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On the Differentiability of Real, Complex and Quaternion Functions

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In the paper, the necessary and sufficient conditions are established for the continuity of functions of several variables, for the differentiability of functions of several real variables, and for the $\mathbf{C}^n$-differentiability of functions of several complex variables. Also, the rules are given for calculating $\mathbf{H}$-derivatives of quaternion functions, and the necessary and sufficient conditions are obtained for the $\mathbf{H}$-differentiability of quaternion functions.

Keywords: Separately strong (angular) partial continuity, nonintense angular partial continuity, angular (strong) partial derivative, angular (strong) gradient, nonintense angular partial derivative, differentiability, total differential, variation in the Hardy (Arzela) sense, indefinite integral, absolutely continuity, Lebesgue's intense point, definite integral with a parameter, $\mathbf{C}^n$-differentiability, Hartog's main theorem, $\mathbf{H}$-derivative

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Introduction

A function of many variables will not have the continuity or differentiability property only because it has the same property with respect to each independent variable.

Functions with this drawback at individual points have been known since the late 19th century, and on the massive set since the 20th century. Namely, the following statement is valid.

Statement A ([1, pp. 432-433]): *There exists the function of two variables which is discontinuous at almost every point of the unit square and at every point of that square continuous with respect to every variable.*

Note that this Tolstov's function does not posses almost everywhere even the property of continuity on the whole (see [2]; [3, pp. 32–39]).

These and analogous problems are studied e.g. in Z. Piotrowski [4].

Here the problem consists in finding out whether there exists or does not exist any notion of a function with respect to an independent variable and whether the fulfillment of this notion for all independent variables will be the necessary and sufficient condition for the continuity and differentiability of the function itself.

In this paper, the discussion concentrates on this problem.

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In formulating the main results, we use the following notation: $x = (x_1, \dots, x_n)$, $x^0 = (x_1^0, \dots, x_n^0)$, $x(x_k^0) = (x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots, x_n)$.

1. The conditions for the continuity

1. The function f is called a **strong partial continuous** with respect to the variable x_k at the point x^0, if the equality

$$\lim_{x \to x^0} [f(x) - f(x(x_k^0))] = 0 \tag{1.1}$$

is fulfilled and f is called **separately strong partial continuous** at the point x^0, if f with respect to every variable is strongly partial continuous at x^0, i.e. equality (1.1) is fulfilled for all $k = 1, 2, \dots, n$.

Theorem 1.1 ([5]; [2]; [3, pp. 20–25])**:** *For the continuity of the function f at the point x^0, it is necessary and sufficient that it possesses separately strong partial continuity at x^0.*

2. The expression

$$f(x) - f(x(x_k^0)) \quad \text{for} \ \ |x_j - x_j^0| \le c_j |x_k - x_k^0|, \quad j \ne k,$$

depending on the variables $x_1, \dots, x_n$, is called an **angular partial increment** of the function f at the point x^0 with respect to the variable x_k, corresponding to the collection $c = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ of positive constants.

The **angular partial continuity** of the function of at the point x^0 with respect to the variable x_k means the fulfillment of the equality

$$\lim_{\substack{x_k \to x_k^0 \\ |x_j - x_j^0| \le c_j |x_k - x_k^0| \\ j \ne k}} [f(x) - f(x(x_k^0))] = 0 \tag{1.2}$$

for every collection $c = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ of positive constants.

The function f is called **separately angular partial continuous** at the point x^0, if with respect to every variable the function f possesses the property of angular partial continuity at the point x^0, i.e. if for all $k = 1, \dots, n$ and for every collection $c = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ of positive constants, equality (1.2) holds.

Theorem 1.2 ([5]; [2] ; [3, pp. 25–27])**:** *For the continuity of the function f at the point x^0, the necessary and sufficient condition is the separately angular partial continuity at x^0.*

3. If in the definition of the angular partial continuity we put $c_j = 1$ for all $j \ne k$, then we have the **nonintense angular partial continuity** at the point x^0 of the function f with respect to the variable x_k.

Theorem 1.3 ([3, pp. 27–28]) **:** *For the continuity of the function f at the point x^0, the necessary and sufficient condition is the separately nonintense angular partial continuity of the function f at the point x^0.*

2. Angular partial derivative and angular gradient

The existence of finite partial derivatives of all orders, i.e. ordinary gradients of the real function f at the point x^0 does not imply the differentiability of the function f at the same point x^0. Even the function, possessing a finite gradient at the point x^0, may be discontinuous at x^0. Such, for example, are at the point $(0,0)$ the most of the functions of two variables indicated in Piotrowski's work [4].

It is remarkable that this fact can be realized at all points of a set, whose plane measure is arbitrarily nearly to total measure.

Statement B ([1, § 4]): *For every number $\mu < 1$ there exists the function F, defined on the square $Q = \{(x,y) \in \mathbf{R}^2;\ \ 0 \le x \le 2,\ 0 \le y \le 1\}$, possessing at all points of the Q finite partial derivatives of all orders, and at the same time F is discontinuous on a certain set $E \subset Q$ of plane measure μ^2.*

We say that the function F has at the point x^0 an **angular partial derivative** with respect to the variable x_k, symbolically $f'_{\widehat{x}_k}(x^0)$, if for every collection $c = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ of positive $n-1$ constants there exists an independent of the c finite limit

$$f'_{\widehat{x}_k}(x^0) = \lim_{\substack{x_k \to x_k^0 \\ |x_j - x_j^0| \le c_j |x_k - x_k^0| \\ j \ne k}} \frac{f(x) - f(x(x_k^0))}{x_k - x_k^0}\,. \tag{2.1}$$

The existence of $f'_{\widehat{x}_k}(x^0)$ implies existence of the partial derivative $f'_{x_k}(x^0)$, and the equality $f'_{x_k}(x^0) = f'_{\widehat{x}_k}(x^0)$. To show this, we have to put in (2.1) $x_j = x_j^0$ for all $j \ne k$.

