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Author for correspondence:
L. Ephremidze
e-mail: le23@nyu.edu

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On explicit Wiener–Hopf factorization of 2×2 matrices in a vicinity of a given matrix

L. Ephremidze[1,2] and I. Spitkovsky[1]

[1]Division of Science and Mathematics, New York University Abu Dhabi (NYUAD), Saadiyat Island, 129188 Abu Dhabi, United Arab Emirates
[2]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 0177 Tbilisi, Georgia

LE, 0000-0002-1000-3841; IS, 0000-0002-1411-3036

As it is known, the existence of the Wiener–Hopf factorization for a given matrix is a well-studied problem. Severe difficulties arise, however, when one needs to compute the factors approximately and obtain the partial indices. This problem is very important in various engineering applications and, therefore, remains to be subject of intensive investigations. In the present paper, we approximate a given matrix function and then explicitly factorize the approximation regardless of whether it has stable partial indices. For this reason, a technique developed in the Janashia–Lagvilava matrix spectral factorization method is applied. Numerical simulations illustrate our ideas in simple situations that demonstrate the potential of the method.

1. Introduction

The Wiener–Hopf factorization of matrix functions

$$G(t) = G^{+}(t)\Lambda(t)G^{-}(t), \quad t \in \mathbb{T}, \tag{1.1}$$

where G^{+} and G^{-} along with their inverses are analytic, respectively, inside and outside of the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$, and the middle factor is a diagonal matrix of the form

$$\Lambda(t) = \mathrm{diag}[t^{\varkappa_1}, t^{\varkappa_2}, \ldots, t^{\varkappa_n}], \quad \varkappa_i \in \mathbb{Z}, \tag{1.2}$$

plays an important role in various branches of mathematics and applied sciences. There is a numerous literature devoted to the theory of this factorization and its applications (e.g. [1–3], and references therein).

In contrast with the scalar situation, however, there is a lack of explicit computational methods for constructing the factors in (1.1) even for general 2×2 matrices, especially in the cases where the partial indices $\varkappa_i$ are different from zero, i.e. (1.2) is not the identity matrix Id. Among several approaches to develop such a computational method we will single out the papers [4,5].

The Janashia–Lagvilava method is a relatively new algorithm for matrix spectral factorization [6,7] which proved to be rather effective [8]. The matrix function (1.1) is positive definite a.e. on $\mathbb{T}$ in this case, which implies that its partial indices are equal to zero. Furthermore, it can be arranged that $G^- = (G^+)^*$ and, therefore, the spectral factorization has the form $G = G^+(G^+)^*$. The essential components of the Janashia–Lagvilava method are the triangular factorization followed by the appropriate approximation and the construction of unitary matrix-functions of a special form. In the present paper, we attempt to apply these tools to the factorization of a certain type of matrix-functions which are not positive definite. (We do not specify the classes to which functions and their factors belong. It will be clear from the described procedures to what extent the process works.)

Let

$$S(t) = \begin{pmatrix} s_{11}(t) & s_{12}(t) \\ s_{21}(t) & s_{22}(t) \end{pmatrix}, \tag{1.3}$$

$t \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, be a matrix function with a factorizable diagonal entry (without loss of generality s_{11}) and factorizable determinant

$$s_{11}(t) = s_{11}^+(t) t^{\kappa_1} s_{11}^-(t) \quad \text{and} \quad \det S(t) = \Delta^+(t) t^{\kappa} \Delta^-(t). \tag{1.4}$$

One can readily observe that such S admits a triangular factorization

$$S(t) = \begin{pmatrix} s_{11}^+(t) & 0 \\ \dfrac{t^{-\kappa_1} s_{21}(t)}{s_{11}^-(t)} & \dfrac{\Delta^+(t)}{s_{11}^+(t)} \end{pmatrix} \begin{pmatrix} t^{\kappa_1} & 0 \\ 0 & t^{\kappa-\kappa_1} \end{pmatrix} \begin{pmatrix} s_{11}^-(t) & \dfrac{t^{-\kappa_1} s_{12}(t)}{s_{11}^+(t)} \\ 0 & \dfrac{\Delta^-(t)}{s_{11}^-(t)} \end{pmatrix}. \tag{1.5}$$

Theorem 1.1. *Let S be as in* (1.3)–(1.5). *Suppose that S_N, $N \geq 1$, is the following approximation of S:*

$$S_N(t) = \begin{pmatrix} s_{11}^+(t) & 0 \\ \varphi_N(t) & \dfrac{\Delta^+(t)}{s_{11}^+(t)} \end{pmatrix} \begin{pmatrix} t^{\kappa_1} & 0 \\ 0 & t^{\kappa-\kappa_1} \end{pmatrix} \begin{pmatrix} s_{11}^-(t) & \psi_N(t) \\ 0 & \dfrac{\Delta^-(t)}{s_{11}^-(t)} \end{pmatrix}, \tag{1.6}$$

where

$$\varphi_N(t) = \sum\nolimits_{k=-N}^{\infty} c_k\{\varphi\} t^k \quad \textit{and} \quad \psi_N(t) = \sum\nolimits_{k=-\infty}^{N} c_k\{\psi\} t^k, \tag{1.7}$$

$\varphi(t) := t^{-\kappa_1} s_{21}(t)/s_{11}^-(t)$, $\psi(t) := t^{-\kappa_1} s_{12}(t)/s_{11}^+(t)$ and $c_k\{f\}$ *stands for the kth Fourier coefficient of a function f. Then one can explicitly construct the Wiener–Hopf factorization of S_N*

$$S_N(t) = S_N^+(t) \Lambda_N(t) S_N^-(t). \tag{1.8}$$

In the future, we hope to extend this theorem to $n \times n$ matrices, since the Janashia–Lagvilava method works for matrices of arbitrary size. The main question for future work, however, arises naturally: what are the relations between the partial indices of S_N and S? So far, we can only conjecture that

$$\Lambda_N = \Lambda \quad \text{for } N \text{ sufficiently large.} \tag{1.9}$$

To be more precise, we will formulate the following statement as:

Conjecture 1.2. *Under the hypothesis of theorem* 1.1, *there exist such choices of the factors S_N^+ and S_N^- in* (1.8) (*as it is well-known unlike the spectral factorization, there is no uniqueness for these factors and there is a certain freedom in their selection*) *that they converge as $N \to \infty$.*

The proof of this conjecture would guarantee that (1.9) holds as well. Preliminary numerical simulations support this conjecture.

