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ON THE BEHAVIOUR OF SERIES, OBTAINED BY TERMWISE INTEGRATION OF DOUBLE TRIGONOMETRIC SERIES

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Abstract. It is shown that for any 2π periodic in each variable function f of two variables, summable on the square $[0,2\pi]^2$, integrating its double trigonometric Fourier series on the rectangle $[0,x]\times[0,y]$ termwise gives a series that converges uniformly on $[0,2\pi]^2$ to the integral $\int_0^x\int_0^y f(t,\tau)\,dt\,d\tau$. Moreover, the convergence of the series $\sum_{m,n=1}^{\infty}\frac{b_{mn}}{mn}$, where b_{mn} is the Fourier coefficient of the function f for the $\sin mx \sin ny$ term, is obtained and its sum is found.

რეზიუმე. ნაშრომში დამტკიცებულია, რომ თითოეული ცვლადით 2π პერიოდული და $[0,2\pi]^2$ კვადრატზე ჯამებადი ყოველი f ფუნქციის ფურიეს ორმაგი ტრიგონომეტრიული მწკრივის $[0,x]\times[0,y]$ მართკუთხედზე წევრობრივი ინტეგრებით მიღებული მწკრივი თანაბრად კრებადია $[0,2\pi]^2$ კვადრატზე $\int_0^x\int_0^y f(t,\tau)\,dt\,d\tau$ ინტეგრალისკენ. გარდა ამისა, დადგენილია $\sum_{m,n=1}^{\infty}\frac{b_{mn}}{mn}$ მწკრივის კრებადობა და ნაპოვნია მისი ჯამი, სადაც b_{mn} აღიშნავს f ფუნქციის ფურიეს კოეფიციენტს $\sin mx \sin ny$ წევრისთვის.

1. Introduction

There are well-known examples of trigonometric Fourier series of two summable on $[0,2\pi]$ functions, of which one series diverges at any point [1, Ch. V, §2], [9, vol. I, Ch. VII, §2] and the other series does not converge in the metric of the space of summable functions on the same interval $[0,2\pi]$ [1, Ch. VIII, §2], [9, vol. I, Ch. VII, §5]. At the same time, the trigonometric Fourier series of all summable functions on $[0,2\pi]$ have the remarkable property stated by the following Lebesgue theorem L.

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Theorem L. [1, Ch. I, §10 and Ch. II, §9], [9, Ch. II, §8 and Miscellaneous Theorems and Examples to Ch. II]. *If f is any 2π periodic function, summable on the segment $[0, 2\pi]$, then the series obtained by integrating its Fourier trigonometric series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ termwise on the segment $[0, x]$, converges uniformly on $[0, 2\pi]$ to the integral $\int_0^x f(t)dt$ (even if the latter series is divergent everywhere) and moreover $\sum_{n=1}^{\infty} \frac{b_n}{n} = \frac{1}{2\pi}\int_0^{2\pi} f(x)(\pi - x)dx$.*

In the present paper, the two assertions of Theorem L are generalized to double Trigonometric Fourier series. In this context, we note that there is a function of two variables that is 2π periodic in each variable and continuous on the square $[0, 2\pi]^2$ and whose double trigonometric Fourier series is divergent everywhere in the sense of Pringsheim[1] [3].

Our proof is based on the following well-known theorem of Hardy:

Theorem H. [6] *If F is a continuous function of two variables that is 2π periodic in each variable and of bounded variation on $[0, 2\pi]^2$ in the sense of Hardy[2], then the double trigonometric Fourier series of F converges uniformly on $[0, 2\pi]^2$ to the function F.*

We should note that various generalizations of Hardy's theorem are given in [4], [5].

2. Auxiliary Results

We begin by the following lemma, whose easy proof is left to the reader.

Lemma. *Let f be a function that is summable on $[0, 2\pi]^2$ and 2π periodic in each variable. If*

$$\int_0^{2\pi}\int_0^{2\pi} f(x,y)\,dx\,dy = 0,$$

[1]A double series is said to converge in the sense of Pringsheim if its partial rectangular sums converge (see, e.g., [9, Ch. XVII].

[2]Let us consider the rectangle $Q = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ c \le y \le d\}$, which is decomposed into rectangles $Q_{ik} = [x_i \le x < x_{i+1},\ y_k \le y < y_{k+1}]$, where $0 \le i \le m-1$, $0 \le k \le n-1$ and $a = x_0 < x_1 < \cdots < x_m = b$, $c = y_0 < y_1 < \cdots < y_n = d$. For the finite function f given on the rectangle Q and for the system $\{Q_{ik}\}$, we compose the sum

$$S_f = \sum_{i=0}^{m-1}\sum_{k=0}^{n-1} |f(x_{i+1}, y_{k+1}) - f(x_{i+1}, y_k) - f(x_i, y_{k+1}) + f(x_i, y_k)|. \tag{1.1}$$

If there exists a finite number $M > 0$ such that $S_f \le M$ for any system $\{Q_{ik}\}$, then f is called a function of bounded variation in the Vitali sense on Q. If the function f of bounded variation in the Vitali sense has an additional property that $f(x,c)$ and $f(a,y)$ are the functions of bounded variations on the intervals $[a,b]$ and $[c,d]$, respectively, then f is called a function of bounded variation in the Hardy sense [7, §254].