The existence of the angular partial derivative does not, in general, follows from existence of the ordinary partial derivative. If $f'_{\widehat{x}_k}(x^0)$ is finite, then the function f with respect to the variable x_k has the property of angular partial continuity at the point x^0.

If there exist finite $f'_{\widehat{x}_k}(x^0)$, $k = 1, \dots, n$, then we call f the function possessing an **angular gradient** at the point x^0 and write

$$\operatorname{anggrad} f(x^0) = (f'_{\widehat{x}_1}(x^0), \dots, f'_{\widehat{x}_n}(x^0)).$$

Theorem 2.1 ([5]; [6]; [3, pp. 60–64])**:** *For the function f to be differentiable at the point x^0, it is necessary and sufficient that* $\operatorname{anggrad} f(x^0)$ *is finite. The total differential $df(x^0)$ of the differentiable at the point x^0 function f admits the following representation*

$$df(x^0) = \sum_{k=1}^{n} f'_{\widehat{x}_k}(x^0)\, dx_k\,.$$

Theorem 2.2 ([6]; [3, p. 65])**:** *For the function f to be differentiable at the point x^0, it is necessary and sufficient that the* ***nonintense angular partial deriva-***

tives

$$D_{\widehat{x}_k} f(x^0) = \lim_{\substack{x_k \to x_k^0 \\ |x_j - x_j^0| \le |x_k - x_k^0| \\ j \ne k}} \frac{f(x) - f(x(x_k^0))}{x_k - x_k^0}$$

are finite for all $k = 1, \ldots, n$.

Corollary 2.3 ([6]; [3, p. 65])**:** *The finiteness of all $D_{\widehat{x}_k} f(x^0)$ implies finiteness of all $f'_{\widehat{x}_k}(x^0)$, and the equality*

$$f'_{\widehat{x}_k}(x^0) = D_{\widehat{x}_k} f(x^0), \quad k = 1, \ldots, n,$$
$$df(x^0) = \sum_{k=1}^{n} D_{\widehat{x}_k} f(x^0)\, dx_k.$$

3. Examples on the differentiability

Using Theorem 2.2, we can establish the differentiability as well as non-differentiability of concrete functions.

On the differentiability we investigate some appearing frequently functions.

Proposition 3.1 ([6]; [3, p. 66])**:** *Suppose the numbers α_j are positive, $j = 1, \ldots, n$. Then the condition*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n > 1$$

is necessary and sufficient for the everywhere continuous function

$$\varphi(x_1, \ldots, x_n) = |x_1|^{\alpha_1} \cdot |x_2|^{\alpha_2} \cdots |x_n|^{\alpha_n}$$

to be differentiable at the point $x^0 = (0, \ldots, 0)$.

In particular, the function $\gamma(x_1, \ldots, x_n) = (|x_1| \cdots |x_n|)^{\alpha}$ is differentiable at the point x^0 if and only if $\alpha > 1/n$.

If $\alpha_1 + \alpha_2 + \cdots + \alpha_n \le 1$, then all $D_{\widehat{x}_k} \varphi(x^0)$ are devoid of existence.

Proposition 3.2 ([3, p. 67])**:** *Suppose the numbers $\beta_j > 1$, $j = 1, \ldots, n$. Then the function*

$$\Phi(x_1, \ldots, x_n) = \begin{cases} \sum\limits_{j=1}^{n} |x_j|^{\beta_j} & \text{for all rational } x_j \\ 0 & \text{at the remaining points} \end{cases} \tag{3.1}$$

is differentiable at the point $x^0 = (0, \ldots, 0)$, $d\Phi(x^0) = 0$ and discontinuous at all the remaining points $(x_1, \ldots, x_n) \ne (0, \ldots, 0)$.

Proposition 3.3 ([3, p. 68]): *The corresponding to the number $q>1$ function*

$$\Psi(x_1,\dots,x_n)=\begin{cases}\Big(\sum\limits_{j=1}^{n}|x_j|\Big)^q & \text{for all rational } x_j\\ 0 & \text{at the remaining points}\end{cases} \tag{3.2}$$

possesses the same properties as the function (3.1).

Proposition 3.4 ([3, p. 68]): *The corresponding to the number $\alpha>0$ function*

$$\omega(x_1,\dots,x_n)=\begin{cases}\Big(\sum\limits_{j=1}^{n}x_j^2\Big)^{\frac{1+\alpha}{2}} & \text{for all rational } x_j\\ 0 & \text{at the remaining points}\end{cases}$$

possesses all properties of functions (3.1) *and* (3.2).

Proposition 3.5 ([3, p. 69]): *The function*

$$g(x_1,x_2)=\begin{cases}x_1x_2\sin\frac{1}{x_1x_2} & \text{for } x_1\cdot x_2\neq 0\\ 0 & \text{for } x_1\cdot x_2=0\end{cases}$$

is differentiable at the point $x^0=(0,0)$, and its gradient $\operatorname{grad} g(x_1,x_2)$ *is indeterminate in the punctured neighborhood of the point x^0.*

Proposition 3.6 ([3, pp. 69–70]) **:** *The function*

$$\psi(x_1,x_2)=\begin{cases}\frac{x_1^2\cdot x_2}{x_1^2+x_2^2} & \text{for } x_1^2+x_2^2>0,\\ 0 & \text{for } x_1=0=x_2\end{cases}$$

possesses the following properties:

1) $\psi(x_1,x_2)$ *is continuous everywhere;*
2) $\operatorname{grad}\psi(x_1,x_2)$ *is finite everywhere;*
3) $\psi(x_1,x_2)$ *is not differentiable at the point* $(0,0)$*;*
4) $\operatorname{grad}\psi(x_1,x_2)$ *is not continuous at the point* $(0,0)$.