2. Notation and preliminary observations

The Lebesgue space of p-integrable complex valued functions defined on $\mathbb{T}$ is denoted by $L_p(\mathbb{T})$ and

$$H_p = H_p(\mathbb{D}) := \left\{ f \in \mathcal{A}(\mathbb{D}) : \sup_{r<1} \int_0^{2\pi} |f(r\,\mathrm{e}^{\mathrm{i}\theta})|^p \,\mathrm{d}\theta < \infty \right\}$$

is the Hardy space of analytic functions in the unit disc $\mathbb{D}$, $0 < p < \infty$ (H_∞ is the space of bounded analytic functions). We assume that functions from H_p are naturally identified with their boundary values and the latter class of functions is denoted by L_p^+. A function $f \in H_p$ is called *outer*, denoted $f \in H_p^O$, if

$$f(z) = c \cdot \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} \log \left| f(\mathrm{e}^{\mathrm{i}\theta}) \right| \mathrm{d}\theta \right), \quad |c| = 1.$$

Let $\mathcal{P}^+$ be the set of polynomials. The set of trigonometric polynomials is denoted by $\mathcal{P}$, i.e. $f \in \mathcal{P}$ if f has only a finite number of non-zero Fourier coefficients. In particular, for integers $m \le n$, let $\mathcal{P}_{[m:n]} := \{f \in \mathcal{P} : c_k\{f\} = 0$ whenever $k < m$ or $k > n\}$ and, for a non-negative integer N, let $\mathcal{P}_N^+ := \mathcal{P}_{[0:N]}$, $\mathcal{P}_N^- := \mathcal{P}_{[-N:0]}$. Obviously, $f \in \mathcal{P}_N^+ \Leftrightarrow \overline{f} \in \mathcal{P}_N^-$, where $\overline{f}$ is the complex conjugate of f. In general, if $\mathcal{S}$ is any set of functions, then $\overline{\mathcal{S}} = \{f : \overline{f} \in \mathcal{S}\}$.

Let $\mathbb{P}^+$ and $\mathbb{P}_0^-$ be the projection operators defined on $L_2(\mathbb{T})$: for $f(t) = \sum_{k=-\infty}^{\infty} c_k t^k$,

$$\mathbb{P}^+[f] = \sum_{k=0}^{\infty} c_k t^k, \quad \text{and} \quad \mathbb{P}_0^-[f] = \sum_{k=1}^{\infty} c_{-k} t^{-k}, \quad \text{so that } \mathbb{P}^+[f] + \mathbb{P}_0^-[f] = f.$$

For a trigonometric polynomial $p(z) = \sum_{k=m}^{n} c_k z^k$, let

$$\widetilde{p}(z) = \sum_{k=m}^{n} \overline{c}_k z^{-k}. \tag{2.1}$$

Note that $\widetilde{p}(t) = \overline{p(t)}$ whenever $t \in \mathbb{T}$.

For a rational function f with Laurent series expansion $f(z) = \sum_{k=-n}^{\infty} c_k z^k$, $n > 0$, in some neighbourhood of the point $z = 0$, let

$$\left.\begin{aligned} \mathbb{V}[f](z) &= \sum_{k=-n}^{-1} c_k z^k, \quad \mathbb{V}_0[f](z) = \sum_{k=-n}^{0} c_k z^k \\ \text{and} \qquad \mathbb{V}_0^+[f](z) &= \sum_{k=1}^{\infty} c_k z^k = f(z) - \mathbb{V}_0[f](z). \end{aligned}\right\} \tag{2.2}$$

If M is a matrix, then $\overline{M}$ denotes the matrix with complex conjugate entries and $M^* := \overline{M}^T$. Furthermore, $\mathbb{C}^{d\times d}$, $L_p(\mathbb{T})^{d\times d}$, etc. denote the set of $d \times d$ matrices with the entries from $\mathbb{C}$, $L_p(\mathbb{T})$, etc. For a polynomial matrix $P = (\sum_{k=m}^{n} p_k^{\{ij\}} t^k)_{i,j=1}^d \in (\mathcal{P}_{[m:n]})^{d\times d}$, let $\|P\| = \sup |p_k^{\{ij\}}|$ and $\|P\|_\infty = \sup_{i,j} \| \sum_{k=m}^{n} p_k^{\{ij\}} t^k \|_\infty$.

A matrix function $F \in H_p(\mathbb{D})^{d\times d}$ is called *outer*, denoted $F \in H_p^O(\mathbb{D})^{d\times d}$, if its determinant belongs to $H_{p/d}^O$. A matrix function $U \in L_\infty(\mathbb{T})^{d\times d}$ is called *unitary* if

$$U(t)U^*(t) = I_d \quad \text{a.e.}, \tag{2.3}$$

where I_d stands for the $d \times d$ identity matrix.

For a matrix function G, which can be factorized in the form (1.1), (1.2), we write for partial indices

$$\mathcal{PI}(G) = (\varkappa_1, \varkappa_2, \ldots, \varkappa_n).$$

We will use the following theorem proved in [6] (see also Theorem 1 in [7] for $n \times n$ matrices).

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Theorem 2.1. *For a matrix function*

$$F(t)=\begin{pmatrix} 1 & 0 \\ \varphi(t) & f^+(t) \end{pmatrix} \in L_2(\mathbb{T})^{2\times 2}, \tag{2.4}$$

where $f^+ \in H_2^O$ and $\varphi \in \mathcal{P}_N^-$, there exists a unique (up to a constant right factor) unitary matrix function of the form

$$U(t)=\begin{pmatrix} \alpha^+(t) & \beta^+(t) \\ -\overline{\beta^+(t)} & \overline{\alpha^+(t)} \end{pmatrix}, \tag{2.5}$$

where α^+, $\beta^+ \in \mathcal{P}_N^+$ and $\det U(t)=1$, such that

$$F(t)U(t) \in L_2^+(\mathbb{T})^{2\times 2}.$$

Remark 2.2. Note that the determinant of $\Phi^+ := FU$ is an outer function, therefore

$$\Phi^+ \in H_2^O(\mathbb{D})^{2\times 2}. \tag{2.6}$$

Remark 2.3. An effective algorithm is provided in [6] (see also [7]) for construction of (2.5) when (2.4) is given in terms of $c_{-N}\{\varphi\}, \dots, c_{-1}\{\varphi\}$, and $c_0\{f^+\}, \dots, c_N\{f^+\}$. Furthermore, one can observe that if these coefficients are from any subfield $\mathbb{F} \subset \mathbb{C}$ which is closed with respect to the complex conjugation, then the coefficients of α^+ and β^+ in (2.5) belong to $\mathbb{F}$ as well.