then the function

$$F(x,y)=\int\limits_0^x\int\limits_0^y f(t,\tau)\,dt\,d\tau - ya(x) - xb(y),$$

where

$$a(x)=\frac{1}{2\pi}\int\limits_0^x dt\int\limits_0^{2\pi} f(t,\tau)\,d\tau \quad \textit{and} \quad b(y)=\frac{1}{2\pi}\int\limits_0^y d\tau\int\limits_0^{2\pi} f(t,\tau)\,dt,$$

is 2π *periodic in each variable. Moreover,* $F(x,y)$ *is zero on the boundary of the square* $[0,2\pi]^2$*, that is*

$$\begin{gathered} F(x,0)=0,\ \ F(0,y)=0,\ \ F(x,2\pi)=0,\ \ F(2\pi,y)=0 \\ \textit{for all } 0\le x\le 2\pi \textit{ and all } 0\le y\le 2\pi. \end{gathered} \tag{2.1}$$

Suppose now that f is a function satisfying the conditions of the lemma and consider the corresponding double trigonometric Fourier series[3]

$$\begin{aligned} f \sim \frac{1}{2}\sum_{m=1}^{\infty}(a_{m0}\cos mx + d_{m0}\sin mx) + \frac{1}{2}\sum_{m=1}^{\infty}(a_{0n}\cos ny + c_{0n}\sin ny) + \\ + \sum_{m,n=1}^{\infty}(a_{mn}\cos mx\cos ny + b_{mn}\sin mx\sin ny + \\ + c_{mn}\cos mx\sin ny + d_{mn}\sin mx\cos ny)\,. \end{aligned} \tag{2.2}$$

Then clearly $\frac{1}{2}a_{m0}$, $\frac{1}{2}d_{m0}$ and $\frac{1}{2}a_{0n}$, $\frac{1}{2}c_{0n}$ are, respectively, the Fourier coefficients of the functions

$$\varphi(x)=\frac{1}{2\pi}\int\limits_0^{2\pi} f(x,y)\,dy \quad \text{and} \quad \psi(y)=\frac{1}{2\pi}\int\limits_0^{2\pi} f(x,y)\,dx \tag{2.3}$$

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$$a_{mn}=\frac{1}{\pi^2}\int\limits_0^{2\pi}\int\limits_0^{2\pi} f(x,y)\cos mx\cos ny\,dx\,dy,$$

$$b_{mn}=\frac{1}{\pi^2}\int\limits_0^{2\pi}\int\limits_0^{2\pi} f(x,y)\sin mx\sin ny\,dx\,dy,$$

$$c_{mn}=\frac{1}{\pi^2}\int\limits_0^{2\pi}\int\limits_0^{2\pi} f(x,y)\cos mx\sin ny\,dx\,dy,$$

$$d_{mn}=\frac{1}{\pi^2}\int\limits_0^{2\pi}\int\limits_0^{2\pi} f(x,y)\sin mx\cos ny\,dx\,dy.$$

both of which are 2π periodic and summable on the segment $[0,2\pi]$. Thus

$$\varphi \sim \frac{1}{2}\sum_{m=1}^{\infty}(a_{m0}\cos mx + d_{m0}\sin mx)$$

and (2.4)

$$\psi \sim \frac{1}{2}\sum_{n=1}^{\infty}(a_{0n}\cos ny + c_{0n}\sin ny)\,.$$

Therefore

$$a(x)=\int_0^x \varphi(t)\,dt, \quad b(y)=\int_0^y \psi(\tau)\,d\tau,$$

$$F(x,y)=\int_0^x\int_0^y f(t,\tau)\,dt\,d\tau - y\int_0^x \varphi(t)\,dt - x\int_0^y \psi(\tau)\,d\tau. \tag{2.5}$$

Moreover, it follows from Theorem L that

$$\int_0^x \varphi(t)\,dt = \frac{1}{2}\sum_{m=1}^{\infty}\int_0^x (a_{m0}\cos mt + d_{m0}\sin mt)\,dt, \tag{2.6}$$

$$\int_0^y \psi(\tau)\,d\tau = \frac{1}{2}\sum_{n=1}^{\infty}\int_0^y (a_{0n}\cos n\tau + c_{0n}\sin n\tau)\,d\tau \tag{2.7}$$

uniformly on $[0,2\pi]$.

3. Main results

Theorem 1. *Relation* (2.2) *implies that*

$$\int_0^x\int_0^y f(t,\tau)\,dt\,d\tau - y\int_0^x \varphi(t)\,dt - x\int_0^y \psi(\tau)\,d\tau =$$

$$= \sum_{m,n=1}^{\infty}\int_0^x\int_0^y (a_{mn}\cos mt\cos n\tau + b_{mn}\sin mt\sin n\tau +$$

$$+c_{mn}\cos mt\sin n\tau + d_{mn}\sin mt\cos n\tau)\,dt\,d\tau \tag{3.1}$$

uniformly on $[0,2\pi]^2$ (*in the sense of Pringsheim*).