4. A strong partial derivative and strong gradient

We say that the function f possesses at the point x^0 a **strong partial derivative** with respect to the variable x_k, symbolically $f'_{[x_k]}(x^0)$, if there exists a finite limit

$$f'_{[x_k]}(x^0)=\lim_{x\to x^0}\frac{f(x)-f(x(x_k^0))}{x_k-x_k^0}\,.$$

We say that the function f has at the point x^0 a **strong gradient**, symbolically $\operatorname{strgrad} f(x^0)$, if for every $k=1,\dots,n$ there exist finite $f'_{[x_k]}(x^0)$, and we write

$$\operatorname{strgrad} f(x^0)=(f'_{[x_k]}(x^0),\dots,f'_{[x_n]}(x^0)).$$

If there exists a strgrad $f(x^0)$, then there likewise exists anggrad $f(x^0)$, and equalities strgrad $f(x^0) =$ anggrad $f(x^0) =$ grad $f(x^0)$ hold.

Consequently, we have

Theorem 4.1 ([5]; [6]; [3, p. 77]) **:** *The existence of a finite* strgrad $f(x^0)$ *implies existence of a total differential* $df(x^0)$ *and*

$$\operatorname{strgrad} f(x^0) = \operatorname{anggrad} f(x^0) = \operatorname{grad} f(x^0).$$

If the grad $f(x)$ is continuous at the point x^0, then we have the equality strgrad $f(x^0) =$ grad $f(x^0)$ ([3, p. 75]).

Besides, the existence of the finite strgrad $f(x^0)$ does not, in general, imply the continuity of grad $f(x)$ at the point x^0.

For, the gradient of the function $g(x_1, x_2)$ from the Proposition 3.5 is not continuous at $(0, 0)$, but strgrad $g(0, 0) = (0, 0)$.

In addition to this, we have

Theorem 4.2 ([3, p. 76])**:** *There exists an absolutely continuous function of two variables which has almost everywhere both finite strong and discontinuous gradients.*

Remark 1 : We should begin the proof of Theorem 4.2 from [3] with the following. There exists a measurable set $e \subset [0, 1]$ such that the sets $e \cap (\alpha, \beta)$ and $([0, 1] \setminus e) \cap (\alpha, \beta)$ have positive measures for all subintervals $(\alpha, \beta) \subset [0, 1]$ ([7, p. 50] or [8, p. 49]). By $\alpha(x)$ we denote the characteristic function of the set e. Analogously, we obtain the function $\beta(y)$.

Proposition 4.3 ([5]; [6]; [3, p. 77]) **:** *The finiteness of* anggrad $f(x^0)$ *or, what is the same, the existence of* $df(x^0)$ *does not imply the existence of* strgrad $f(x^0)$.

The function $\lambda(x_1, x_2) = |x_1 \cdot x_2|^{2/3}$ is differentiable at the point $x^0 = (0.0)$ (see Proposition 4.3), but strgrad $\lambda(x^0)$ does not exist.

Afterwards, G.G. Oniani established that the existence of a finite strong gradient is essentially stronger property than the differentiability.

Theorem 4.4 ([11]; [12]) **:** *For arbitrary $n \geq 2$ there exists a continuous function $f : [0, 1]^n \to \mathbf{R}$ such that:*

1. *f is differentiable almost everywhere,*
2. *f devoid of finite a strong gradient almost everywhere.*

The following theorem is improvement of Theorem 4.4.

Theorem 4.5 ([10, Theorem 4])**:** *For arbitrary $n \geq 2$ there exists a continuous function $f : [0, 1]^n \to \mathbf{R}$ that is differentiable almost everywhere, but everywhere devoid of finite a strong gradient.*

As is known, functions of bounded variationss in the Hardy or Arzela sense possess the differentiability property almost everywhere, i.e. have finite angular gradients almost everywhere.

As to the existence of a strong gradient, functions of these classes behave differently.

Theorem 4.6 ([9]; [10])**:** *Every function $f : [0, 1]^n \to \mathbf{R}$ of bounded variation in the Hardy sense has a finite strong gradient almost everywhere.*

Theorem 4.7 ([10, Theorem 3])**:** *For arbitrary $n \geq 2$ there exists a continuous function $f : [0,1]^n \to \mathbf{R}$ of bounded variation in the Arzela sense that everywhere devoid of finite the strong gradient.*

5. Classification of functions by various gradients

Theorem 5.1 ([3, p. 80])**:** *A class with continuous at the point x^0 gradients of functions is contained strictly in a class with finite at the point x^0 strong gradients of functions, and the latter is contained strictly in a class of functions with finite at the point x^0 angular gradients. This class coincides with the class of differentiable at x^0 functions.*

Remark 1: The notions of angular and strong gradients were generalized by Leri Bantsuri, who introduced the notion of a gradient with respect to the basis and established, in particular, the relationship between the differentiablity and the existence of the gradient which he has introduced ([13], [14]).