Remark 2.4. For simplicity, the uniqueness condition

$$U(1)=I_2 \tag{2.7}$$

is applied when U is being constructed.

Remark 2.5. Application of theorem 2.1 to the matrix function

$$F(t)=F^{**}(t)=\begin{pmatrix} 1 & \overline{\varphi(t)} \\ 0 & \overline{f^+(t)} \end{pmatrix}^*,$$

yields the construction of a unitary matrix function

$$V(t)=\begin{pmatrix} \alpha^-(t) & -\overline{\beta^-(t)} \\ \beta^-(t) & \overline{\alpha^-(t)} \end{pmatrix}$$

such that α^-, $\beta^- \in \mathcal{P}_N^-$, $\det V(t)=1$, and $V(t)F^*(t) \in \overline{H_2^O}(\mathbb{D})^{2\times 2}$.

3. Some auxiliary statements

We will use the following theorem which is implicitly proved in [9–11].

Theorem 3.1. *Let*

$$U(t)=\begin{pmatrix} \alpha^+(t) & -\overline{\beta^+(t)} \\ \beta^+(t) & \overline{\alpha^+(t)} \end{pmatrix}, \quad \alpha^+, \beta^+ \in \mathcal{P}_N^+, \tag{3.1}$$

be a unitary matrix function with determinant 1 such that

$$|\alpha^+(0)|+|\beta^+(0)|>0. \tag{3.2}$$

Then the partial indices of U are equal to 0 and there exists a (unique) Wiener–Hopf factorization of U which has the following explicit form:

$$U(t)=\begin{pmatrix} \alpha^+(t) & \phi^-(t)\alpha^+(t)-\overline{\beta^+(t)} \\ \beta^+(t) & \phi^-(t)\beta^+(t)+\overline{\alpha^+(t)} \end{pmatrix}\begin{pmatrix} 1 & -\phi^-(t) \\ 0 & 1 \end{pmatrix}, \tag{3.3}$$

where $\phi^- \in \mathcal{P}_{[-N:-1]}$. Furthermore (see (2.1) and (2.2)),

$$\phi^-(z)=-\mathbb{V}[\widetilde{\alpha^+}(z)/\beta^+(z)] \quad \text{or} \quad \phi^-(z)=\mathbb{V}[\widetilde{\beta^+}(z)/\alpha^+(z)] \tag{3.4}$$

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if, respectively, $\beta^+(0)\neq 0$ *or* $\alpha^+(0)\neq 0$.

Remark 3.2. Because of (3.2), at least one of the equations in (3.4) is applicable for computing ϕ^-. In particular, if $\beta^+(0)\neq 0$ and $\widetilde{\alpha^+}(z)/\beta^+(z)=\sum_{k=-N}^{\infty}c_kz^k$ in a (punctured) neighbourhood of 0, then

$$\phi^-(z)=-\frac{\widetilde{\alpha^+}(z)}{\beta^+(z)}-\sum_{k=0}^{\infty}c_kz^k=:-\frac{\widetilde{\alpha^+}(z)}{\beta^+(z)}-A^+(z). \tag{3.5}$$

Formulae (3.4), however, might not be reliable in practical computations if both $\alpha^+(0)$ and $\beta^+(0)$ are very close to 0.

Remark 3.3. Although $\|U\|_\infty\leq 1$, it may happen that L_∞ norms of the factors in (3.3) are large. Note, however, that for any $\phi^+\in L_\infty^+(\mathbb{T})$, we can factorize U as

$$U(t)=\begin{pmatrix}\alpha^+(t) & \phi(t)\alpha^+(t)-\overline{\beta^+(t)}\\ \beta^+(t) & \phi(t)\beta^+(t)+\overline{\alpha^+(t)}\end{pmatrix}\begin{pmatrix}1 & -\phi(t)\\ 0 & 1\end{pmatrix}, \tag{3.6}$$

where $\phi=\phi^-+\phi^+$. When we incorporate theorem 3.1 in our algorithm, we need only the first factor in (3.3) to belong to $H_\infty^O(\mathbb{D})^{d\times d}$ and we can relax the condition $\phi^-\in\mathcal{P}_{[-N:-1]}$ to $\mathbb{P}_0^-[\phi]\in\mathcal{P}_{[-N:-1]}$. This might be useful in reducing the norm of factors of U in (3.6).

Proof. Equation (3.3) can be checked by direct multiplication. All determinants of matrix functions in (3.3) are equal to 1 and the right factor is anti-analytic. Hence the proof will be completed as soon as we show that

$$\phi^-(z)\alpha^+(z)-\widetilde{\beta^+}(z)\in\mathcal{P}_N^+\quad\text{and}\quad\phi^-(z)\beta^+(z)+\widetilde{\alpha^+}(z)\in\mathcal{P}_N^+. \tag{3.7}$$

To this end, observe first of all that both functions in (3.7) automatically belong to $\mathcal{P}_{[-N:N]}$. Because of (3.5) (assuming that $\beta^+(0)\neq 0$; the proof is similar if $\alpha^+(0)\neq 0$) we have that

$$\begin{aligned}\phi^-(z)\alpha^+(z)-\widetilde{\beta^+}(z)&=-\frac{\widetilde{\alpha^+}(z)}{\beta^+(z)}\alpha^+(z)-\widetilde{\beta^+}(z)-A^+(z)\alpha^+(z)\\&=-\frac{\widetilde{\alpha^+}(z)\alpha^+(z)+\beta^+(z)\widetilde{\beta^+}(z)}{\beta^+(z)}-A^+(z)\alpha^+(z)\\&=-\frac{1}{\beta^+(z)}-A^+(z)\alpha^+(z)\end{aligned}$$

is analytic in the neighbourhood of 0. Consequently, the first inclusion in (3.7) holds.

For the second one, we have that

$$\phi^-(z)\beta^+(z)+\widetilde{\alpha^+}(z)=-\frac{\widetilde{\alpha^+}(z)}{\beta^+(z)}\beta^+(z)+\widetilde{\alpha^+}(z)-A^+(z)\beta^+(z)=-A^+(z)\beta^+(z)$$

is also analytic in the neighbourhood of 0 and hence (3.7) holds.

