $\nu(S) = \mu(S)$ for each measurable set $S \subset B$. Obviously, if we have a countable collection of sets $\{B_n\}_{n=1}^{\infty}$ and $\mu = \nu$ on B_n for each n, then $\nu = \mu$ on $\cup_{n=1}^{\infty} B_n$.

If $\lambda > \lambda_{\min}$, then $G_\lambda \neq \boldsymbol{R}$, by virtue of the definition (16), and hence its connected components are finite. Thus, Lemma 5 implies that $\nu = \mu$ on each connected component of G_λ and hence on G_λ. Consequently $\nu = \mu$ on $G_{\lambda_{\min}} = \cup_{n=1}^{\infty} G_{\lambda_n}$, where $\lambda_n \searrow \lambda_{\min}$, and Theorem 1 follows in the case where $G_{\lambda_{\min}} = \boldsymbol{R}$.

Consider the case where $G_{\lambda_{\min}}$ is a proper subset of $\boldsymbol{R}$. One can then take $\lambda_0 = \lambda_{\min}$ in Lemma 5. We need to show that $\nu = \mu$ on $\boldsymbol{T} \backslash G_{\lambda_{\min}}$.

Suppose $G_{\lambda_{\min}} = \cup_{n=1}^{\infty} (a_n, b_n\rangle$. Since $M_+\nu(x) \geq \lambda_{\min}$ for each $x \in \boldsymbol{R}$ and $a_n \notin G_{\lambda_{\min}} \Leftrightarrow M_+\nu(a_n) \leq \lambda_{\min}$, we have $M_+\nu(a_n) = \lambda_{\min}$. Thus, applying Lemma 1 to the measures ν and μ, we get

$$\nu[a_n, b_n\rangle = \lambda_{\min}(b_n - a_n) = \mu[a_n, b_n\rangle, \tag{51}$$

which also implies that

$$\nu\big(\cup_{n=1}^{\infty} [a_n, b_n\rangle\big) = \lambda_{\min} m\big(\cup_{n=1}^{\infty} [a_n, b_n\rangle\big) = \mu\big(\cup_{n=1}^{\infty} [a_n, b_n\rangle\big). \tag{52}$$

If we let x tend to a from the right in (50), we get $\nu(a,b\rangle = \mu(a,b\rangle$ for each finite connected component of $G_{\lambda_{\min}}$. Thus

$$\nu(a_n, b_n\rangle = \mu(a_n, b_n\rangle. \tag{53}$$

It follows from (51) and (53) that $\nu\{a_n\} = \mu\{a_n\}$ for each n. Thus it remains to show that $\nu = \mu$ on B, where

$$B = \boldsymbol{T} \backslash \big(\cup_{n=1}^{\infty} [a_n, b_n\rangle\big), \tag{54}$$

and let us show that $\nu = \lambda_{\min} \cdot m = \mu$ on B, i.e.

$$\nu(S) = \int_S \lambda_{\min} dm = \mu(S) \text{ for each } S \subset B. \tag{55}$$

By Proposition 2,

$$\nu(\boldsymbol{T}) = \lambda_{\min} \cdot 2\pi \tag{56}$$

and it follows from (52), (54) and (56) that

$$\nu(B) = \lambda_{\min} m(B). \tag{57}$$

Since $M_+\nu(x) = \lambda_{\min}$ for $x \in B$, we have $\nu[x,y) \le \lambda_{\min}(y-x)$ for each $y \in (x, x+2\pi]$ and hence, by virtue of Lemma 3,

$$\nu(S) \le \int_S \lambda_{\min} dm \text{ for each measurable set } S \subset B. \tag{58}$$

If there were the strict inequality for some $S \subset B$, say

$$\nu(S) < \int_S \lambda_{\min} dm, \tag{59}$$

then subtracting (59) from (57), we would get

$$\nu(B \backslash S) > \int_{B\backslash S} \lambda_{\min} dm,$$

which contradicts (58). Thus the first equality in (55) holds.

The same consideration is true for the measure μ, so that (55) holds.

5. An auxiliary function Ψ.

To prove Lemma 5, we introduce an auxiliary function Ψ which is uniquely determined in terms of the maximal function $M_+\nu$ and we represent $\nu(x,b\rangle$ in terms of Ψ. Equality (50) follows from the fact that the same representation is naturally correct for $\mu(x,b\rangle$ as well.