6. Differentiability of an indefinite integral and of an absolutely continuous functions

Let the function of two variables f be summable on the rectangle $Q = \{(x,y) \in \mathbf{R}^2 : a \leq x \leq b,\ c \leq y \leq d\}$, $f \in L(Q)$. Consider for the function f the indefinite double integral

$$F(x,y) = \int_a^x \int_c^y f(t,\tau)\,dt\,d\tau. \tag{6.1}$$

The following problems are quite natural.
I. Does the indefinite double integral (6.1), have or have no total differential almost everywhere?
II. If F has a total differential, then at what points and how the set of such points is connected with the function f?

The answer to problem I will be given here, and problem II will be considered in Section 8.

Theorem 6.1 ([5, Theorem 6.7]; [15]; [3, pp. 102–104])**:** *Indefinite integral* (6.1) *has a total differential at almost all points $(x,y) \in Q$ for every function $f \in L(Q)$.*

Theorem 6.2 ([5]; [15]; [3, p. 104])**:** *At every point $(x_0,y_0) \in Q$ of differentiability of the indefinite integral* (6.1) *with $f \in L(Q)$ we have*

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{|h|+|k|} \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} f(t,\tau)\,dt\,d\tau = 0. \tag{6.2}$$

In particular, equality (6.2) *is fulfilled at almost all points $(x_0,y_0) \in Q$.*

Theorem 6.3 ([5]; [3, p. 105])**:** *Let the indefinite integral* (6.1) *for $f \in L(Q)$ have in the neighborhood of the point $(x_0,y_0) \in Q$ finite F'_x, F'_y and $F''_{x,y}$. Then for the function F to be differentiable at the point (x_0,y_0), it is necessary and sufficient* *that*

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{h\cdot k}{|h|+|k|} F''_{x,y}(x_0+\theta_1 h, y_0+\theta_2 k) = 0, \quad 0<\theta_1,\theta_2<1.$$

If, in addition, $F''_{x,y} = f$ in the neighborhood of the point (x_0, y_0), then for the differentiability of the function F at (x_0, y_0) it is necessary and sufficient that

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{h\cdot k}{|h|+|k|} f(x_0+\theta_1 h, y_0+\theta_2 k) = 0, \quad 0<\theta_1,\theta_2<1.$$

Theorem 6.4 ([5]; [15]; [3, pp. 107–108])**:** *Every absolutely continuous on the Q function has a total differential almost everywhere on the Q. Its partial and mixed partial derivatives are summable on the Q functions.*

Note that Theorem 6.4 is not true for separately absolutely continuous functions!

7. The finiteness of a strong gradient of an indefinite integral and an absolutely continuous function

Theorem 7.1 ([5, Theorem 6.6]; [3, p. 109])**:** *Let the function f be summable on the rectangle $Q = \{(x,y) \in \mathbf{R}^2 : a \le x \le b,\ c \le y \le d\}$. Then the corresponding indefinite integral*

$$F(x,y) = \int_a^x \int_c^y f(t,\tau)\, dt\, d\tau$$

possesses the following properties:

1) *for almost every $x_0 \in [a,b]$ and for every $y_0 \in [c,d]$ the $F'_{[x]}(x_0,y_0)$ is finite, and*

$$F'_{[x]}(x,y) = \int_c^{y_0} f(x_0,\tau)\, d\tau;$$

2) *for every $x^0 \in [a,b]$ and for almost every $y_0 \in [c,d]$ the $F'_{[y]}(x_0,y_0)$ is finite, and*

$$F'_{[y]}(x,y) = \int_a^{x_0} f(t,y_0)\, dt;$$

3) *at almost every point $(x_0,y_0) \in Q$ the* strgrad $F(x_0,y_0)$ *is finite.*

Theorem 7.2 ([5, Theorem 6.8]; [3, pp. 111–112])**:** *For every summable on the rectangle $Q = \{(x,y) \in \mathbf{R}^2 : a \le x \le b,\ c \le y \le d\}$ function f the following statements are valid:*

1) *there exists a measurable set $e_1 \subset [a,b]$ with $|e_1| = b-a$, such that at every point (x_0,y_0) with $x^0 \in e_1$ and $y \in [c,d]$ the integral $\int_c^{y_0} f(x_0,\tau)\, d\tau$ is finite,*

and

$$\lim_{\substack{h\to 0\\ y\to y_0}} \frac{1}{h}\int_{x_0}^{x_0+h}\int_c^{y_0} f(t,\tau)\,dt\,d\tau = \int_c^{y_0} f(x_0,\tau)\,d\tau; \tag{7.1}$$

2) *there exists a measurable set* $e_2 \subset [c,d]$ *with* $|e_2| = d-c$, *such that at every point* (x_0, y_0) *with* $x^0 \in [a,b]$ *and* $y_0 \in e_2$ *the integral* $\int_a^{x_0} f(t,y_0)\,dt$ *is finite, and*

$$\lim_{\substack{k\to 0\\ x\to x_0}} \frac{1}{k}\int_{y_0}^{y_0+k}\int_a^{x_0} f(t,\tau)\,dt\,d\tau = \int_a^{x_0} f(t,y_0)\,dt; \tag{7.2}$$

3) *equalities* (7.1) *and* (7.2) *are fulfilled simultaneously at the points* $(x_0, y_0) \in E$, *where* $E = e_1 \times e_2$, $|E| = |Q|$.