As for the uniqueness, let

$$U=\Phi_1^+\begin{pmatrix}1 & \phi_1^-\\ 0 & 1\end{pmatrix}=\Phi_2^+\begin{pmatrix}1 & \phi_2^-\\ 0 & 1\end{pmatrix}$$

be two factorizations of U, where $\Phi_1^+,\Phi_2^+\in(\mathcal{P}^+)^{2\times 2}$ and $\phi_1^-,\phi_2^-\in\mathcal{P}_{[-N:-1]}$. Then

$$\begin{pmatrix}1 & \phi_1^--\phi_2^-\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & \phi_1^-\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \phi_2^-\\ 0 & 1\end{pmatrix}^{-1}=(\Phi_1^+)^{-1}\Phi_2^+.$$

Consequently, $\phi_1^--\phi_2^-\in\mathcal{P}_{[-N:-1]}\cap\mathcal{P}^+$ (since $\det\Phi_1^+=1$) and hence $\phi_1^--\phi_2^-=0$. ■

Remark 3.4. The simple form of explicit formulae for factorization of unitary matrix function (3.1) provided above enables us to arrive at the following important conclusion: if $\mathbb{F}\subset\mathbb{C}$ is any subfield of complex numbers which is closed with respect to the complex conjugation, say $\mathbb{F}=\mathbb{Q}$, and the coefficients of the entries of U belong to $\mathbb{F}$, then the coefficients of the factors belong to

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the same field $\mathbb{F}$. (Obviously, this phenomenon does not occur for general polynomial functions even in the scalar case.) This property is used in the last section where some examples of the numerical factorization are provided. On the other hand, if the matrices of type (3.1) arise during the approximation processes as it is described in the paper, their coefficients can always be chosen to be rational numbers.

The following lemma is proved in [12] (see also [13] and [2, p. 12]). We formulate it along with its proof for the convenience of reference and because it is rather short and straightforward.

Lemma 3.5. *Let $n, m \in \mathbb{Z}$ and $h(t) = \sum_{k=-\infty}^{\infty} c_k t^k$, where $\sum_{k=-\infty}^{\infty} |c_k| < \infty$, and let*

$$T(t) = \begin{pmatrix} t^n & h(t) \\ 0 & t^m \end{pmatrix}. \tag{3.8}$$

Then we have $\mathcal{PI}(T) = (n, m)$ if $m \le n+1$ and

$$\mathcal{PI}(T) = \mathcal{PI}\begin{pmatrix} t^n & h_{n,m}(t) \\ 0 & t^m \end{pmatrix} \quad \text{if } m > n+1,$$

where $h_{n,m}(t) = \sum_{k=n+1}^{m-1} c_k t^k$.

Proof. We have

$$\begin{pmatrix} 1 & h^+(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^n & H(t) \\ 0 & t^m \end{pmatrix} \begin{pmatrix} 1 & h^-(t) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t^n & h^-(t)t^n + H(t) + h^+(t)t^m \\ 0 & t^m \end{pmatrix}.$$

Therefore, if

$$h^+(t) := \sum_{k=0}^{\infty} c_{k+m} t^k, \quad h^-(t) := \sum_{k=-\infty}^{\min(0, m-n-1)} c_{k+n} t^k, \quad \text{and} \quad H(t) = \begin{cases} 0, & \text{if } m \le n+1 \\ h_{n,m}(t), & \text{if } m > n+1 \end{cases},$$

then we have

$$\begin{aligned} H(t) + h^-(t)t^n + h^+(t)t^m &= H(t) + \sum_{k=-\infty}^{\min(0, m-n-1)} c_{k+n} t^{k+n} + \sum_{k=0}^{\infty} c_{k+m} t^{k+m} \\ &= H(t) + \sum_{k=-\infty}^{\min(n, m-1)} c_k t^k + \sum_{k=m}^{\infty} c_k t^k \\ &= \sum_{k=-\infty}^{\infty} c_k t^k = h(t). \end{aligned}$$

Thus the lemma holds. ■

The following simple algorithm presented in [12,13] can be used for explicit factorization of triangular matrices (3.8) in a non-trivial case $m > n+1$.

Theorem 3.6 ([13, §1]). *Let*

$$G(t) = \begin{pmatrix} 1 & \phi(t) \\ 0 & t^m \end{pmatrix}, \tag{3.9}$$

where $m \in \mathbb{N}$ and $\phi \in \mathcal{P}^+_{m-1}$. Then the Wiener–Hopf factorization (1.1) of (3.9) can be written in the form

$$G^+ = (-1)^{\ell-1} \begin{pmatrix} t^{-\nu_\ell}(B_{\ell-1} - \phi A_{\ell-1}) & t^{\nu_\ell - m}(-B_\ell + \phi A_\ell) \\ -t^{m-\nu_\ell} A_{\ell-1} & t^{\nu_\ell} A_\ell \end{pmatrix},$$

$$G^- = \begin{pmatrix} A_\ell & B_\ell \\ A_{\ell-1} & B_{\ell-1} \end{pmatrix}, \quad \Lambda(t) = \mathrm{diag}[t^{\nu_\ell}, t^{m-\nu_\ell}],$$

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where A_k and B_k are defined by the following recursive formulae:

$$A_{-1}=1,\quad A_0=\psi_0,\quad A_i=\psi_i A_{i-1}+A_{i-2}$$
$$B_{-1}=0,\quad B_0=1,\quad B_i=\psi_i B_{i-1}+B_{i-2},\quad i=1,2,\ldots$$
$$\phi_0=\phi,\quad \phi_{k+1}=\mathbb{V}_0^+[1/\phi_k],\quad \psi_k=\mathbb{V}_0[1/\phi_k],$$

and ℓ and ν_ℓ are defined by the following relations: let q_k denote the order of the pole of ψ_k at the point $t=0$ ($q_k=\infty$ if $\phi_k\equiv 0$). Then ℓ is the smallest integer for which $2(q_0+q_1+\cdots+q_\ell)+q_{\ell+1}\geq m$ and

$$\nu_\ell=q_0+q_1+\cdots+q_\ell\,.$$

Remark 3.7. Although theoretically sound and simple, the algorithm based on theorem 3.6 is sensitive to round-off errors. This flaw is in the nature of the problem as partial indices are equal to the ranks of certain Toeplitz matrices in this situation [14]. The algorithm works perfectly, however, if the Fourier coefficients of ϕ are rational and computations are performed in symbolic arithmetic. The simplicity of formulae in theorem 3.6 allows us to factorize matrices with a large degree of m by symbolic tools. This approach, however, has its own limitations when the growth of nominators and denominators in fractions become uncontrolled.

4. Computational procedures

The Wiener–Hopf factorization of S_N will be performed in several steps. We assume that $f^+(t):=\Delta^+(t)/s_{11}^+(t)\in H_2^O$, $f^-(t):=\Delta^-(t)/s_{11}^-(t)\in\overline{H_2^O}$, and $\varphi,\psi\in L_2(\mathbb{T})$, so that the convergence in (1.7) holds in L_2-norm.