Fix any $x \in (a,b)$ for which

$$\lambda_x := M_+\nu(x) < +\infty \tag{60}$$

(recall that $\lambda_x > \lambda_0$) and, for each $\lambda \in [\lambda_0, \lambda_x)$, let $(a_\lambda, b_\lambda\rangle$ be the connected component of G_λ containing x. Define the map $\Psi : [\lambda_0, \lambda_x] \mapsto \boldsymbol{R}$ by the equalities:

$$\Psi(\lambda) = b_\lambda \text{ for } \lambda_0 \le \lambda < \lambda_x, \quad \text{and } \Psi(\lambda_x) = x. \tag{61}$$

Note that $\Psi(\lambda_0) = b$ by the hypothesis of the lemma.

Obviously, the sets $(a_\lambda, b_\lambda\rangle$ are embedded, so that Ψ is the decreasing function on $[\lambda_0, \lambda_x]$, $\lambda_0 \le \lambda_1 < \lambda_2 \le \lambda_x \Rightarrow (a_{\lambda_1}, b_{\lambda_1}\rangle \supset (a_{\lambda_2}, b_{\lambda_2}\rangle \Rightarrow \Psi(\lambda_1) \ge \Psi(\lambda_2)$. Note also that

$$\Psi(\lambda) > x \text{ for } \lambda_0 \le \lambda < \lambda_x, \tag{62}$$

since if $\Psi(\lambda) = x$ for some $\lambda < \lambda_x = M_+\nu(x)$, then $(a_\lambda, x]$ would be the connected component of $\{M_+\nu > \lambda\}$ which is a contradiction, by virtue of (9), (10) and (60).

Let

$$\Psi(\lambda\pm) := \lim_{\lambda' \to \lambda\pm} \Psi(\lambda'), \quad \lambda \in [\lambda_0, \lambda_x],$$

where it is assumed that

$$\Psi(\lambda_x+) = \Psi(\lambda_x) = x \text{ and } \Psi(\lambda_0-) = \Psi(\lambda_0) = b. \tag{63}$$

Let D be the set of discontinuity points of Ψ,

$$D = \{\lambda \in [\lambda_0, \lambda_x] : \Psi(\lambda-) \ne \Psi(\lambda+)\},$$

(D is a countable set), let C be the set where Ψ does not decrease strictly,

$$C = \{\lambda \in [\lambda_0, \lambda_x] : \exists \lambda' \in [\lambda_0, \lambda_x] \backslash \{\lambda\} \text{ such that } \Psi(\lambda') = \Psi(\lambda)\}$$

(C consists of a countable union of disjoint intervals) and let

$$E = (\lambda_0, \lambda_x) \backslash (C \cup D).$$

For $B \subset [\lambda_0, \lambda_x]$, let $I(B) = \{y : \exists \lambda \in [\lambda_0, \lambda_x] \text{ such that } y = \Psi(\lambda)\}$.

By virtue of (62), we have

$$x \notin I(C). \tag{64}$$

Note also that if $\Psi(\lambda) = b$ for some $\lambda \in (\lambda_0, \lambda_x)$, then $\lambda \in C$. So that, by virtue of the definition of E,

$$x, b \notin I(E). \tag{65}$$

Because of the monotonicity of Ψ, we have

$$(x, b) \subset \cup_{\lambda \in D}[\Psi(\lambda+), \Psi(\lambda-)] \cup I(E) \cup I(C) \subset [x, b] \tag{66}$$

and the sets

$$(\Psi(\lambda+), \Psi(\lambda-)),\ \lambda \in D,\ I(E) \text{ and } I(C), \text{ are pairwise disjoint.} \tag{67}$$

Besides

$$[\Psi(\lambda+), \Psi(\lambda-)] \cap I(E) = \emptyset \text{ for each } \lambda \in D \tag{68}$$

and

$$\begin{gathered} y \in [\Psi(\lambda_1+), \Psi(\lambda_1-)] \cap [\Psi(\lambda_2+), \Psi(\lambda_2-)], \text{ where } \lambda_1 > \lambda_2, \Longrightarrow \\ \Psi(\lambda_1-) = \Psi(\lambda_2+) = y \Longrightarrow \Psi(\lambda) = y \text{ for each } \lambda \in (\lambda_2, \lambda_1) \Longrightarrow y \in I(C). \end{gathered} \tag{69}$$

We need some further properties of the function Ψ.