Proof. According to Theorem H, if the function $F(x,y)$ is defined as in (2.5), then[4]

$$F(x,y) = \frac{1}{4} A_{00} + \frac{1}{2} \sum_{m=1}^{\infty} (A_{m0} \cos mx + D_{m0} \sin mx) + \\ + \frac{1}{2} \sum_{n=1}^{\infty} (A_{0n} \cos ny + C_{0n} \sin ny) + \\ + \sum_{m,n=1}^{\infty} (A_{mn} \cos mx \cos ny + B_{mn} \sin mx \sin ny + \\ + C_{mn} \cos mx \sin ny + D_{mn} \sin mx \cos ny) \quad (3.2)$$

uniformly on $[0,2\pi]^2$. Putting here $x=0$ and $y=0$, we obtain

$$\frac{1}{4} A_{00} = -\frac{1}{2} \sum_{m=1}^{\infty} A_{m0} - \frac{1}{2} \sum_{n=1}^{\infty} A_{0n} - \sum_{m,n=1}^{\infty} A_{mn}. \quad (3.3)$$

The coefficients in (3.2) are calculated by a theorem due to Tolstov [4, pp. 48–49] which for the function F defined by the equality (2.5) is formulated as follows:

(1) the equality

$$F'_x(x,y) = \int_0^y f(x,\tau)\,d\tau - y\varphi(x) - \int_0^y \psi(\tau)\,d\tau \quad (3.4)$$

holds for almost all x and all y;

(2) we have

$$F'_y(x,y) = \int_0^x f(t,y)\,dt - \int_0^x \varphi(t)\,dt - x\psi(y) \quad (3.5)$$

for all x and almost all y;

(3) the relations

$$F''_{xy}(x,y) = f(x,y) - \varphi(x) - \psi(y) = F''_{yx}(x,y) \quad (3.6)$$

[4]Writing the function F from equality (2.5) in the form

$$F(x,y) = \int_0^x \int_0^y w(t,\tau)\,dt\,d\tau$$

from equality (1.1) we have

$$S_F = \sum_{i=0}^{m-1} \sum_{k=0}^{n-1} \Big| \int_{Q_{ik}} w(t,\tau)\,dt\,d\tau \Big| \le \sum_{i=0}^{m-1} \sum_{k=0}^{n-1} \int_{Q_{ik}} |w(t,\tau)|\,dt\,d\tau = \int_Q |w(t,\tau)|\,dt\,d\tau < +\infty.$$

Moreover, the function $F(x,0)=0$ and $F(0,y)=0$ are functions of bounded variation on $[0,2\pi]$.

holds for almost all (x,y).

In the first place we will prove the following equalities:

$$A_{mn}=\frac{b_{mn}}{mn}\,,\quad B_{mn}=\frac{a_{mn}}{mn}\,,\quad C_{mn}=-\frac{d_{mn}}{mn}\,,\quad D_{mn}=-\frac{c_{mn}}{mn} \tag{3.7}$$
$$(m,n=1,2,\dots).$$

We have

$$\pi^2A_{mn}=\int\limits_0^{2\pi}\Big[\int\limits_0^{2\pi}F(x,y)\cos mx\,dx\Big]\cos ny\,dy.$$

According to the equality (3.4) we have

$$\int\limits_0^{2\pi}F(x,y)\cos mx\,dx=$$
$$=-\frac{1}{m}\int\limits_0^{2\pi}\Big[\int\limits_0^{y}f(x,\tau)\,d\tau-y\varphi(x)-\int\limits_0^{y}\psi(\tau)\,d\tau\Big]\sin mx\,dx.$$

Therefore

$$\pi^2mA_{mn}=-\int\limits_0^{2\pi}\Big\{\int\limits_0^{2\pi}\Big[\int\limits_0^{y}f(x,\tau)\,d\tau-y\varphi(x)-$$
$$-\int\limits_0^{y}\psi(\tau)\,d\tau\Big]\sin mx\,dx\Big\}\cos ny\,dy.$$

But

$$\int\limits_0^{2\pi}\Big[\int\limits_0^{y}f(x,\tau)\,d\tau-y\varphi(x)-\int\limits_0^{y}\psi(\tau)\,d\tau\Big]\sin mx\,dx=$$
$$=\int\limits_0^{2\pi}\Big[\int\limits_0^{y}f(x,\tau)\,d\tau\Big]\sin mx\,dx-y\int\limits_0^{2\pi}\varphi(x)\sin mx\,dx-$$
$$-\int\limits_0^{2\pi}\Big[\int\limits_0^{y}\psi(\tau)\,d\tau\Big]\sin mx\,dx$$

and

$$\int\limits_0^{2\pi}\varphi(x)\sin mx\,dx=\frac{\pi}{2}\,d_{m0}\quad\Big(\text{since }\ \frac{1}{2}\,d_{m0}=\frac{1}{\pi}\int\limits_0^{2\pi}\varphi(x)\sin mx\,dx\Big).$$

Thus

$$\pi^2 m A_{mn} = -\int_0^{2\pi}\left\{\int_0^{2\pi}\left[\int_0^y f(x,\tau)\,d\tau\right]\sin mx\,dx\right\}\cos ny\,dy + \\ + \frac{\pi}{2}\,d_{m0}\int_0^{2\pi} y\cos ny\,dy + \int_0^{2\pi}\left\{\int_0^{2\pi}\left[\int_0^y \psi(\tau)\,d\tau\right]\sin mx\,dx\right\}\cos ny\,dy.$$