Step 1: Let $\varphi_N^\pm:=\mathbb{P}^\pm[\varphi_N]$ and $\psi_N^\pm:=\mathbb{P}^\pm[\psi_N]$. Note the φ_N^+ and ψ_N^- are independent of N because of definition (1.7). Therefore, we assume $\varphi^+=\varphi_N^+$ and $\psi^-=\psi_N^-$. Then

$$S_N=\begin{pmatrix} s_{11}^+ & 0\\ \varphi^+ & 1\end{pmatrix}\begin{pmatrix} 1 & 0\\ \varphi_N^- & f^+\end{pmatrix}\begin{pmatrix} t^{\kappa_1} & 0\\ 0 & t^{\kappa-\kappa_1}\end{pmatrix}\begin{pmatrix} 1 & \psi_N^+\\ 0 & f^-\end{pmatrix}\begin{pmatrix} s_{11}^- & \psi^-\\ 0 & 1\end{pmatrix},\tag{4.1}$$

Step 2: After applying theorem 2.1 (and remark 2.5), we get from (4.1)

$$S_N(t)=\begin{pmatrix} s_{11}^+ & 0\\ \varphi^+ & 1\end{pmatrix}\Phi_N^+(t)U_N^*(t)\begin{pmatrix} t^{\kappa_1} & 0\\ 0 & t^{\kappa-\kappa_1}\end{pmatrix}V_N^*(t)\Psi_N^-(t)\begin{pmatrix} s_{11}^- & \psi^-\\ 0 & 1\end{pmatrix},\tag{4.2}$$

where

$$U_N(t)=\begin{pmatrix} \alpha_N^+(t) & \beta_N^+(t)\\ -\overline{\beta_N^+(t)} & \overline{\alpha_N^+(t)}\end{pmatrix}\quad\text{and}\quad V_N(t)=\begin{pmatrix} \alpha_N^-(t) & -\overline{\beta_N^-(t)}\\ \beta_N^-(t) & \overline{\alpha_N^-(t)}\end{pmatrix}\tag{4.3}$$

are unitary matrix functions with determinant 1 and $\alpha_N^\pm,\beta_N^\pm\in\mathcal{P}_N^\pm$, while (see (2.6))

$$\Phi_N^+\in H_2^O(\mathbb{D})^{2\times 2}\quad\text{and}\quad \Psi_N^-\in\overline{H_2^O}(\mathbb{D})^{2\times 2}.$$

Hence the factorization problem for S_N is reduced to the Wiener–Hopf factorization of

$$W_N(t):=U_N^*(t)\Lambda_0(t)V_N^*(t),\tag{4.4}$$

where U_N and V_N are defined by (4.3) and $\Lambda_0(t)=\operatorname{diag}(t^{\kappa_1},t^{\kappa-\kappa_1})$.

Step 3: According to (4.4),

$$W_N(t)=\begin{pmatrix} \overline{\alpha_N^+(t)} & -\beta_N^+(t)\\ \overline{\beta_N^+(t)} & \alpha_N^+(t)\end{pmatrix}\begin{pmatrix} t^{\kappa_1} & 0\\ 0 & t^{\kappa-\kappa_1}\end{pmatrix}\begin{pmatrix} \overline{\alpha_N^-(t)} & \overline{\beta_N^-(t)}\\ -\beta_N^-(t) & \alpha_N^-(t)\end{pmatrix}.\tag{4.5}$$

First, we group non-diagonal unitary matrices together

$$\begin{aligned} W_N(t) &= \begin{pmatrix} \overline{\alpha_N^+(t)} & -\beta_N^+(t) \\ \overline{\beta_N^+(t)} & \alpha_N^+(t) \end{pmatrix} \begin{pmatrix} \overline{\alpha_N^-(t)} & \overline{\beta_N^-(t)} t^{2\kappa_1-\kappa} \\ -\beta_N^-(t) t^{\kappa-2\kappa_1} & \alpha_N^-(t) \end{pmatrix} \begin{pmatrix} t^{\kappa_1} & 0 \\ 0 & t^{\kappa-\kappa_1} \end{pmatrix} \\ &=: \begin{pmatrix} a_N(t) & -\overline{b_N(t)} \\ b_N(t) & \overline{a_N(t)} \end{pmatrix} \begin{pmatrix} t^{\kappa_1} & 0 \\ 0 & t^{\kappa-\kappa_1} \end{pmatrix}, \end{aligned}$$

where

$$a_N(t) = \overline{\alpha_N^+(t)}\,\overline{\alpha_N^-(t)} + \beta_N^+(t)\beta_N^-(t) t^{\kappa-2\kappa_1} \in \mathcal{P}_{[-N+\nu_-:N+\nu_+]}$$

and

$$b_N(t) = \overline{\beta_N^+(t)}\,\overline{\alpha_N^-(t)} - \alpha_N^+(t)\beta_N^-(t) t^{\kappa-2\kappa_1} \in \mathcal{P}_{[-N+\nu_-:N+\nu_+]}.$$

Here, $\nu_- := \min(0, \kappa - 2\kappa_1)$, $\nu_+ := \max(0, \kappa - 2\kappa_1)$.

Let $\mathcal{N}$ be the least positive integer for which both functions $t^{\mathcal{N}} a_N(t)$ and $t^{\mathcal{N}} b_N(t)$ are polynomials (most probably $\mathcal{N} = N + \max\{0, 2\kappa_1 - \kappa\}$). Then we have

$$W_N(t) = \begin{pmatrix} t^{\mathcal{N}} a_N(t) & -t^{-\mathcal{N}}\overline{b_N(t)} \\ t^{\mathcal{N}} b_N(t) & t^{-\mathcal{N}}\overline{a_N(t)} \end{pmatrix} \begin{pmatrix} t^{\kappa_1-\mathcal{N}} & 0 \\ 0 & t^{\kappa-\kappa_1+\mathcal{N}} \end{pmatrix}. \tag{4.6}$$

We emphasize that $t^{\mathcal{N}} a_N(t), t^{\mathcal{N}} b_N(t)$ are polynomials and at least one of these functions differs from zero at the origin.