Let $G_+ := \{\lambda \in [\lambda_0, \lambda_x) : \Psi(\lambda') < \Psi(\lambda+) \text{ for each } \lambda' \in (\lambda, \lambda_x)\}$ and $G_- := \{\lambda \in [\lambda_0, \lambda_x) : \Psi(\lambda') > \Psi(\lambda-) \text{ for each } \lambda' \in (\lambda_0, \lambda)\}$. Note that

$$\Psi(\lambda\pm) \notin I(C) \Longrightarrow \lambda \in G_\pm, \tag{70}$$

but the converse implication is not correct.

LEMMA 6. *If $\lambda \in G_+$, then*

$$M_+\nu(\Psi(\lambda+)) \le \lambda. \tag{71}$$

PROOF. For each $\varepsilon > 0$ and $\delta > 0$, there is $\lambda' \in (\lambda, \lambda+\varepsilon)$ such that $\Psi(\lambda') \in (\Psi(\lambda+)-\delta, \Psi(\lambda+))$ and since $(a_{\lambda'}, \Psi(\lambda')\rangle$ is the connected component of $\{M_+\nu > \lambda'\}$, it follows that there exists $y \in [\Psi(\lambda'), \Psi(\lambda+)) \subset (\Psi(\lambda+)-\delta, \Psi(\lambda+))$ such that $M_+\nu(y) \le \lambda'$. Thus, by virtue of Proposition 1(i), we have $M_+\nu(\Psi(\lambda+)) \le \lambda + \varepsilon$ and since ε is arbitrary (71) follows. □

LEMMA 7. *If $\lambda \in G_-$, then*

$$M_+\nu(\Psi(\lambda-)) \ge \lambda. \tag{72}$$

PROOF. By the hypothesis, for each $\lambda' \in (\lambda_0, \lambda)$ we have $\Psi(\lambda') > \Psi(\lambda-)$, so that $[x, \Psi(\lambda-)] \subset (a_{\lambda'}, \Psi(\lambda')) \subset \{M_+\nu > \lambda'\}$ and consequently $M_+\nu(\Psi(\lambda-)) > \lambda'$. This implies (72). □

COROLLARY 2. *For each $\lambda \in E$,*

$$M_+\nu(\Psi(\lambda)) = \lambda. \tag{73}$$

PROOF. Since $\lambda \notin C$, we have $\Psi(\lambda) \notin I(C)$. Since Ψ is continuous at λ, i.e. $\Psi(\lambda+) = \Psi(\lambda-) = \Psi(\lambda)$, and (70) holds, we have $\lambda \in G_\pm$. Thus we can apply Lemmas 6 and 7, and (73) follows from (71) and (72). □

LEMMA 8. *Let $y \in I(C)\backslash\{b\}$ and $\langle\alpha, \beta\rangle = \Psi^{-1}\{y\}$ ($\langle\alpha, \beta\rangle \subset [\lambda_0, \lambda_x]$ is a non-degenerated interval). Then either $M_+\nu(y) = \infty$ or $M_+\nu(y) = \alpha$.*

PROOF. If $\alpha = \lambda_0$, then

$$M_+\nu(y) \geq \alpha \tag{74}$$

since $y \in \{M_+\nu > \lambda_0\}$. If $\alpha > \lambda_0$, then for each $\lambda \in (\lambda_0, \alpha)$ we have $\Psi(\lambda) > y$ and since $[x, y] \subset (a_\lambda, \Psi(\lambda)) \subset \{M_+\nu > \lambda\}$, we have $M_+\nu(y) > \lambda$. Thus (74) is always correct. Assume now that $M_+\nu(y) < \infty$ and let us show that

$$M_+\nu(y) > \alpha \tag{75}$$

cannot be true. This completes the proof of the lemma.