Since

$$\int_0^{2\pi} y\cos ny\,dy = 0, \tag{3.8}$$

we obtain

$$\pi^2 m A_{mn} = -\int_0^{2\pi}\left\{\int_0^{2\pi}\left[\int_0^y f(x,\tau)\,d\tau\right]\cos ny\,dy\right\}\sin mx\,dx + \\ + \int_0^{2\pi}\left\{\int_0^{2\pi}\left[\int_0^y \psi(\tau)\,d\tau\right]\cos ny\,dy\right\}\sin mx\,dx \equiv -I_1 + I_2.$$

We have

$$I_1 = \int_0^{2\pi}\left\{\frac{1}{n}\,\sin ny\cdot\int_0^y f(x,\tau)\,d\tau\Big|_{y=0}^{y=2\pi} - \frac{1}{n}\int_0^{2\pi} f(x,y)\sin ny\,dy\right\}\sin mx\,dx = \\ = -\frac{1}{n}\int_0^{2\pi}\int_0^{2\pi} f(x,y)\sin mx\sin ny\,dx\,dy = -\frac{1}{n}\,\pi^2 b_{mn}\,,$$

$$I_2 = \int_0^{2\pi}\left\{\frac{1}{n}\,\sin ny\cdot\int_0^y \psi(\tau)\,d\tau\Big|_{y=0}^{y=2\pi} - \frac{1}{n}\int_0^{2\pi}\psi(y)\sin ny\,dy\right\}\sin mx\,dx = \\ = -\frac{1}{n}\int_0^{2\pi}\left\{\int_0^{2\pi}\psi(y)\sin ny\,dy\right\}\sin mx\,dx = \\ = -\frac{1}{n}\int_0^{2\pi}\left\{\frac{\pi}{2}\,c_{0n}\right\}\sin mx\,dx = 0.$$

Therefore

$$A_{mn} = \frac{1}{mn}\,b_{mn} \quad (m,n = 1,2,\dots). \tag{3.9}$$

The other equalities from (3.7) are proved in an analogous manner. This means that the double series from (3.2) has the form

$$\sum_{m,n=1}^{\infty} \frac{1}{mn}\big[b_{mn}\cos mx\cos ny + a_{mn}\sin mx\sin ny-$$
$$- d_{mn}\cos mx\sin ny - c_{mn}\sin mx\cos ny\big]. \qquad (3.10)$$

Let us now prove the equalities $(m,n=1,2,\dots)$

$$\begin{aligned} A_{m0} &= \frac{1}{m}\,\beta_m - \pi\,\frac{d_{m0}}{m}\,, \quad D_{m0} = \pi\,\frac{d_{m0}}{m} - \frac{1}{m}\,\alpha_m,\\ A_{0n} &= \frac{1}{n}\,\delta_n - \pi\,\frac{c_{0n}}{n}\,, \qquad C_{0n} = \pi\,\frac{a_{0n}}{n} - \frac{1}{n}\,\gamma_n, \end{aligned} \qquad (3.11)$$

where

$$\begin{aligned} \alpha_m &= \frac{1}{\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi} yf(x,y)\cos mx\,dx\,dy,\\ \beta_m &= \frac{1}{\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi} yf(x,y)\sin mx\,dx\,dy,\\ \gamma_n &= \frac{1}{\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi} xf(x,y)\cos ny\,dx\,dy,\\ \delta_n &= \frac{1}{\pi^2}\int_0^{2\pi}\!\!\int_0^{2\pi} xf(x,y)\sin ny\,dx\,dy. \end{aligned} \qquad (3.12)$$

Since

$$\pi^2 A_{m0} = \int_0^{2\pi}\Big[\int_0^{2\pi} F(x,y)\cos mx\,dx\Big]dy$$

and

$$\int_0^{2\pi} F(x,y)\cos mx\,dx =$$
$$= -\frac{1}{m}\int_0^{2\pi}\Big[\int_0^{y} f(x,\tau)\,d\tau - y\varphi(x) - \int_0^{y}\psi(\tau)\,d\tau\Big]\sin mx\,dx,$$

we obtain

$$-\pi^2 mA_{m0} = \int_0^{2\pi}\Big\{\int_0^{2\pi}\Big[\int_0^{y} f(x,\tau)\,d\tau - y\varphi(x) - \int_0^{y}\psi(\tau)\,d\tau\Big]\sin mx\,dx\Big\}dy =$$

$$= \int_0^{2\pi} \left\{ \int_0^{2\pi} \left[\int_0^y f(x,\tau)\,d\tau - y\varphi(x) - \int_0^y \psi(\tau)\,d\tau \right] dy \right\} \sin mx\,dx =$$

$$= \int_0^{2\pi} \left\{ y \left[\int_0^y f(x,\tau)\,d\tau - y\varphi(x) - \int_0^y \psi(\tau)\,d\tau \right] \Bigg|_{y=0}^{y=2\pi} = \right.$$

$$\left. - \int_0^{2\pi} y[f(x,y) - \varphi(x) - \psi(y)]dy \right\} \sin mx\,dx =$$

$$= \int_0^{2\pi} \left\{ 2\pi \left[\int_0^{2\pi} f(x,y)\,dy - 2\pi\varphi(x) - \int_0^{2\pi} \psi(y)\,dy \right] - \right.$$

$$\left. - \int_0^{2\pi} [yf(x,y) - y\varphi(x) - y\psi(y)]\,dy \right\} \sin mx\,dx \equiv J_1 - J_2.$$