Step 4: According to theorem 3.1, matrix function (4.6) can be factorized as

$$\begin{aligned} W_N(t) &= \begin{pmatrix} t^{\mathcal{N}} a_N(t) & t^{\mathcal{N}} \phi_N^-(t) a_N(t) - t^{-\mathcal{N}}\overline{b_N(t)} \\ t^{\mathcal{N}} b_N(t) & t^{\mathcal{N}} \phi_N^-(t) b_N(t) + t^{-\mathcal{N}}\overline{a_N(t)} \end{pmatrix} \begin{pmatrix} 1 & -\phi_N^-(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{\kappa_1-\mathcal{N}} & 0 \\ 0 & t^{\kappa-\kappa_1+\mathcal{N}} \end{pmatrix} \\ &=: \Phi_0^+(t) \begin{pmatrix} t^{\kappa_1-\mathcal{N}} & -\phi_N^-(t) t^{\kappa-\kappa_1+\mathcal{N}} \\ 0 & t^{\kappa-\kappa_1+\mathcal{N}} \end{pmatrix}. \end{aligned}$$

Thus, due to (4.2), (4.4) and (4.6), we have

$$S_N(t) = \begin{pmatrix} s_{11}^+ & 0 \\ \varphi^+ & 1 \end{pmatrix} \Phi_N^+(t) \Phi_0^+(t) \begin{pmatrix} t^{\kappa_1-\mathcal{N}} & -\phi_N^-(t) t^{\kappa-\kappa_1+\mathcal{N}} \\ 0 & t^{\kappa-\kappa_1+\mathcal{N}} \end{pmatrix} \Psi_N^-(t) \begin{pmatrix} s_{11}^- & \psi^- \\ 0 & 1 \end{pmatrix}. \tag{4.7}$$

Step 5: Using lemma 3.5 and theorem 3.6, we can explicitly factorize the middle triangular matrix in (4.7). Indeed

$$\begin{aligned} \begin{pmatrix} t^{\kappa_1-\mathcal{N}} & -\phi_N^-(t) t^{\kappa-\kappa_1+\mathcal{N}} \\ 0 & t^{\kappa-\kappa_1+\mathcal{N}} \end{pmatrix} &= t^{\kappa_1-\mathcal{N}} \begin{pmatrix} 1 & -\phi_N^-(t) t^{\kappa-2(\kappa_1-\mathcal{N})} \\ 0 & t^{\kappa-2(\kappa_1-\mathcal{N})} \end{pmatrix} \\ &= t^{\kappa_1-\mathcal{N}} \times T^+(t)\, \mathrm{diag}[t^{\nu_\ell}, t^{\kappa-2(\kappa_1-\mathcal{N})-\nu_\ell}] T^-(t) \\ &= T^+(t)\, \mathrm{diag}[t^{\nu_\ell+\kappa_1-\mathcal{N}}, t^{\kappa-(\kappa_1-\mathcal{N})-\nu_\ell}] T^-(t). \end{aligned}$$

Therefore, the factorization of (4.7) is completed.

To summarize, the factorization of S_N is reduced to factorization of W_N and this can be done in $O(N^2)$ operations. However, as it is observed by numerical simulations presented in the next section, the Fourier coefficients of intermediate functions ϕ_N^- may increase rapidly together with N. This might influence the accuracy of the final result. Remark 6 describes some suggestions on how to deal with this problem in the future.

5. Numerical simulations

The proposed algorithm works most efficiently in situations where off-diagonal entries $t^{-\kappa_1} s_{21}/s_{11}^-$ and $t^{-\kappa_1} s_{12}/s_{11}^+$ in factorization (1.5) have a small number of, respectively, negative and positive indexed non-zero Fourier coefficients, i.e. in the situations where $c_{-k}\{s_{21}/s_{11}^+\} = 0$ and $c_k\{s_{12}/s_{11}^-\} =$

0 whenever $k > N$ for a reasonably small positive integer N. As an illustrative example, we demonstrate the factorization of Laurent polynomial matrix $S(t) =$

$$\begin{bmatrix} -18t^{-3}+39t^{-2}-57t^{-1}+84-60t+42t^2-24t^3 & 12t^{-1}-45t+75t^2-117t^3+99t^4-36t^5 \\ -9t^{-5}+63t^{-4}-62t^{-3}+65t^{-2}-56t^{-1}+18-6t & -21t^{-3}-11t^{-2}-39t^{-1}+153-117t+15t^2+3t^3 \end{bmatrix}$$

which belongs to the above class since its lower-upper factorization of form (1.5) is $S(t) =$

$$\begin{bmatrix} 6-9t+9t^2-12t^3 & 0 \\ 3t^{-2}-19t^{-1}+6-3t & -9+3t+3t^2+12t^3 \end{bmatrix} \times \begin{bmatrix} -3t^{-3}+2t^{-2}-2t^{-1}+2 & 2t^{-1}+3-6t+3t^2 \\ 0 & 3t^{-3}-t^{-2}-2t^{-1}+1 \end{bmatrix}.$$

We can factorize the matrix even though it has non-stable partial indices. After performing Steps 1 and 2, we get (see remark 2.3) $S(t) =$

$$\begin{bmatrix} \frac{108}{53}+\frac{81}{53}t-\frac{471}{106}t^2+\frac{198}{53}t^3-\frac{1071}{106}t^4+\frac{66}{53}t^5 & -\frac{99}{53}+\frac{567}{106}t-\frac{387}{53}t^2+\frac{909}{106}t^3-\frac{324}{53}t^4+\frac{72}{53}t^5 \\ -\frac{876}{53}+\frac{695}{106}t-\frac{375}{53}t^2+\frac{429}{106}t^3 & -\frac{1203}{106}+\frac{408}{53}t+\frac{873}{106}t^2+\frac{234}{53}t^3 \end{bmatrix}$$
$$\times \begin{bmatrix} -\frac{11}{106}t^{-2}+\frac{54}{265}t^{-1}+\frac{297}{530}t+\frac{18}{53}t^2 & \frac{33}{106}t^{-2}-\frac{162}{265}t^{-1}+\frac{99}{530}t+\frac{6}{53}t^2 \\ -\frac{6}{53}t^{-2}-\frac{99}{530}t^{-1}+\frac{162}{265}t-\frac{33}{106}t^2 & \frac{18}{53}t^{-2}+\frac{297}{530}t^{-1}+\frac{54}{265}t-\frac{11}{106}t^2 \end{bmatrix}$$
$$\times \begin{bmatrix} -2.7t^{-5}+1.8t^{-4}-2.1t^{-3}+2t^{-2}-0.2t^{-1}+0.2 & 2.7t^{-3}+2.4t^{-2}-6.7t^{-1}+3.6 \\ -0.9t^{-5}+0.6t^{-4}+0.3t^{-3}+0.6t^{-1}-0.6 & 0.9t^{-3}+0.8t^{-2}+0.1t^{-1}-0.8 \end{bmatrix}$$

and we have to factorize the middle matrix which we denote by W as it is done in (4.5). To avoid the round-off errors, which might influence the accurate computation of partial indices by the algorithm depending on theorem 3.6, we use the symbolic computations and obtain

$$W(t) = \begin{bmatrix} \frac{33}{53}+\frac{20}{53}t & -\frac{4}{5}-\frac{891}{530}t^2-\frac{54}{53}t^3 \\ \frac{36}{53}-\frac{55}{159}t & \frac{11}{15}-\frac{486}{265}t^2+\frac{99}{106}t^3 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \begin{bmatrix} -\frac{1}{6}t^{-3} & \frac{1}{2}t^{-3}+3 \\ -\frac{1}{3} & 1 \end{bmatrix}$$