Since $\nu\{y\} \leq 0$ (see (10)), if (75) were correct, then $[y, y+\delta]$ would be a subset of $\{M_+\nu > \lambda\}$ for some $\lambda \in (\alpha, \min(\beta, M_+\nu(y))$ and $\delta > 0$, by virtue of Proposition 1(ii). This contradicts the fact that $(a_\lambda, b_\lambda\rangle = (a_\lambda, y\rangle$ is the connected component of $\{M_+\nu > \lambda\}$. □

LEMMA 9. *Let $y \in [x, b)$ be such that $M_+\nu(y) = \inf\{\lambda : \Psi(\lambda) = y\} =: \lambda_y$ and $\Psi(\lambda_y-) = y$. Then $\nu\{y\} = 0$.*

PROOF. Since $M_+\nu(y) = \lambda_y \leq \lambda_x < \infty$, the impossibility of the inequality $\nu\{y\} > 0$ follows from (10).

Since $y \neq b$ and $\Psi(\lambda_0-) = b$, we have $\lambda_y \neq \lambda_0$. If $\lambda \in (\lambda_0, \lambda_y)$, then $\Psi(\lambda) > y$ and, for each $\tau > 0$, there exists $y' \in [\Psi(\lambda), \Psi(\lambda) + \tau]$ such that $M_+\nu(y') \leq \lambda$. Letting λ tend to λ_y from the left, $\Psi(\lambda)$ converges to y, by the hypothesis, so that we can conclude $\liminf_{y' \to y+} M_+\nu(y') \leq \lambda_y$. The impossibility of $\nu\{y\} < 0$ follows now from Proposition 1(iii). □

Lemma 10. *For* $\lambda \in [\lambda_0, \lambda_x)$, *let*

$$M_+\nu(\Psi(\lambda)) = \infty \tag{76}$$

and

$$\beta = \sup\{\lambda' : \Psi(\lambda') = \Psi(\lambda)\}. \tag{77}$$

Then $\beta \in D$.

Proof. Assume $\beta \notin D$. Then it follows from the definition (77) that

$$\Psi(\beta+) = \Psi(\beta) = \Psi(\beta-) = \Psi(\lambda). \tag{78}$$

If $\beta = \lambda_x$, then $\Psi(\beta) = x$, by (61), and (78) implies that $M_+\nu(\Psi(\lambda)) = M_+\nu(\Psi(\beta)) = M_+\nu(x) = \lambda_x < \infty$ (see (60)), which contradicts (76). Thus the lemma follows in this case.

Consider the case $\beta < \lambda_x$. It follows from the definition (77) and the relation (78) that $\beta \in G_+$. Thus, by virtue of Lemma 6, we get $M_+\nu(\Psi(\beta+)) \le \beta$. This contradicts (76) since $\Psi(\beta+) = \Psi(\lambda)$, by (78). $\square$

Ψ is strictly decreasing on E. Hence Ψ^{-1} exists on $I(E)$. By virtue of Corollary 2, we have

$$M_+\nu(y) = \Psi^{-1}(y) \tag{79}$$

for each $y \in I(E)$. We can identify ν on E by the following

Lemma 11. *For each measurable set* $S \subset I(E)$, *we have*

$$\nu(S) = \int_S \Psi^{-1} dm. \tag{80}$$

Proof. Let $y \in I(E)$. By virtue of (79), we have

$$\frac{\nu[y,z]}{z-y} \le \Psi^{-1}(y) \tag{81}$$

for each $z \in (y, y+2\pi)$. Let us now show that, for each $z \in (x, y)$,

$$\frac{\nu[z,y]}{y-z} \geq \Psi^{-1}(y). \tag{82}$$

Let $\Psi^{-1}(y) = \lambda$ and consider $(a_\lambda, b_\lambda\rangle = (a_\lambda, y\rangle$. Because of the definition of E, for each $\varepsilon > 0$ there exists $\lambda' \in (\lambda-\varepsilon, \lambda)$ such that $y < b_{\lambda'} = \Psi(\lambda') < b_\lambda + \varepsilon = \Psi(\lambda)+\varepsilon$. Thus $x \in (a_\lambda, b_\lambda] \subset (a_{\lambda'}, b_{\lambda'}\rangle$. By virtue of (13), for each $z \in (a_\lambda, b_\lambda)$, we have

$$\frac{\nu[z, b_{\lambda'}\rangle}{b_{\lambda'} - z} \geq \lambda'. \tag{83}$$

If we let ε tend to 0, then $\lambda' \to \lambda$ and $b_{\lambda'} \to b_\lambda$, so that, passing to the limit in (83), we get $\frac{\nu[z,b_\lambda]}{b_\lambda - z} \geq \lambda$, which is the same as (82) since $\Psi(\lambda) = b_\lambda = y$.