To formulate this and the subsequent theorems in short, we introduce the following measurable sets:

A) $$E_1 = \bigcup_{x_0\in e_1} m(x_0), \quad |E_1| = |Q|,$$

where the measurable set $e_1 \subset [a,b]$ with $|e_1| = b-a$ is adopted from statement 1) of Theorem 7.2, and the vertical closed interval $m(x_0)$ is defined by the equality

$$m(x_0) = \{(x_0,y) : c \le y \le d\};$$

B) $$E_2 = \bigcup_{y_0\in e_2} n(y_0), \quad |E_2| = |Q|,$$

where the measurable set $e_2 \subset [c,d]$ with $|e_2| = d-c$ is adopted from statement 2) of Theorem 7.2, and the horizontal closed interval $n(y_0)$ is defined by the equality

$$n(y_0) = \{(x,y_0) : a \le x \le b\}.$$

Now Theorem 7.2 can be rephrased as follows.

Theorem 7.3 ([3, p. 112])**:** *For every function* $f \in L(Q)$, *equalities* (7.1) *and* (7.2) *take place at the points* $(x_0,y_0) \in E_1$ *and* $(x_0,y_0) \in E_2$, *respectively. Equalities* (7.1) *and* (7.2) *are fulfilled simultaneously at the points* $(x_0,y_0) \in E_3$, *where* $E_3 = E_1 \cap E_2$, $|E_3| = |Q|$.

Theorem 7.4 ([5]; [3, pp. 112–113])**:** *For every function* $f \in L(Q)$ *the following statements take place:*

1) *at the points* $(x_0,y_0) \in E_1$ *the equality*

$$\lim_{(h,k)\to(0,0)} \frac{1}{h}\int_{x_0}^{x_0+h}\int_{y_0}^{y_0+k} f(t,\tau)\,dt\,d\tau = 0 \tag{7.3}$$

holds;

2) *at the points* $(x_0, y_0) \in E_2$ *the equality*

$$\lim_{(h,k)\to(0,0)} \frac{1}{k} \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} f(t,\tau)\,dt\,d\tau = 0 \tag{7.4}$$

is valid;

3) *equalities* (7.3) *and* (7.4) *are fulfilled simultaneously at the points* $(x_0, y_0) \in E_3$*, where* $E_3 = E_1 \cap E_2$*,* $|E_3| = |Q|$*;*

4) *at the points* $(x_0, y_0) \in E_3$ *the equality*

$$\lim_{(h,k)\to(0,0)} \frac{h+k}{hk} \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} f(t,\tau)\,dt\,d\tau = 0 \tag{7.5}$$

holds.

Remark 1 ([3, p. 113])**:** If $S(x,y) \in L(Q)$ is Sak's function, then the expression

$$\frac{1}{hk} \int_{x}^{x+h} \int_{y}^{y+k} S(t,\tau)\,dt\,d\tau \tag{7.6}$$

has the strong supper limit $+\infty$ at every point $(x,y) \in Q$. At the same time, Theorem 7.4 shows that tending of expression (7.6) to $+\infty$ is subordinate to equalities (7.3) and (7.4) at the points $(x_0, y_0) \in E_1 \cap E_2$, i.e.

$$\frac{1}{hk} \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} S(t,\tau)\,dt\,d\tau = O\left(\frac{1}{\max(h,k)}\right).$$

Theorem 7.5 ([3, pp. 113–114])**:** *To every absolutely continuous on the rectangle* Q *function* Φ *there corresponds a triple of functions* $\varphi \in L(Q)$*,* $g \in L([a,b])$ *and* $h \in L([c,d])$*, such that the following statements take place:*

1) *for almost every* $x_0 \in [a,b]$ *and for every* $y_0 \in [c,d]$ *there exists the finite* $\Phi'_{[x]}(x_0, y_0)$*, and*

$$\Phi'_{[x]}(x_0, y_0) = \int_{c}^{y_0} \varphi(x_0, y)\,dy + g(x_0);$$

2) *for every* $x_0 \in [a,b]$ *and for almost every* $y_0 \in [c,d]$ *there exists the finite* $\Phi'_{[y]}(x_0, y_0)$*, and*

$$\Phi'_{[y]}(x_0, y_0) = \int_{a}^{x_0} \varphi(x, y_0)\,dx + h(y_0);$$

3) *at almost every point* $(x_0, y_0) \in Q$ *the strgrad*$\Phi(x_0, y_0)$*,* $\Phi''_{x,y}(x_0, y_0)$ *and* $\Phi''_{y,x}(x_0, y_0)$ *are finite, and*

$$\Phi''_{x,y}(x_0, y_0) = \varphi(x_0, y_0) = \Phi''_{y,x}(x_0, y_0).$$

An $n \geq 2$-dimensional analogue of statement 3) of Theorem 7.1 is

Theorem 7.6 ([16]; [17]; [3, p. 111])**:** *For every $n \geq 2$ and $f \in L(0,1)^n$ the indefinite integral of f, at almost every point is differentiable, moreover, has a finite strong gradient.*

8. Lebesgue's intense points and finiteness at these points of a strong gradient of an indefinite integral

Definition 8.1 ([5]; [3, p. 115])**:** Let the function f belong to the space $L^p(Q)$ for some $p \geq 1$.

The point $(x_0, y_0) \in Q$ is called **jointly Lebesgue's intense point** (of p-th degree) **of the function** f, symbolically $(x_0, y_0) \in \operatorname{int} L^p_{x,y}(f)$, if the following two conditions are fulfilled:

$$\lim_{(h,k)\to(0,0)} \frac{1}{h} \int_{x_0}^{x_0+h} \left| \int_c^{y_0+k} f(x,y)\,dy - \int_c^{y_0} f(x_0,y)\,dy \right|^p dx = 0, \tag{8.1}$$

$$\lim_{(h,k)\to(0,0)} \frac{1}{k} \int_{y_0}^{y_0+k} \left| \int_a^{x_0+h} f(x,y)\,dy - \int_a^{x_0} f(x,y_0)\,dx \right|^p dy = 0. \tag{8.2}$$

When equality (8.1) is fulfilled, then the point (x_0, y_0) is called **Lebesgue's intense point with respect to the variable** x (of p-th degree) **of the function** f, symbolically $(x_0, y_0) \in \operatorname{int} L^p_x(f)$.