(see remarks 3.4 and 3.7). The final result of the factorization is

$$S(t) = \begin{bmatrix} 6t-9t^2+9t^3-12t^4 & -3+2.7t-1.8t^2-12.9t^3+27.9t^4-24.3t^5+32.4t^6 \\ -18+7t+t^2 & 4.9+0.4t+60.3t^2-18.9t^3-2.7t^4 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}$$
$$\times \begin{bmatrix} 0.5t^{-6}-\frac{91}{30}t^{-5}+\frac{32}{15}t^{-4}+\frac{17}{30}t^{-3}+1.8t^{-1}-1.8 & \frac{7}{6}t^{-4}+1.7t^{-3}+2.4t^{-2}+0.3t^{-1}-2.4 \\ t^{-3}-\frac{2}{3}t^{-2}+\frac{2}{3}t^{-1}-\frac{2}{3} & \frac{7}{3}t^{-1}-2 \end{bmatrix}.$$

The application of symbolic computations reveals that one can factorize (and obtain the result in the symbolic form) any matrix function which has the following form:

$$S(t) = \begin{pmatrix} f_{11}^{+} & 0 \\ Qf_{21}^{+} & f_{22}^{+} \end{pmatrix} \begin{pmatrix} f_{11}^{-} & Pf_{12}^{-} \\ 0 & f_{22}^{-} \end{pmatrix}, \tag{5.1}$$

where $f_{11}^{+}, f_{22}^{+}, \overline{f_{11}^{-}}, \overline{f_{22}^{-}} \in H_2^{O}$, $f_{21}^{+}, \overline{f_{12}^{-}} \in H_2$, $P(t) = p_0 + p_1 t + p_2 t^2$ and $Q(t) = q_0 + q_1 t^{-1} + q_2 t^{-2}$ (assuming that the Fourier coefficients of these functions are real and the coefficients p_k and q_k are parameters; see the electronic supplementary material). Furthermore, depending on this factorization, one can formulate the sufficient conditions for (5.1) to have partial indices $(2, -2)$

$$c_{-1}\{Qf_{21}^{+}(f_{22}^{+})^{-1}\} = 0, \quad c_1\{Pf_{12}^{-}(f_{22}^{-})^{-1}\} = 0 \quad \text{and} \quad c_{-2}\{Qf_{21}^{+}(f_{22}^{+})^{-1}\} \cdot c_2\{Pf_{12}^{-}(f_{22}^{-})^{-1}\} = -1,$$

and the sufficient conditions to have partial indices $(1, -1)$:

$$c_{-1}\{Qf_{21}^{+}(f_{22}^{+})^{-1}\} \neq 0, \quad c_1\{Pf_{12}^{-}(f_{22}^{-})^{-1}\} = 0 \quad \text{and} \quad c_{-2}\{Qf_{21}^{+}(f_{22}^{+})^{-1}\} \cdot c_2\{Pf_{12}^{-}(f_{22}^{-})^{-1}\} = -1,$$

or

$$c_{-1}\{Qf_{21}^{+}(f_{22}^{+})^{-1}\}=0,\quad c_1\{Pf_{12}^{-}(f_{22}^{-})^{-1}\}\neq 0,$$

$$c_{-2}\{Qf_{21}^{+}(f_{22}^{+})^{-1}\}\cdot c_2\{Pf_{12}^{-}(f_{22}^{-})^{-1}\}=-1,$$

and

$$c_1\{Pf_{12}^{-}(f_{22}^{-})^{-1}\}\neq c_{-2}\{Qf_{21}^{+}(f_{22}^{+})^{-1}\}-c_2\{Pf_{12}^{-}(f_{22}^{-})^{-1}\}$$

which most likely are necessary as well (see the electronic supplementary material).

In the next example, we test empirically conjecture 1.2 formulated in the Introduction in the simplest possible situation. For this reason, we take random Laurent polynomial matrices S of degree 1 with zero partial indices and $\kappa_1=0$ in (1.4). More specifically, we take two random matrices $\mathbf{A}_i(t)=A_i+B_it$, $i=1,2$, where entries of matrices $A_i, B_i\in\mathbb{R}^{2\times 2}$ are randomly chosen from the interval $[-1;1]$. Then we perform the spectral factorization of the matrices $S_i(t)=\mathbf{A}_i(t)\mathbf{A}_i^*(t)$ as $S_i(t)=S_i^+(t)S_i^-(t)$, $i=1,2$, where $S_i^-=(S_i^+)^*$, according to the algorithm developed in [8] and construct

$$S(t)=S_1^+(t)S_2^-(t). \tag{5.2}$$

Representation (5.2) is a Wiener–Hopf factorization of S and we are guaranteed that partial indices of S are equal to zero in this way. We further consider only such random matrices S for which the index κ_1 of s_{11} is equal to zero.

After constructing S we consider its triangular factorization and the approximation S_N, $N\geq 1$, as it is described in theorem 1.1. When we factorize S_N according to the algorithm described in the paper, we get

$$S_N(t)=S_N^+(t)S_N^-(t). \tag{5.3}$$

We are not guaranteed, however, that $S_N^{\pm}\to S^{\pm}$ as $N\to\infty$. Furthermore, $S_N^{\pm}$ might not be convergent at all.

The situation can be resolved by introducing normalizing factors K_N and representing factorization (5.3) in the form

$$S_N(t)=S_N^+(t)K_N\cdot K_N^{-1}S_N^-(t).$$

For this reason, we do the following: Let

$$S_N(t)=\sigma_N^+(t)W_N(t)\sigma_N^-(t)$$

be the representation of S_N in form (4.2) (see also (4.4)) and let (see (1.6))

$$M=\lim_{N\to\infty}\begin{pmatrix} s_{11}^+(1) & 0\\ \varphi_N(1) & \dfrac{\Delta^+(1)}{s_{11}^+(1)}\end{pmatrix}=\begin{pmatrix} s_{11}^+(1) & 0\\ \varphi(1) & \dfrac{\Delta^+(1)}{s_{11}^+(1)}\end{pmatrix}.$$

Then $\lim_{N\to\infty}\sigma_N^+(1)=M$ also since we have $U_N(1)=I_2$ (see (2.7)) for the additional factor U_N.