By virtue of (81) and (82), we can use Corollary 1 to conclude that (80) holds. □

For $\lambda \in D \backslash \{\lambda_0\}$, let

$$\langle \Psi(\lambda+), \Psi(\lambda-)\rangle := \begin{cases} [\Psi(\lambda+), \Psi(\lambda-)] \text{ if } M_+\nu(\Psi(\lambda+)) \leq \lambda \text{ and } M_+\nu(\Psi(\lambda-)) \geq \lambda, \\ [\Psi(\lambda+), \Psi(\lambda-)) \text{ if } M_+\nu(\Psi(\lambda+)) \leq \lambda \text{ and } M_+\nu(\Psi(\lambda-)) < \lambda, \\ (\Psi(\lambda+), \Psi(\lambda-)] \text{ if } M_+\nu(\Psi(\lambda+)) > \lambda \text{ and } M_+\nu(\Psi(\lambda-)) \geq \lambda, \\ (\Psi(\lambda+), \Psi(\lambda-)) \text{ if } M_+\nu(\Psi(\lambda+)) > \lambda \text{ and } M_+\nu(\Psi(\lambda-)) < \lambda, \end{cases} \tag{84}$$

and if $\lambda_0 \in D$, let

$$\langle \Psi(\lambda_0+), \Psi(\lambda_0-)\rangle := (\Psi(\lambda_0+), b\rangle. \tag{85}$$

LEMMA 12. *For each $\lambda \in D$, we have*

$$\nu\langle \Psi(\lambda+), \Psi(\lambda-)\rangle = \lambda(\Psi(\lambda-) - \Psi(\lambda+)). \tag{86}$$

PROOF. First consider the case $\lambda = \lambda_0 \in D$. Since $\Psi(\lambda_0+) \in (a,b)$, we have $M_+\nu(\Psi(\lambda_0+)) > \lambda_0$. Thus, there exists $\delta > 0$ such that

$$\inf_{y \in [\Psi(\lambda_0+)-\delta, \Psi(\lambda_0+)]} M_+\nu(y) > \lambda_0, \tag{87}$$

by Proposition 1(i). If $\lambda \in (\lambda_0, \lambda_x]$, then $\Psi(\lambda) \leq \Psi(\lambda_0+)$ and since $(a_\lambda, \Psi(\lambda)\rangle$ is a connected component of $\{M_+\nu > \lambda\}$, for each $\tau > 0$, there exists $y \in$

$[\Psi(\lambda), \Psi(\lambda) + \tau]$ such that $M_+\nu(y) \le \lambda$. Letting λ tend to λ_0 from the right ($\Psi(\lambda) \to \Psi(\lambda_0+)$ in this case) and taking into account (87), we conclude that $\Psi(\lambda) = \Psi(\lambda_0+)$ whenever $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ for some $\varepsilon > 0$, and $\liminf_{y \to \Psi(\lambda_0+)+} M_+\nu(y) \le \lambda$ for each $\lambda > \lambda_0$. Since $M_+\nu$ is greater than λ_0 on $(a, b\rangle$, it follows that $\liminf_{y \to \Psi(\lambda_0+)+} M_+\nu(y) = \lambda_0$. We are now able to use Lemma 2 in order to conclude that $\nu(\Psi(\lambda_0+), b\rangle = \lambda_0(b - \Psi(\lambda_0+))$ which is the equality (86) for $\lambda = \lambda_0$, by (85).

Consider now the case $\lambda \in D \backslash \{\lambda_0\}$.

If $M_+\nu(\Psi(\lambda+)) \le \lambda$, then $\nu[\Psi(\lambda+), y) \le \lambda(y - \Psi(\lambda+))$ for each $y \in (\Psi(\lambda+), \Psi(\lambda+) + 2\pi]$. Hence

$$M_+\nu(\Psi(\lambda+)) \le \lambda \Rightarrow \begin{cases} \nu[\Psi(\lambda+), \Psi(\lambda-)] \le \lambda(\Psi(\lambda-) - \Psi(\lambda+)), \\ \nu[\Psi(\lambda+), \Psi(\lambda-)) \le \lambda(\Psi(\lambda-) - \Psi(\lambda+)). \end{cases} \tag{88}$$