When equality (8.2) is fulfilled, then the point (x_0, y_0) is called **Lebesgue's intense point with respect to the variable** y (of p-th degree) **of the function** f, symbolically $(x_0, y_0) \in \operatorname{int} L^p_y(f)$.

Theorem 8.2 ([3, p. 115])**:** *Let the function f belong to the space $L^p(Q)$ for some $p \geq 1$. The following statements take place:*

1) *there exists a measurable set $e_1^* \subset [a,b]$ with $|e_1^*| = b - a$, such that the set of all points (x_0, y_0) with $x_0 \in e_1^*$ and $y_0 \in [c,d]$ forms the set $\operatorname{int} L^p_x(f)$, $|\operatorname{int} L^p_x(f)| = |Q|$;*
2) *there exists a measurable set $e_2^* \subset [c,d]$ with $|e_2^*| = d - c$, such that the set of all points (x_0, y_0) with $x_0 \in [a,b]$ and $y_0 \in e_2^*$ forms the set $\operatorname{int} L^p_y(f)$, $|\operatorname{int} L^p_y(f)| = |Q|$;*
3) *the set of all points (x_0, y_0) with $x_0 \in e_1^*$ and $y_0 \in e_2^*$ forms the set $\operatorname{int} L^p_{x,y}(f)$, $|\operatorname{int} L^p_{x,y}(f)| = |Q|$.*

Theorem 8.3 ([5]; [3, p. 118])**:** *Let the function $f \in L(Q)$. Then the corresponding indefinite integral*

$$F(x,y) = \int_a^x \int_c^y f(t,\tau)\,dt\,d\tau$$

possesses the following properties:

1) *at every point $(x_0, y_0) \in \operatorname{int} L_x(f)$ the $F'_{[x]}(x_0, y_0)$ is finite and*

$$F'_{[x]}(x_0, y_0) = \int_c^{y_0} f(x_0, \tau)\,d\tau,$$

or what is the same,

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{h}\int_{x_0}^{x_0+h}\int_{c}^{y_0+k} f(t,\tau)\,dt\,d\tau = \int_{c}^{y_0} f(x_0,\tau)\,d\tau;$$

2) *at every point* $(x_0, y_0) \in \operatorname{int} L_y(f)$ *the* $F'_{[y]}(x_0, y_0)$ *is finite and*

$$F'_{[y]}(x_0, y_0) = \int_{a}^{x_0} f(t, y_0)\,dt,$$

or what is the same,

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{1}{k}\int_{y_0}^{y_0+k}\int_{a}^{x_0+h} f(t,\tau)\,dt\,d\tau = \int_{a}^{x_0} f(t,y_0)\,dt;$$

3) *at every point* $(x_0, y_0) \in \operatorname{int} L_{x,y}(f)$ *the* strgrad $F(x_0, y_0)$ *is finite, in particular, there exists* $dF(x_0, y_0)$.

Theorem 8.4 ([5]; [3, p. 119])**:** *For every function* $f \in L^p(Q)$ *with* $p \geq 1$ *the following statements are valid:*

1) *at every point* $(x_0, y_0) \in \operatorname{int} L^p_x(f)$ *the equality*

$$\lim_{(h,k)\to(0,0)} \frac{1}{h}\int_{x_0}^{x_0+h} \left| \int_{y_0}^{y_0+k} f(x,y)\,dy \right|^p dx = 0 \tag{8.3}$$

holds;

2) *at every point* $(x_0, y_0) \in \operatorname{int} L^p_y(f)$ *we have*

$$\lim_{(h,k)\to(0,0)} \frac{1}{k}\int_{y_0}^{y_0+k} \left| \int_{x_0}^{x_0+h} f(x,y)\,dx \right|^p dy = 0; \tag{8.4}$$

3) *at every point* $(x_0, y_0) \in \operatorname{int} L^p_{x,y}(f)$ *equalities* (8.3) *and* (8.4) *are fulfilled simultaneously.*

9. A generalization of classical theorems on derivatives on an indefinite integral and of the definite integral with a parameter

We mean the following Lebesgue's and Ch. J. de la Valle'e Poussin's theorems.

Theorem L: *Let* ψ *be a summable function on a closed interval* $[a, b]$*, and suppose that*

$$\Psi(x) = \int_{a}^{x} \psi(t)\,dt.$$

Then for almost all $x \in [a, b]$ *the equality*

$$\Psi'(x) = \psi(x) \tag{9.1}$$

is fulfilled.

Theorem VP: *Let a function $f(x,y)$ be summable with respect to x on a closed interval $[a,b]$ for every fixed y from a closed interval $[c,d]$. Consider a finite on $[c,d]$ function –* ***a definite integral with parameter*** *y –*

$$\Phi(y) = \int_a^b f(x,y)\,dx.$$

Suppose that the following conditions are fulfilled:

(A) *$f(x,y)$ is a function absolutely continuous with respect to y on $[c,d]$ for every fixed $x \in [a,b]$;*
(B) *partial derivative f'_y with respect to y is a summable function on a closed rectangle $Q = \{(x,y) \in \mathbf{R}^2 : a \le x \le b,\ c \le y \le d\}$.*

Then

$$\Phi'(y) = \int_a^b f'_y(x,y)\,dx \tag{9.2}$$

for almost all $y \in [c,d]$.