Furthermore

$$J_1 = 2\pi \int_0^{2\pi}\int_0^{2\pi} f(x,y) \sin mx\,dx\,dy - 4\pi^2 \int_0^{2\pi} \varphi(x) \sin mx\,dx -$$

$$- 2\pi \int_0^{2\pi} \left(\int_0^{2\pi} \psi(y)\,dy \right) \sin mx\,dx = 2\pi \cdot \pi^2 d_{m0} - 4\pi^2 \cdot \frac{\pi}{2}\,d_{m0} = 0,$$

$$J_2 = \int_0^{2\pi} \left\{ \int_0^{2\pi} yf(x,y)\,dy - \varphi(x) \int_0^{2\pi} y\,dy - \int_0^{2\pi} y\psi(y)\,dy \right\} \sin mx\,dx =$$

$$= \int_0^{2\pi}\int_0^{2\pi} yf(x,y) \sin mx\,dx\,dy - 2\pi^2 \int_0^{2\pi} \varphi(x) \sin mx\,dx -$$

$$- \left(\int_0^{2\pi} y\psi(y)\,dy \right) \int_0^{2\pi} \sin mx\,dx =$$

$$= \int_0^{2\pi}\int_0^{2\pi} yf(x,y) \sin mx\,dx\,dy - \pi^3 d_{m0}\,.$$

We have thereby established the first equality in (3.11), whereas the other equalities are proved analogously.

We now use (3.3) to represent the right side of (3.2) as a sum $A+B+C$:

$$A=-\frac{1}{2}\sum_{m=1}^{\infty}A_{m0}+\frac{1}{2}\sum_{m=1}^{\infty}(A_{m0}\cos mx+D_{m0}\sin mx)=$$
$$=-\frac{1}{2}\sum_{m=1}^{\infty}\frac{\beta_m-\pi d_{m0}}{m}+$$
$$+\frac{1}{2}\sum_{m=1}^{\infty}\left[(\beta_m-\pi d_{m0})\frac{\cos mx}{m}+(\pi a_{m0}-\alpha_m)\frac{\sin mx}{m}\right]=$$
$$=-\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}(\beta_m-\pi d_{m0})+\frac{1}{2}\sum_{m=1}^{\infty}\left[(\beta_m-\pi d_{m0})\left(\frac{1}{m}-\int_0^x\sin mt\,dt\right)+\right.$$
$$\left.+(\pi a_{m0}-\alpha_m)\int_0^x\cos mt\,dt\right]=$$
$$=\frac{1}{2}\sum_{m=1}^{\infty}\left[(\pi d_{m0}-\beta_m)\int_0^x\sin mt\,dt+(\pi a_{m0}-\alpha_m)\int_0^x\cos mt\,dt\right]. \tag{3.13}$$

Since

$$\frac{1}{\pi}\int_0^{2\pi}yf(x,y)\,dy\sim(\alpha_m,\beta_m)\ \text{ and }\ \frac{1}{\pi}\int_0^{2\pi}xf(x,y)\,dx\sim(\gamma_n,\delta_n), \tag{3.14}$$

one can apply Theorem L to obtain

$$\int_0^x\left(\frac{1}{\pi}\int_0^{2\pi}\tau f(t,\tau)\,d\tau\right)dt-\frac{\alpha_0}{2}\,x=\frac{1}{2}\sum_{m=1}^{\infty}\int_0^x(\alpha_m\cos mt+\beta_m\sin mt)\,dt,$$
$$\int_0^y\left(\frac{1}{\pi}\int_0^{2\pi}tf(t,\tau)\,dt\right)d\tau-\frac{\gamma_0}{2}\,y=\frac{1}{2}\sum_{n=1}^{\infty}\int_0^y(\gamma_n\cos n\tau+\delta_n\sin n\tau)\,d\tau$$

uniformly on $[0,2\pi]$, where

$$\alpha_0=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}yf(x,y)\,dx\,dy,$$
$$\gamma_0=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}xf(x,y)\,dx\,dy. \tag{3.15}$$

Since both series in the last two equalities are convergent, the series (3.13) can be expressed as the sum of two series. Thus,

$$A = \frac{1}{2}\,\pi \sum_{m=1}^{\infty} \int\limits_0^x (a_{m0} \cos mt + d_{m0} \sin mt) dt -$$
$$- \int\limits_0^x \left(\frac{1}{\pi} \int\limits_0^{2\pi} \tau f(t,\tau)\, d\tau \right) dt + \frac{\alpha_0}{2}\, x\,. \tag{3.16}$$

Similarly, we find that

$$B = \frac{1}{2}\,\pi \sum_{n=1}^{\infty} \int\limits_0^y (a_{0n} \cos n\tau + c_{0n} \sin n\tau) d\tau -$$
$$- \int\limits_0^y \left(\frac{1}{\pi} \int\limits_0^{2\pi} t f(t,\tau)\, dt \right) d\tau + \frac{\gamma_0}{2}\, y\,. \tag{3.17}$$