If $M_+\nu(\Psi(\lambda+)) > \lambda$ (note that $\lambda \ne \lambda_x$ and $\Psi(\lambda+) \ne x$ in this case, because of (60)), then there are $\varepsilon > 0$ and $\delta > 0$ such that $M_+\nu(y) > \lambda + \varepsilon$ for each $y \in (\Psi(\lambda+) - \delta, \Psi(\lambda+)]$, by virtue of Proposition 1(i), and $\Psi(\lambda') \in (\Psi(\lambda+) - \delta, \Psi(\lambda+)]$ for each $\lambda' \in (\lambda, \lambda + \varepsilon)$. Thus $(a_{\lambda'}, b_{\lambda'}\rangle \supset (\Psi(\lambda+) - \delta, \Psi(\lambda+)]$ and $b_{\lambda'} = \Psi(\lambda')$ cannot belong to $(\Psi(\lambda+) - \delta, \Psi(\lambda+))$, so that $\Psi(\lambda') = \Psi(\lambda+)$ for each $\lambda' \in (\lambda, \lambda + \varepsilon)$. This implies that there exists a convergent from the right to $\Psi(\lambda+)$ sequence $\{y_n\}_{n=1}^{\infty}$ such that $M_+\nu(y_n) \le \lambda'$, since $(a_{\lambda'}, \Psi(\lambda')\rangle = (a_{\lambda'}, \Psi(\lambda+)]$ is a connected component of $\{M_+\nu > \lambda'\}$. Passing to the limit, as $n \to \infty$, in the inequality $\nu[y_n, y) \le \lambda'(y - y_n)$, where $y \in (y_n, \Psi(\lambda+) + 2\pi]$, we get $\nu(\Psi(\lambda+), y) \le \lambda'(y - \Psi(\lambda+))$, and since the last inequality holds for each $\lambda' \in (\lambda, \lambda + \varepsilon)$, we have $\nu(\Psi(\lambda+), y) \le \lambda(y - \Psi(\lambda+))$. Thus

$$M_+\nu(\Psi(\lambda+)) > \lambda \Rightarrow \begin{cases} \nu(\Psi(\lambda+), \Psi(\lambda-)] \le \lambda(\Psi(\lambda-) - \Psi(\lambda+)), \\ \nu(\Psi(\lambda+), \Psi(\lambda-)) \le \lambda(\Psi(\lambda-) - \Psi(\lambda+)). \end{cases} \tag{89}$$

If $M_+\nu(\Psi(\lambda-)) \ge \lambda$, then for each $\lambda' \in (\lambda_0, \lambda)$ we have $[\Psi(\lambda+), \Psi(\lambda-)] \subset (a_{\lambda'}, \Psi(\lambda')\rangle$, by virtue of the definition of function Ψ. Thus $\nu[y, \psi(\lambda')\rangle \ge \lambda'(\psi(\lambda') - y)$ for each $y \in [\Psi(\lambda+), \Psi(\lambda-))$, by (13). Letting λ' tend to λ from the left, we get $\nu[y, \psi(\lambda-)] \ge \lambda(\psi(\lambda-) - y)$. Thus

$$M_+\nu(\Psi(\lambda-)) \ge \lambda \Rightarrow \begin{cases} \nu[\Psi(\lambda+), \Psi(\lambda-)] \ge \lambda(\Psi(\lambda-) - \Psi(\lambda+)), \\ \nu(\Psi(\lambda+), \Psi(\lambda-)] \ge \lambda(\Psi(\lambda-) - \Psi(\lambda+)). \end{cases} \tag{90}$$

If $M_+\nu(\Psi(\lambda-)) < \lambda$, then for each $\lambda' \in (M_+\nu(\Psi(\lambda-)), \lambda)$ we have $\Psi(\lambda') =$