Next, we invoke the fact that $W_N(1)=I_2$ and define the matrix K_N in such a way that

$$W_N^+(1)K_N=I_2,$$

where

$$W_N(t)=W_N^+(t)W_N^-(t)$$

is the factorization of W_N computed according to the steps of the proposed algorithm. After introducing such normalization, the factorization of S_N has the form

$$S_N(t)=\sigma_N^+(t)W_N^+(t)K_N\cdot K_N^{-1}W_N^-(t)\sigma_N^-(t)=:S_N^+(t)K_N\cdot K_N^{-1}S_N^-(t).$$

Let

$$S(t)=S_1^+(t)\big(S_1^+(1)\big)^{-1}M\cdot M^{-1}S_1^+(1)S_2^-(t)=:S^+(t)S^-(t)$$

(see (5.2)) be the normalized factorization of S which has the property that $S^+(1)=M$. We should expect that

$$S_N^+(t)K_N\to S^+(t).$$

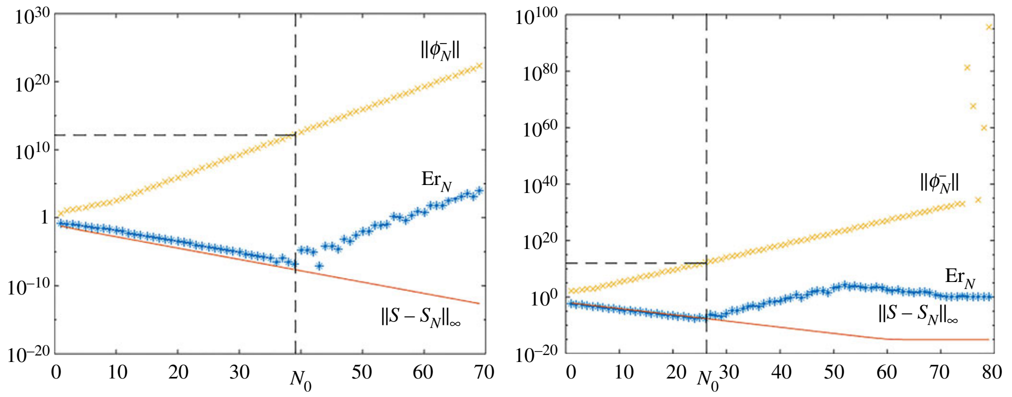

Figure 1. Two typical $\mathrm{Er}_N(S)$, $N = 2, 3, \ldots$, sequences along with $\|\phi_N^-\|$ and $\|S - S_N\|_\infty$ norms. (Online version in colour.)

Table 1. Number n of matrices S for which $\inf_{1<N\leq 100} \mathrm{Er}_N(S) \in [10^{-l}, 10^{-m})$.

$[10^{-l}, 10^{-m})$	$10^{-8}, 10^{-7}$	$10^{-7}, 10^{-6}$	$10^{-6}, 10^{-5}$	$10^{-5}, 10^{-4}$	$10^{-4}, 10^{-3}$	$10^{-3}, 10^{-2}$
n	6	48	37	5	2	2

Table 2. Number n of matrices S for which $N_0(S) \in [l, m]$.

$[l, m]$	5–8	9–12	13–17	18–25	26–35	36–55	56–75	75–99	100≤
n	9	19	20	17	15	8	5	2	5

Since we know that factors of S are matrix polynomials of the same degree as S (see [15]), we discard extra powers of computed factors S_N^+ and S_N^-, the coefficients of which are very close to 0 anyway. In other words, it is assumed that S_N^+ and S_N^- are of degree 1 while formally their degrees are $2N+1$ and $N+1$, respectively (according to the computational steps of the algorithm).

We checked empirically the norms

$$\mathrm{Er}_N(S) := \|S_N^+(t)K_N - S^+(t)\|, \text{ where } N = 2, 3, \ldots, 100, \tag{5.4}$$

for 100 random matrices S. For each S, we have

$$r_{\min} := 1.0 \times 10^{-8} \leq \inf_{1<N\leq 100} \mathrm{Er}_N(S) \leq r_{\max} := 7.3 \times 10^{-3} \tag{5.5}$$

which is rather close to 0 (more informative data about the distribution of $\inf_{1<N\leq 100} \mathrm{Er}_N(S)$ in the range $[r_{\min}, r_{\max}]$ is given in table 1). Furthermore, $\mathrm{Er}_N(S)$ decreases monotonically until N reaches some critical value $N_0 = N_0(S)$ which differs from matrix to matrix. (The distribution of $N_0(S)$ in the range $[2, 100]$ is given in table 2. On very rare occasions, it has been observed that the monotonicity of $\mathrm{Er}_N(S)$, $n = 2, 3, \ldots, N_0$, fails because of large outliers. This is due to difficulties arising during the computations mentioned in remark 3.2.) However, after N surpasses N_0, the sequence $\mathrm{Er}_N(S)$, $N = N_0 + 1, \ldots$, increases and the computed results are repelled from their correct values. This can be explained by the fact that the coefficients of ϕ_N^- in the factorization of W_N (according to Step 4) become very large and the standard Matlab arithmetic on such numbers is unreliable. (Indeed, it is empirically observed that the increase in error starts for those values of N for which these coefficients approach the range $[10^{13}, 10^{15}]$; see the dashed lines in figure 1.) Nevertheless, the closeness to zero of (5.4) for reasonable values of N suggests that conjecture 1.2 formulated in Introduction might be true.

Two samples from the sequences $\mathrm{Er}_N(S)$, $n=2,3,\ldots,100$, which are more or less typical representatives, along with corresponding values of norms $\|\phi_N^-\|$ and $\|S-S_N\|_\infty$ are plotted in figure 1. Note that $\|S_N^+(t)K_N - S^+(t)\|$ is as small as $\|S-S_N\|_\infty$ within the range $N\in[2,N_0]$.

Data accessibility. This article has no additional data.

Authors' contributions. Both authors equally contributed to the design of the proposed algorithms. The first author conducted the numerical experiments and drafted the paper; the second author played the main role in interpreting the data and revising the paper. Both authors approve the final version and agree to be accountable for all aspects of the work.

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