Finally,

$$C = - \sum_{m,n=1}^{\infty} A_{mn} + \sum_{m,n=1}^{\infty} \left[\frac{b_{mn}}{mn} \cos mx \cos ny + \frac{a_{mn}}{mn} \sin mx \sin ny - \right.$$
$$\left. - \frac{d_{mn}}{mn} \cos mx \sin ny - \frac{c_{mn}}{mn} \sin mx \cos ny \right] =$$
$$= - \sum_{m,n=1}^{\infty} A_{mn} + \sum_{m,n=1}^{\infty} \left[b_{mn} \left(\frac{1}{m} - \int\limits_0^x \sin mt\, dt \right) \left(\frac{1}{n} - \int\limits_0^y \sin n\tau\, d\tau \right) + \right.$$
$$+ a_{mn} \int\limits_0^x \cos mt\, dt \int\limits_0^y \cos n\tau\, d\tau -$$
$$- d_{mn} \left(\frac{1}{m} - \int\limits_0^x \sin mt\, dt \right) \int\limits_0^y \cos n\tau\, d\tau -$$
$$\left. - c_{mn} \int\limits_0^x \cos mt\, dt \left(\frac{1}{n} - \int\limits_0^y \sin n\tau\, d\tau \right) \right] =$$
$$= - \sum_{m,n=1}^{\infty} \frac{b_{mn}}{mn} + \sum_{m,n=1}^{\infty} \left\{ \int\limits_0^x \int\limits_0^y \big[a_{mn} \cos mt \cos n\tau + b_{mn} \sin mt \sin m\tau + \right.$$
$$+ c_{mn} \cos mt \sin n\tau + d_{mn} \sin mt \cos n\tau \big] dt\, d\tau +$$

$$+\frac{b_{mn}}{mn}-b_{mn}\cdot\frac{1}{n}\int\limits_0^x \sin mt\,dt-b_{mn}\cdot\frac{1}{m}\int\limits_0^y \sin n\tau\,d\tau-$$

$$-c_{mn}\cdot\frac{1}{n}\int\limits_0^x \cos mt\,dt-d_{mn}\cdot\frac{1}{m}\int\limits_0^y \cos n\tau\,d\tau\bigg\}.$$

Therefore,

$$C=\sum_{m,n=1}^{\infty}\bigg\{\int\limits_0^x\int\limits_0^y\big[a_{mn}\cos mt\cos n\tau+b_{mn}\sin mt\sin n\tau+$$

$$+c_{mn}\cos mt\sin n\tau+d_{mn}\sin mt\cos n\tau\Big]dt\,d\tau-$$

$$-\frac{1}{n}\int\limits_0^x (c_{mn}\cos mt+b_{mn}\sin nt)dt-$$

$$-\frac{1}{m}\int\limits_0^y (d_{mn}\cos n\tau+b_{mn}\sin n\tau)d\tau\bigg\}. \quad (3.18)$$

Applying now (3.2) and (3.16)–(3.18) gives

$$F(x,y)=\pi\cdot\frac{1}{2}\sum_{m=1}^{\infty}\int\limits_0^x (a_{m0}\cos mt+d_{m0}\sin mt)dt+$$

$$+\pi\cdot\frac{1}{2}\sum_{n=1}^{\infty}\int\limits_0^y (a_{0n}\cos n\tau+c_{0n}\sin n\tau)d\tau+$$

$$+\sum_{m,n=1}^{\infty}\bigg\{\int\limits_0^x\int\limits_0^y\big[a_{mn}\cos mt\cos n\tau+b_{mn}\sin mt\sin n\tau+$$

$$+c_{mn}\cos mt\sin n\tau+d_{mn}\sin mt\cos n\tau\big]dt\,d\tau-$$

$$-\bigg[\int\limits_0^x\bigg(\frac{c_{mn}}{n}\cos mt+\frac{b_{mn}}{n}\sin mt\bigg)dt+$$

$$+\int\limits_0^y\bigg(\frac{d_{mn}}{m}\cos n\tau+\frac{b_{mn}}{m}\sin n\tau\bigg)d\tau\bigg]\bigg\}-$$

$$-\int\limits_0^x\bigg(\frac{1}{\pi}\int\limits_0^{2\pi}\tau f(t,\tau)\,d\tau\bigg)dt+\frac{\alpha_0}{2}\,x-$$

$$-\int_0^y \left(\frac{1}{\pi}\int_0^{2\pi} tf(t,\tau)\,dt\right)d\tau + \frac{\gamma_0}{2}\,y. \tag{3.19}$$

Then it follows from (2.5)–(2.7) and (3.19) that

$$\begin{aligned}
\int_0^x\int_0^y f(t,\tau)\,dt\,d\tau &= (\pi+y)\,\frac{1}{2}\sum_{m=1}^{\infty}\int_0^x (a_{m0}\cos mt + d_{m0}\sin mt)dt + \\
&+ (\pi+x)\,\frac{1}{2}\sum_{n=1}^{\infty}\int_0^y (a_{0n}\cos n\tau + c_{0n}\sin n\tau)d\tau + \\
&+ \sum_{m,n=1}^{\infty}\Bigg\{\int_0^x\int_0^y \big[a_{mn}\cos mt\cos n\tau + b_{mn}\sin mt\sin n\tau + \\
&\qquad + c_{mn}\cos mt\sin n\tau + d_{mn}\sin mt\cos n\tau\big]\,dt\,d\tau - \\
&- \Bigg[\int_0^x \left(\frac{c_{mn}}{n}\cos mt + \frac{b_{mn}}{n}\sin mt\right)dt + \\
&\qquad + \int_0^y \left(\frac{d_{mn}}{m}\cos n\tau + \frac{b_{mn}}{m}\sin n\tau\right)d\tau\Bigg]\Bigg\} - \\
&- \int_0^x \left(\frac{1}{\pi}\int_0^{2\pi} \tau f(t,\tau)\,d\tau\right)dt - \int_0^y \left(\frac{1}{\pi}\int_0^{2\pi} tf(t,\tau)\,dt\right)d\tau + \\
&\qquad + \frac{\alpha_0}{2}\,x + \frac{\gamma_0}{2}\,y.
\end{aligned} \tag{3.20}$$