$\Psi(\lambda-)$, since otherwise $\Psi(\lambda-) \in (a_{\lambda'}, \Psi(\lambda'))$ and $M_+\nu(\Psi(\lambda-)) > \lambda'$, which is a contradiction. For each $\lambda' \in (M_+\nu(\Psi(\lambda-)), \lambda)$, we have $[\Psi(\lambda+), \Psi(\lambda-)) \subset (a_{\lambda'}, \Psi(\lambda')\rangle = (a_{\lambda'}, \Psi(\lambda-))$ and $\nu[y, \Psi(\lambda-)) \geq \lambda'(\Psi(\lambda-) - y)$ for each $y \in [\Psi(\lambda+), \Psi(\lambda-))$, by (13). If we let λ' tend to λ, we get $\nu[y, \Psi(\lambda-)) \geq \lambda(\Psi(\lambda-)-y)$. Thus

$$M_+\nu(\Psi(\lambda-)) < \lambda \Rightarrow \begin{cases} \nu[\Psi(\lambda+), \Psi(\lambda-)) \geq \lambda(\Psi(\lambda-) - \Psi(\lambda+)), \\ \nu(\Psi(\lambda+), \Psi(\lambda-)) \geq \lambda(\Psi(\lambda-) - \Psi(\lambda+)). \end{cases} \tag{91}$$

If we now combine the relations (88)–(91), we get the desired equality (86). $\square$

6. Proof of the main lemma.

Since G_{λ_0} is periodic and differs from $\boldsymbol{R}$, the interval $(a, b\rangle$ is finite. Obviously, if we prove (50) for each x from a dense subset of (a, b), say $\{M_+\nu < +\infty\} \cap (a, b)$ (see (6)), then we get (50) for each $x \in (a, b)$. So we can assume that x is fixed and (60) holds. Let us construct the function Ψ considered in Section 5 for that x. Set

$$\langle x, b\rangle := \begin{cases} [x, b\rangle \text{ if } \lambda_x \in D, \\ (x, b\rangle \text{ if } \lambda_x \notin D. \end{cases} \tag{92}$$

We need to observe that $\langle x, b\rangle$ is the union of disjoint sets

$$\langle x, b\rangle = \cup_{\lambda \in D} \langle \Psi(\lambda+), \Psi(\lambda-)\rangle \cup I(E) \cup Q, \tag{93}$$

where

$$Q := \big\{ y \in I(C) \backslash \{b\} : M_+\nu(y) = \inf\{\lambda : \Psi(\lambda) = y\} =: \lambda_y,\ \Psi(\lambda_y-) = y \big\} \subset I(C). \tag{94}$$

Recall first of all that function Ψ satisfies the relations (66)–(69).

We prove the following two statements:

(i) If $y \in \{x, b\}$, then y cannot belong to the more than one part of the right-hand side of (93) and

$$y \in \langle x, b\rangle \Longleftrightarrow y \in \cup_{\lambda \in D} \langle \Psi(\lambda+), \Psi(\lambda-)\rangle \cup I(E) \cup Q.$$

(ii) Each $y \in (x, b)$ belongs to one and only one part of the right-hand side of (93).

By taking into account the inclusions in (66) and (94), the relations (i) and (ii) imply the validity of decomposition (93).

Since (65) and (69) hold and $x, b \notin Q$ (see (64) and (94)), the first part of claim (i) is clear.

If $x \in \langle x, b\rangle$, i.e. $\lambda_x \in D$ (see (92)), then $\Psi(\lambda_x+) = x \in \langle \Psi(\lambda_x+), \Psi(\lambda_x-)\rangle$ by virtue of (63), (60) and the definition (84). If $x \notin \langle x, b\rangle$, i.e. $\lambda_x \notin D$, then x have no possibility to belong to $\cup_{\lambda\in D}[\Psi(\lambda+), \Psi(\lambda-)]$, by (62).

If $b \in (x, b\rangle$, i.e. $(x, b\rangle = (x, b]$, then $\nu\{b\} > 0$ (see (9)) and since $M_+\nu(b) = +\infty$ and $\Psi(\lambda_0) = b$, we have $\beta = \sup\{\lambda : \Psi(\lambda) = b\} \in D$, by virtue of Lemma 10. Thus $b = \Psi(\beta-)$ and $b \in \langle \Psi(\beta+), \Psi(\beta-)\rangle$ either by the definition (84), if $\beta > \lambda_0$, or by the definition (85), if $\beta = \lambda_0$. Hence b belongs to the right-hand side of (93). If $b \notin (x, b\rangle$, i.e. $(x, b\rangle = (x, b)$, then $b \notin \langle \Psi(\lambda_0+), \Psi(\lambda_0-)\rangle$, by (85), in the case $\lambda_0 \in D$, and $b \notin \langle \Psi(\lambda+), \Psi(\lambda-)\rangle$ for each $\lambda \in D\backslash\{\lambda_0\}$, by (84) (since $M_+\nu(b) \le \lambda_0 < \lambda$). Thus $b \notin \cup_{\lambda\in D}\langle \Psi(\lambda+), \Psi(\lambda-)\rangle \cup I(E) \cup Q$.

