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THE SMOOTHNESS OF FUNCTIONS OF TWO VARIABLES AND DOUBLE TRIGONOMETRIC SERIES

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Abstract. The notion of smoothness (according to Riemann) is introduced for functions of two variables and some of their properties are established. As an application we prove the uniform smoothness of an everywhere continuous sum of a double trigonometric series in the complex form which is obtained by twice term-by-term integration, over every variable rectangle $[0,x]\times[0,y]\subset[0,2\pi]\times[0,2\pi]$ of a double trigonometric series in the complex form absolutely converging at some point. An analogous consideration is given to a double trigonometric series in the real form, the absolute values of whose coefficients form a converging series.

Keywords and phrases: Smoothness, unilateral and symmetrical differentiability of functions of two variables, smoothness of the sum of a double trigonometric series.

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0. Introduction

According to Riemann, a function $\varphi(x)$ defined in the neighborhood of a point $x_0\in\mathbb{R}$ is called smooth at x_0 (the term was introduced by Zygmund [5]) if the equality

$$\lim_{h\to 0}\frac{\varphi(x_0+h)+\varphi(x_0-h)-2\varphi(x_0)}{h}=0 \tag{0.1}$$

is fulfilled.

Riemann showed that the twice term-by-term integration of a trigonometric series in real form with coefficients converging to zero gives a function that satisfies equality (0.1) for all $x_0\in\mathbb{R}$ ([4, p. 245], even uniformly [1, p. 184], [5], [6, p. 320]).

A detailed investigation of Riemann-smooth functions with various applications to different classes of functions and to trigonometric series was carried out by Zygmund [5].

1. The smoothness of functions of two variables

Definition 1.1. A function $f(x,y)$ defined in a neighborhood of a point $(x_0,y_0)\in\mathbb{R}^2$ is called smooth at the point (x_0,y_0) if the following equality is fulfilled:

$$\lim_{(h,k)\to(0,0)}\frac{f(x_0+h,y_0+k)+f(x_0-h,y_0-k)-2f(x_0,y_0)}{|h|+|k|}=0. \tag{1.1}$$

If $f(x,y)$ is a smooth function at every point of some open set $E\subset\mathbb{R}^2$, then f is called smooth in E. If f is continuous and satisfies condition (1.1) uniformly with respect to all points $(x_0,y_0)\in E$, then f is called uniformly smooth in E.

It is obvious that

1) if a function $f(x,y)$ is smooth at a point (x_0,y_0), then the partial functions $f(x,y_0)$ and $f(x_0,y)$ of one variable are smooth at the points x_0 and y_0, respectively.

2) If the functions $a(x)$ and $b(y)$ are smooth at the points x_0 and y_0, respectively, then their sum $f(x,y)=a(x)+b(y)$ is smooth at (x_0,y_0).

Theorem 1.2. *A function of two variables which is differentiable at some point is smooth at the same point. The converse statement is not true.*

Corollary 1.3. If the functions $a(x)$ and $b(y)$ are differentiable at the points x_0 and y_0, respectively, then the product $\phi(x,y)=a(x)\cdot b(y)$ is a smooth function at the point (x_0,y_0).

Theorem 1.4. *Let a function $f(x,y)$ be summable on the rectangle $[a,b]\times[c,d]$. Then the function*

$$F(x,y)=\int_a^x\int_c^y f(t,\tau)\,dt\,d\tau \tag{1.2}$$

is continuous everywhere and is smooth at almost all interior points (x_0,y_0) of this rectangle.

Theorem 1.5. *Let (x_0,y_0) be the point at which the function F defined by equality* (1.2) *is differentiable. Then the function*

$$\Phi(x,y)=\int_{x_0}^x\int_{y_0}^y f(t,\tau)\,dt\,d\tau \tag{1.3}$$

is smooth at the point (x_0,y_0).

2. The differentiability of a smooth function of two variables at a point of extremum

Though a function of two variables at the point of smoothness may be nondifferentiable (see Theorem 1.2), there may nevertheless occur a case where smoothness implies differentiability.

Theorem 2.1. *If a smooth function $f(x,y)$ at a point (x_0,y_0) has a maximum or a minimum at (x_0,y_0), then $f(x,y)$ at (x_0,y_0) has zero angular partial derivatives* [2] $f'_{\hat{x}}(x_0,y_0)=0$, $f'_{\hat{y}}(x_0,y_0)=0$ *and therefore* $df(x_0,y_0)=0$.

3. The smoothness and symmetrical differentiability of functions of two variables

From the differentiability of a function of two variables we have its symmetrical differentiability without the converse statement [3]. Let us now prove that the symmetrical differentiability implies that the considered function is differentiable when it is smooth.

Definition 3.1. ([3]) A function $\varphi(x,y)$ is called symmetrically differentiable at a point (x_0,y_0) if there exist finite constants A and B with the property

$$\lim_{(h,k)\to(0,0)}\frac{\varphi(x_0+h,y_0+k)-\varphi(x_0-h,y_0-k)-2Ah-2Bk}{|h|+|k|}=0. \tag{3.1}$$

Proposition 3.2. *Let a function $f(x,y)$ be smooth at a point (x_0,y_0). Then for f to be differentiable at (x_0,y_0) it is necessary and sufficient that f be symmetrically differentiable at (x_0,y_0).*

Corollary 3.3. The everywhere smooth and almost everywhere nondifferentiable function $\varphi(x,y)$ indicated in the proof of Theorem 1.2 is even symmetrically nondifferentiable almost everywhere.