In particular, putting $y=0$ and $x=0$ respectively in (3.20), we obtain

$$\begin{aligned}
\sum_{m,n=1}^{\infty}\int_0^x \left(\frac{c_{mn}}{n}\cos mt + \frac{b_{mn}}{n}\sin mt\right)dt &= \\
= \frac{\pi}{2}\sum_{m=1}^{\infty}\int_0^x (a_{m0}\cos mt + d_{m0}\sin mt)dt &- \\
- \int_0^x \left(\frac{1}{\pi}\int_0^{2\pi} \tau f(t,\tau)\,d\tau\right)dt + \frac{\alpha_0}{2}\,x &
\end{aligned} \tag{3.21}$$

and

$$\sum_{m,n=1}^{\infty}\int_0^y\left(\frac{d_{mn}}{m}\cos n\tau+\frac{b_{mn}}{m}\sin n\tau\right)d\tau=$$

$$=\frac{\pi}{2}\sum_{n=1}^{\infty}\int_0^y(a_{0n}\cos n\tau+c_{0n}\sin n\tau)d\tau-$$

$$-\int_0^y\left(\frac{1}{\pi}\int_0^{2\pi}tf(t,\tau)\,dt\right)d\tau+\frac{\gamma_0}{2}\,y. \tag{3.22}$$

Hence (3.20) can be rewritten as follows:

$$\int_0^x\int_0^y f(t,\tau)\,dt\,d\tau=y\,\frac{1}{2}\sum_{m=1}^{\infty}\int_0^x(a_{m0}\cos mt+d_{m0}\sin mt)dt+$$

$$+\,x\,\frac{1}{2}\sum_{n=1}^{\infty}\int_0^y(a_{0n}\cos n\tau+c_{0n}\sin n\tau)d\tau+$$

$$+\sum_{m,n=1}^{\infty}\int_0^x\int_0^y\big[a_{mn}\cos mt\cos n\tau+b_{mn}\sin mt\sin n\tau+$$

$$+\,c_{mn}\cos mt\sin n\tau+d_{mn}\sin mt\cos n\tau\big]\,dt\,d\tau. \tag{3.23}$$

We now immediately see, using (2.6) and (2.7), that (3.23) and (3.1) are equivalent. This completes the proof of Theorem 1. □

Theorem 2. [5] *If f is any 2π periodic in each variable function, summable on the square $[0,2\pi]^2$, and if*

$$f\sim\frac{a_{00}}{4}+\frac{1}{2}\sum_{m=1}^{\infty}(a_{m0}\cos mx+d_{m0}\sin mx)+\frac{1}{2}\sum_{n=1}^{\infty}(a_{0n}\cos ny+c_{0n}\sin ny)+$$

$$+\sum_{m,n=1}^{\infty}\big(a_{mn}\cos mx\cos ny+b_{mn}\sin mx\sin ny+$$

$$+\,c_{mn}\cos mx\sin ny+d_{mn}\sin mx\cos ny\big)\,, \tag{3.24}$$

then the equality

$$\int_0^x\int_0^y f(t,\tau)\,dt\,d\tau=\frac{a_{00}}{4}\,xy+\frac{y}{2}\sum_{m=1}^{\infty}\int_0^x(a_{m0}\cos mt+d_{m0}\sin mt)dt+$$

[5]This result was announced in the author's paper [2].

$$+\frac{x}{2}\sum_{n=1}^{\infty}\int_0^y (a_{0n}\cos n\tau + c_{0n}\sin n\tau)d\tau +$$

$$+\sum_{m,n=1}^{\infty}\int_0^x\int_0^y \big[a_{mn}\cos mt\cos n\tau + b_{mn}\sin mt\sin n\tau +$$

$$+ c_{mn}\cos mt\sin n\tau + d_{mn}\sin mt\cos n\tau\big]\,dt\,d\tau \qquad (3.25)$$

is fulfilled uniformly on $[0,2\pi]^2$*.*

Proof. It is clear that the series on the right side of (3.24) without the first term $\frac{a_{00}}{4}$ is the Fourier series of the function $f_1(x,y) = f(x,y) - \frac{a_{00}}{4}$ and that $f_1(x,y)$ satisfies the conditions of the lemma. The desired result now follows from (3.23) on replacing $f(x,y)$ by $f_1(x,y)$. □

4. Corollaries

Taking into account (3.7) and (3.8), (3.3) can be rewritten in the following form:

$$\frac{1}{4}\,A'_{00} + \frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}\,\beta_m - \frac{\pi}{2}\sum_{m=1}^{\infty}\frac{1}{m}\,d_{m0} +$$

$$+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,\delta_n - \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,c_{0n} + \sum_{m,n=1}^{\infty}\frac{b_{mn}}{mn} = 0, \quad (4.1)$$

where A'_{00} is calculated for the function $f_1 = f - \frac{1}{4}\,a_{00}$, provided that f does not satisfy the condition of the lemma. To calculate the coefficient, we must use the functions

$$a_1(x) = a(x) - \frac{1}{4}\,a_{00}x,\quad b_1(y) = b(y) - \frac{1}{4}\,a_{00}y,\quad F_1(x,y) = F(x,y) + \frac{1}{4}\,a_{00}xy$$

and then we get the equality $A'_{00} = A_{00} + a_{00}\pi^2$.