The proof of the claim (i) is completed and let us now prove (ii). This is evident in the case $y \in \cup_{\lambda\in D}(\Psi(\lambda+), \Psi(\lambda-)) \cup I(E)$, by (67) and (68). Consider the remaining cases where $y \in (x, b)$ is either the endpoint of an interval $(\Psi(\lambda+), \Psi(\lambda-))$ for some $\lambda \in D$ or it belongs to $I(C)$ (see (66); recall that $x \notin I(E)$ in these cases, by (67) and (68)).

If y is the endpoint of an interval $(\Psi(\lambda_y+), \Psi(\lambda_y-))$ for some $\lambda_y \in D$ and $y \notin I(C)$, then, by virtue of (70), Lemma 6 or 7 and the definition (84), $y \in \langle \Psi(\lambda_y+), \Psi(\lambda_y-)\rangle$. At the same time $y \notin [\Psi(\lambda+), \Psi(\lambda-)]$ for each $\lambda \in D\backslash\{\lambda_y\}$, by (69).

If now $y \in I(C) \cap (x, b)$ and $\langle \alpha, \beta\rangle = \Psi^{-1}\{y\}$, then we have only the following three possibilities (see Lemma 8):

i) $M_+\nu(y) = \alpha$ and $\lim_{\lambda\to\alpha-} \Psi(\lambda) = y$. Then $y \in Q$, by the definition (94). Besides, we have $\alpha \notin D$, $y = \Psi(\beta-)$ and if $\beta \in D$, then $y \notin \langle \Psi(\beta+), \Psi(\beta-)\rangle$, by (84). Thus $y \notin \cup_{\lambda\in D}\langle \Psi(\lambda+), \Psi(\lambda-)\rangle$.

ii) $M_+\nu(y) = \alpha$ and $\lim_{\lambda\to\alpha-} \Psi(\lambda) > y$. Then $y \notin Q$, $\alpha \in D$, $\Psi(\alpha+) = y$ and $y \in \langle \Psi(\alpha+), \Psi(\alpha-)\rangle$, by (84). Besides, if $\beta \in D$, then $y = \Psi(\beta-) \notin \langle \Psi(\beta+), \Psi(\beta-)\rangle$, by (84). Thus $y \notin \cup_{\lambda\in D\backslash\{\alpha\}}\langle \Psi(\lambda+), \Psi(\lambda-)\rangle$.

iii) $M_+\nu(y) = \infty$. Then $\beta \in D$, by virtue of Lemma 10, and $y = \Psi(\beta-) \in \langle \Psi(\beta+), \Psi(\beta-)\rangle$, by (84). Besides, if $\alpha \in D$, we have $y = \Psi(\alpha+) \notin \langle \Psi(\alpha+), \Psi(\alpha-)\rangle$, by (84). Thus $y \notin \cup_{\lambda\in D\backslash\{\beta\}}\langle \Psi(\lambda+), \Psi(\lambda-)\rangle$.

We have considered all the possible situations to ensure that (93) holds.

Now if $\lambda_x \notin D$, then $\Psi(\lambda_x-) = x$, and $y = x$ satisfies the conditions of Lemma 9, because of (62) and (60). Hence $\nu\{x\} = 0$ in this case. Consequently, we have (see (92))

$$\nu\langle x, b\rangle = \nu[x, b\rangle. \tag{95}$$

Lemma 9 assures us as well that each point of Q has measure 0 and, since Q is a countable set, we have

$$\nu(Q) = 0. \tag{96}$$

Combining (95), (93), (96), (86) and (80), we get

$$\nu[x,b\rangle = \sum_{\lambda \in D} \lambda(\Psi(\lambda-) - \Psi(\lambda+)) + \int_{I(E)} \Psi^{-1} dm.$$

This is the desired expression of the left-hand side of (50) in terms of Ψ. Obviously, the same representation is correct for the right-hand side of (50) as well, so that the equality holds and Lemma 5 is proved.

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