4. The smoothness and unilateral differentiability of functions of two variables

Let $U(O)$ and $U^0(O)=U(O)\setminus\{O\}$ denote the neighborhood and the punctured neighborhood of the point $O=(0,0)$. We use the following sets ([2, p. 43]):

$$A_1^+=\{(h,k)\in U(O):h>0\},\quad A_2^+=\{(0,k)\in U(O):k>0\},$$
$$A_1^-=\{(h,k)\in U(O):h<0\},\quad A_2^-=\{(0,k)\in U(O):k<0\},$$
$$A_{12}^+=A_1^+\cup A_2^+,\quad A_{12}^-=A_1^-\cup A_2^-.$$

It is obvious that $A_{12}^+\cap A_{12}^-=\varnothing$ and $A_{12}^+\cup A_{12}^-=U^0(O)$.

Let us introduce the following two definitions.

Definition 4.1. A function $f(x,y)$ is called right-differentiable at the point $p_0=(x_0,y_0)$ if the equality

$$\lim_{\substack{(h,k)\to(0,0)\\ (h,k)\in A_{12}^+}}\frac{f(x_0+h,y_0+k)-f(x_0,y_0)-A^+h-B^+k}{|h|+|k|}=0 \tag{4.1}$$

is fulfilled for some finite numbers A^+ and B^+, and the linear function A^+h+B^+k for $(h,k)\in A_{12}^+$ is called a right-differential of f at the point p_0, denoted by $d^+f(p_0)$ and we write

$$d^+f(p_0)=A^+h+B^+k. \tag{4.2}$$

Definition 4.2. A function $f(x,y)$ is called left-differentiable at the point $p_0=(x_0,y_0)$ if there exist finite numbers A^- and B^- such that the equality

$$\lim_{\substack{(h,k)\to(0,0)\\ (h,k)\in A_{12}^-}}\frac{f(x_0+h,y_0+k)-f(x_0,y_0)-A^-h-B^-k}{|h|+|k|}=0 \tag{4.3}$$

is fulfilled, and the linear function A^-h+B^-k is called a left-differential of f at the point p_0, denoted by $d^-f(p_0)$, for $(h,k)\in A_{12}^-$, and we write

$$d^-f(p_0)=A^-h+B^-k. \tag{4.4}$$

The next two propositions are obvious.

Proposition 4.3. *A differentiable at a point p_0 function $f(x,y)$ is bilaterally differentiable at p_0 and the equalities $d^+f(p_0)=df(p_0)$, $d^-f(p_0)=df(p_0)$,*

$$A^+=A^-=f'_x(p_0),\quad B^+=B^-=f'_y(p_0) \tag{4.5}$$

are fulfilled.

Proposition 4.4. *If a function $f(x,y)$ is bilaterally differentiable at a point p_0 and the equalities $A^+ = A^-$ and $B^+ = B^-$ are fulfilled, then f is differentiable at p_0 and*

$$A^+ = f'_{\hat{x}}(p_0) = A^-, \quad B^+ = f'_{\hat{y}}(p_0) = B^-. \tag{4.6}$$

We have

Theorem 4.5. *A smooth at some point $p_0 = (x_0, y_0)$ function $f(x,y)$ is differentiable at p_0 if and only if it is unilaterally differentiable at the point p_0.*

5. The smoothness of sums of double trigonometric series

5A. The complex case. Assume that there is a double trigonometric series in the complex form

$$\sum_{m,n=-\infty}^{\infty} c_{mn} e^{i(mx+ny)}., \tag{5.1}$$

Theorem 5.1. *Let series (5.1) converge absolutely at some point from the square $I = [0, 2\pi]^2$. Then by twice term-by-term integration of series (5.1) over every variable rectangle $[0,x]\times[0,y] \subset I$ we obtain function $\Omega(x,y)$ which is an everywhere continuous and uniformly smooth function on the square I.*

5B. The real case. Let us consider a double trigonometric series in the real form

$$\frac{1}{4} + \frac{1}{2}\sum_{m=1}^{\infty}(a_{m0}\cos mx + d_{m0}\sin mx) + \frac{1}{2}\sum_{n=1}^{\infty}(a_{0n}\cos ny + c_{0n}\sin ny)$$
$$+ \sum_{m,n=1}^{\infty}(a_{mn}\cos mx\cos ny + b_{mn}\sin mx\sin ny + c_{mn}\cos mx\sin ny$$
$$+ d_{mn}\sin mx\cos ny). \tag{5.2}$$

It is assumed that

$$\sum_{m=1}^{\infty}(|a_{m0}| + |d_{m0}|) < \infty, \quad \sum_{n=1}^{\infty}(|a_{0n}| + |c_{0n}|)) < \infty, \tag{5.3}$$

$$\sum_{m,n=1}^{\infty}(|a_{mn}| + |b_{mn}| + |c_{mn}| + |d_{mn}|)) < \infty. \tag{5.4}$$

Theorem 5.2. *Let the coefficients of the double trigonometric series (5.2) in the real form satisfy conditions (5.3) and (5.4). Then by twice term-by-term integration of series (5.2) over every variable rectangle $[0,x]\times[0,y] \subset I$ we obtain the function $\omega(x,y)$ which is everywhere continuous and uniformly smooth function on the square $I = [0,2\pi]\times[0,2\pi]$.*

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