Note that each one-dimensional series in the left-side of (4.1) is convergent by the following well-known equality

$$\sum_{n=1}^{\infty}\frac{b_n}{n} = \frac{1}{2\pi}\int_0^{2\pi}\lambda(u)(\pi - u)\,du,$$

where

$$\lambda \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nu + b_n\sin nu).$$

Specializing to the present situation and using (3.14), we obtain

$$\sum_{m=1}^{\infty}\frac{1}{m}\,\beta_m=\frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{1}{\pi}\int_0^{2\pi}\tau f(t,\tau)\,d\tau\Big)(\pi-t)\,dt\,,\tag{4.2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\,\delta_n=\frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{1}{\pi}\int_0^{2\pi}tf(t,\tau)\,dt\Big)(\pi-\tau)\,d\tau\,,\tag{4.3}$$

$$\sum_{m=1}^{\infty}\frac{1}{m}\,d_{m0}=\frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{1}{\pi}\int_0^{2\pi}f(t,\tau)\,d\tau\Big)(\pi-t)\,dt\,,\tag{4.4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\,c_{0n}=\frac{1}{2\pi}\int_0^{2\pi}\Big(\frac{1}{\pi}\int_0^{2\pi}f(t,\tau)\,dt\Big)(\pi-\tau)\,d\tau\,.\tag{4.5}$$

Corollary 1. *In the circumstances above, we have:*

$$\sum_{m,n=1}^{\infty}\frac{b_{mn}}{mn}=-\frac{1}{4}\,(A_{00}+a_{00}\pi^2)-\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}\,\beta_m+\frac{\pi}{2}\sum_{m=1}^{\infty}\frac{1}{m}\,d_{m0}-$$
$$-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,\delta_n+\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n}\,c_{0n}\,,\quad A_{00}=\frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi}F(x,y)\,dx\,dy.\tag{4.6}$$

Corollary 2. *For any function of two variables that is 2π periodic in each variable and summable on the square $[0,2\pi]^2$, we have:*

$$\sum_{m,n=1}^{\infty}\Big(\frac{c_{mn}}{mn}\,\sin mx-\frac{b_{mn}}{mn}\,\cos mx\Big)=-\sum_{m,n=1}^{\infty}\frac{b_{mn}}{mn}+$$
$$+\frac{1}{2}\sum_{m=1}^{\infty}(A_{m0}\cos mx+D_{m0}\sin mx-A_{m0}),\quad 0\le x\le 2\pi,\tag{4.7}$$

$$\sum_{m,n=1}^{\infty}\Big(\frac{d_{mn}}{mn}\,\sin ny-\frac{b_{mn}}{mn}\,\cos ny\Big)=-\sum_{m,n=1}^{\infty}\frac{b_{mn}}{mn}+$$
$$+\frac{1}{2}\sum_{n=1}^{\infty}(A_{0n}\cos ny+C_{0n}\sin ny-A_{0n}),\quad 0\le y\le 2\pi.\tag{4.8}$$

Proof. By the relation (3.14), the right hand-side of (3.21) equals to

$$\frac{\pi}{2}\sum_{m=1}^{\infty}\int_0^{x}(a_{m0}\cos mt+d_{m0}\sin mt)dt-$$

$$-\frac{1}{2}\sum_{m=1}^{\infty}\int_0^x(\alpha_m\cos mt+\beta_m\sin mt)dt=$$

$$=\frac{1}{2}\sum_{m=1}^{\infty}\int_0^x[(\pi a_{m0}-\alpha_m)\cos mt+(\pi d_{m0}-\beta_m)\sin mt]dt=$$

$$=\frac{1}{2}\sum_{m=1}^{\infty}\int_0^x(mD_{m0}\cos mt-mA_{m0}\sin mt)dt=$$

$$=\frac{1}{2}\sum_{m=1}^{\infty}(A_{m0}\cos mx+D_{m0}\sin mx-A_{m0}).$$

But the left hand-side of (3.21) can be written in the form

$$\sum_{m,n=1}^{\infty}\left(\frac{c_{mn}}{mn}\sin mx-\frac{b_{mn}}{mn}\cos mx+\frac{b_{mn}}{mn}\right),$$

whence it follows – since the series $\sum_{m,n=1}^{\infty}\frac{b_{mn}}{mn}$ is convergent – that the equation (4.7) holds. In a similar way, one can shows that the equation (4.8) also holds. □

Corollary 3. *The equation* (3.25) *can be used to determine functions from their Fourier coefficients almost everywhere on* $[0,2\pi]^2$*. In order to do this, it suffices to integrate the double trigonometric Fourier series whose coefficients are the given ones term-by-term over* $[0,x]\times[0,y]$*. Then the mixed partial derivative of the function so obtained equals to the given function almost everywhere* (*also see* (3.6)).

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