We have the following generalization of Theorem L and Theorem VP.

Theorem 9.1 ([18]; [3, pp. 120–123])**:** *Let assumptions* (A) *and* (B) *of Theorem VP be fulfilled. Then the function –* ***an indefinite integral with parameter*** *y –*

$$F(x,y) = \int_a^x f(t,y)\,dt \tag{9.3}$$

possesses the following properties:

(i) *there exists $e_1 \subset [a,b]$ such that $|e_1| = b-a$, $F'_{[x]}(x_0,y_0)$ is finite for $(x_0,y_0) \in e_1 \times [c,d]$, and*

$$F'_{[x]}(x_0,y_0) = f(x_0,y_0); \tag{9.4}$$

(ii) *there exists $e_2 \subset [c,d]$ such that $|e_2| = d-c$, $F'_{[y]}(x_0,y_0)$ is finite for $(x_0,y_0) \in [a,b] \times e_2$, and*

$$F'_{[y]}(x_0,y_0) = \int_a^{x_0} f'_y(t,y_0)\,dt; \tag{9.5}$$

(iii) *the* strgrad *$F(x_0,y_0)$ is finite at almost all points $(x_0,y_0) \in Q$, in particular, there exists the total differential $dF(x_0,y_0)$.*

If in equality (9.3) we put $x = b$, then equality (9.5) for $x^0 = b$ takes the form of equality (9.2) because the derivative of the function of one variable is, in fact, its strong partial derivative with respect to the same variable, if we consider this function as the function of two variables, constant with respect to the second variable. Thus equality (9.5) is the generalization of equality (9.2).

Equality (9.1) is obtained analogously from equality (9.4) if the function f in equality (9.3) is assumed to be independent of the variable y.

10. A criterion of $\mathbf{C}^n$-differentiability

Theorem 10.1 ([19])**:** *A function f is $\mathbf{C}^n$-differentiable at a point $z \in \mathbf{C}^n$, if and only if the condition*

$$f'_{\widehat{x}_k}(z) + if'_{\widehat{y}_k}(z) = 0$$

or, equivalently

$$D_{\widehat{x}_k} f(z) + iD_{\widehat{y}_k} f(z) = 0$$

holds for all $k = 1, \ldots, n$, where $z = (z_1, \ldots, z_n)$ and $z_k = x_k + y_k$.

Hartog's Main Theorem([19, p. 17]). *A function f holomorphic (analytic) with respect to each variable in an open set $G \subset \mathbf{C}^n$ is $\mathbf{C}^n$-holomorphic ($\mathbf{C}^n$-analytic) in G.*

11. On the H-differentiability*

Definition 11.1 ([20])**:** A quaternion function $f(z)$, $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, defined on some neighborhood of a point $z^0 = x_0^0 + x_1^0 i_1 + x_2^0 i_2 + x_3^0 i_3$, is called **H**-differentiable at z^0 if there exists two sequences of quaternions $A_k(z^0)$ and $B_k(z^0)$ such that $\sum\limits_k A_k(z^0)B_k(z^0)$ is finite and that the increment $f(z^0+h) - f(z^0)$ of the function f can be represented as

$$f(z^0 + h) - f(z^0) = \sum_k A_k(z^0)B_k(z^0) + \omega(z^0, h),$$

where

$$\lim_{h \to 0} \frac{|\omega(z^0, h)|}{|h|} = 0.$$

In this case, the quaternion $\sum\limits_k A_k(z^0)B_k(z^0)$ is called the **H**-derivative of the function f at the point z^0 and is denoted $f'(z^0)$. Thus

$$f'(z^0) = \sum_k A_k(z^0)B_k(z^0). \tag{11.1}$$

The uniqueness of the **H**-derivative follows from the fact that the right-hand part of (11.1), if it exists, is just the partial derivative $f'_{x_0}(z^0)$ of f at z^0 with respect to its real variable x_0 (see [6], equality (2)).

*For the history of differentiability of quaternion functions see [20] and [21], p. 385.

The basic elementary quanternion functions z^n, e^z, $\cos z$, $\sin z$ are **H**-differentiable and fulfilled the following equalities $(z^n)' = nz^{n-1}$, $(e^z)' = e^z$, $(\cos z)' = -\sin z$, $(\sin z)' = \cos z$.

The rules for calculating **H**-derivatives are identical to those derived in a standard calculus course: $(cf)'(z) = cf'(z)$, $(fc)'(z) = f'(z)c$, $(f + \varphi)'(z) = f'(z)+\varphi'(z)$, $(f\varphi)'(z) = f'(z)\varphi(z)+f(z)\varphi'(z)$, $(1/\varphi)'(z) = -1/\varphi(z)\cdot\varphi'(z)\cdot 1/\varphi(z)$, $(f \cdot 1/\varphi)'(z) == f'(z) \cdot 1/\varphi(z) - f(z) \cdot 1/\varphi(z) \cdot \varphi'(z) \cdot 1/\varphi(z)$, $(1/\varphi \cdot f)'(z) = -1/\varphi(z) \cdot \varphi'(z) \cdot 1/\varphi(z) \cdot f(z) + 1/\varphi(z) \cdot f'(z)$.

Right now we formulate a relationship between **H**-differentiability of a quaternion function $f(z) = u_0(z) + u_1(z)i_1 + u_2(z)i_2 + u_3(z)i_3$ of a quaternion variable $z = x_0 + x_1i_1 + x_2i_2 + x_3i_3$ and the existence of the differential $df(z)$ (with respect to real variables x_0, x_1, x_2, x_3).

Since the partial angular derivatives are the derivatives with respect to real variables (see the Section 2), the condition of differentiability for real, complex and quaternion functions are expressed in the same form.

It then follows that for the differentiability of a quaternion function f at a point $z = x_0+x_1i_1+x_2i_2+x_3i_3$, a necessary and sufficient condition is the existence finite partial angular derivatives $f'_{\widehat{x}_k} = (u_0)'_{\widehat{x}_k}+i_1(u_2)'_{\widehat{x}_k}+i_2(u_2)'_{\widehat{x}_k}+i_3(u_3)'_{\widehat{x}_k}$, $k = 0, 1, 2, 3$.

Moreover, when f is differentiable at z, the following equalities hold for its differential $df(z)$:

$$df(z) = f'_{\widehat{x}_0}(z)dx_0 + f'_{\widehat{x}_1}(z)dx_1 + f'_{\widehat{x}_2}(z)dx_2 + f'_{\widehat{x}_3}(z)dx_3,$$
$$df(z) = du_0(z) + i_1du_1(z) + i_2du_2(z) + i_3du_3(z).$$

Theorem 11.2 ([22])**:** *If a quaternion function f is* **H***-differentiable at a point $z = x_0 + x_1i_1 + x_2i_2 + x_3i_3$, then f is differentiable at the same point z and its partial angular derivatives $f'_{\widehat{x}_0}(z)$, $f'_{\widehat{x}_1}(z)$, $f'_{\widehat{x}_2}(z)$, $f'_{\widehat{x}_3}(z)$ can be expressed in terms of the* **H***-derivative $f'(z) = \sum_k A_k(z)B_k(z)$ as follows:*

$$f'_{\widehat{x}_0}(z) = \sum_k A_k(z)B_k(z) = f'(z), \tag{11.2}$$

$$f'_{\widehat{x}_1}(z) = \sum_k A_k(z)i_1B_k(z), \tag{11.3}$$

$$f'_{\widehat{x}_2}(z) = \sum_k A_k(z)i_2B_k(z), \tag{11.4}$$

$$f'_{\widehat{x}_3}(z) = \sum_k A_k(z)i_3B_k(z). \tag{11.5}$$

Moreover, we have

$$df(z) = \sum_k A_k(z)\, dz\, B_k(z). \tag{11.6}$$

Remark 1: Equation (11.6) can be interpreted as follows. As in the classical case, the differential $df(z)$ of an **H**-differentiable function f is linear with respect to the differential dz of the independent variable z.

Theorem 11.3 ([22])**:** *If a quaternion function f is differentiable at a point z and its partial angular derivatives $f'_{\widehat{x}_0}(z)$, $f'_{\widehat{x}_1}(z)$, $f'_{\widehat{x}_2}(z)$ and $f'_{\widehat{x}_3}(z)$ can be expressed in the forms* (11.2)–(11.5) *for some quaternions $A_k(z)$ and $B_k(z)$, then f is $\mathbf{H}$-differentiable at the point z and*

$$f'(z) = \sum_k A_k(z)B_k(z).$$

We can combine Theorems 11.2 and 11.3 to obtain the following theorem

Theorem 11.4 ([22])**:** *The existence of the differential $df(z)$ of a quaternion function f and its representability in the form*

$$df(z) = \sum_k A_k(z)\, dz\, B_k(z) \tag{11.7}$$

is equivalent to the existence of the derivative $f'(z)$ and its representability in the form

$$f'(z) = \sum_k A_k(z)B_k(z).$$

Corollary 11.5 ([22])**:** *When $x_2 = 0 = x_3$ and $u_2 = 0 = u_3$, then one has a complex function $f(z) = u(z) + iv(z)$ of a complex variable $z = x + iy$. In this case, Eq.* (11.7) *has the form $df(z) = c(z)dz = c(z)dx + ic(z)dy$, where $c(z) = \sum_k A_k(z)B_k(z)$, from which we obtain the equalities $f'_{\widehat{x}}(z) = c(z)$ and $f'_{\widehat{y}}(z) = ic(z)$. Thus, we have*

$$f'_{\widehat{x}}(z) + if'_{\widehat{y}}(z) = 0. \tag{11.8}$$

Note that Eq.(11.8) is a necessary and sufficient condition for the complex function f to be $\mathbf{C}^1$-differentiable at the point z (see [19], Theorem 3.1, when $n = 1$). Moreover, we have obtained the well known equalities $f'(z) = f'_{\widehat{x}}(z)$ and $f'(z) = -if'_{\widehat{y}}(z)$ for the derivative $f'(z)$.

Corollary 11.6 ([22])**:** *For a quaternion $z = x_0 + x_1i_1 + x_2i_2 + x_3i_3$, we have $dz^n = z^{n-1} \cdot dz + z^{n-2}dz \cdot z + z^{n-3}dz \cdot z^2 + \cdots + zdz \cdot z^{n-2} + dz \cdot z^{n-1}$ for all $n = 0, 1, 2, \ldots$.*

Corollary 11.7 ([22])**:** *For the partial derivatives of the functions $f_n(z) = z^n$, $n = 0, 1, 2, \ldots,$ with respect to real variables x_k, $k = 0, 1, 2, 3$, we have $(z^n)'_{x_k} = z^{n-1} \cdot i_k + z^{n-2} \cdot i_k \cdot z + \cdots + z \cdot i_k \cdot z^{n-2} + i_k \cdot z^{n-1}$.*

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