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$\mathbb{C}^2$-DIFFERENTIABILITY OF QUATERNION FUNCTIONS AND THEIR REPRESENTATION BY INTEGRALS AND SERIES

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Abstract. In the paper, the necessary and sufficient conditions are established for a quaternion function to be $\mathbb{C}^2$-differentiable or $\mathbb{C}^2$-holomorphic. The representations of $\mathbb{C}^2$-holomorphic quaternion functions by double integrals and double power series are obtained.

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1. Introduction

We consider a quaternion function $u = f(z)$ of the quaternion variable z, where $z = \sum_{k=0}^{3} x_k i_k$, $u(z) = \sum_{k=0}^{3} u_k(z) i_k$ and $i_0 = 1$, $i_1^2 = i_2^2 = i_3^2 = -1$, $i_1 i_2 = i_3 = -i_2 i_1$, $i_2 i_3 = i_1 = -i_3 i_2$, $i_3 i_1 = i_2 = -i_1 i_3$. After introducing the complex variables $z_1 = x_0 + x_1 i_1$ and $z_2 = x_2 + x_3 i_1$, the quaternion z takes the form

$$z = z_1 + z_2 i_2 \tag{1.1}$$

or, briefly, $z = (z_1, z_2)$. Hence the four-dimensional real Euclidean space $\mathbb{R}^4$ is identified with the two-dimensional complex space $\mathbb{C}^2$ having points $z = (z_1, z_2)$.

The conjugate quaternion $\overline{z} = x_0 - x_1 i_1 - x_2 i_2 - x_3 i_3$ will have the form $\overline{z} = \overline{z}_1 - z_2 i_2$, where $\overline{z}_1 = x_0 - x_1 i_1$. We also have the equality

$$z_2 i_2 = i_2 \overline{z}_2. \tag{1.2}$$

Therefore $\overline{z_1 + z_2 i_2} = \overline{z}_1 - i_2 \overline{z}_2$. The equality $z = 0$ is equivalent to two equalities $z_1 = 0$ and $z_2 = 0$.

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The product of the quaternion $z = z_1 + z_2 i_2$ by the quaternion $w = w_1 + w_2 i_2$, which we denote by zw, is defined by the formula [1, p. 37] $zw = (z_1 w_1 - \overline{w}_2 z_2) + (w_2 z_1 + z_2 \overline{w}_1) i_2$. In particular, for complex variables z_1 and z_2 we have

$$z_1 z_2 = z_2 z_1, \quad z_1 \in \mathbb{C}^1, \quad z_2 \in \mathbb{C}^1. \tag{1.3}$$

The set of all points $z = (z_1, z_2) \in \mathbb{C}^2$ with the property $\|z - z^0\| < \delta$, where $\|z\| = \|z_1\| + \|z_2\|$, $\|z_1\| = |x_0| + |x_1|$, $\|z_2\| = |x_2| + |x_3|$, is called the δ-neighborhood of a point $z^0 = (z_1^0, z_2^0) \in \mathbb{C}^2$ denoted by $U(z^0, \delta)$. We denote by the symbol $U(z^0)$ the neighborhood of a point z^0 in general.

Analogously to equality (1.1), the function $u = f(z)$ takes the form

$$f = f_1 + f_2 i_2, \tag{1.4}$$

where

$$f_1(z_1, z_2) = u_0(z_1, z_2) + i_1 u_1(z_1, z_2)$$

and

$$f_2(z_1, z_2) = u_2(z_1, z_2) + i_1 u_3(z_1, z_2).$$

2. Differentiability of Quaternion Functions

In this paper we establish some properties of quaternion functions $f = f_1 + f_2 i_2$ with respect to the complex variables z_1 and z_2. For this, we use the necessary and sufficient condition of existence at a point $z^0 = x_0^0 + x_1^0 i_1 + x_2^0 i_2 + x_3^0 i_3$ of the differential $df(z^0)$ (with respect to the collection (x_0, x_1, x_2, x_3) of real variables). This condition means the finiteness of the angular gradient (i.e. the finiteness of all its components) of the function f at a point z^0 and is written as

$$\operatorname{anggrad} df(z^0) = (f'_{\widehat{x}_0}(z^0), f'_{\widehat{x}_1}(z^0), f'_{\widehat{x}_2}(z^0), f'_{\widehat{x}_3}(z^0)). \tag{2.1}$$

This $\operatorname{anggrad} f(z^0)$ is a particular case of the general case where the function $F(t)$, $t = (t_1, \dots, t_n)$, given in a neighborhood of a point $t^0 = (t_1^0, \dots, t_n^0) \in R^n$ has the finite angular partial derivative [2, p. 24; 3, p. 61] with respect to each t_k

$$F'_{\widehat{t}_k}(t^0) = \lim_{\substack{t_k \to t_k^0 \\ |t_j - t_j^0| \le c_j |t_k - t_k^0| \\ j \neq k}} \frac{F(t) - F(t(t_k^0))}{t_k - t_k^0} \tag{2.2}$$

where $t(t_k^0) = (t_1, \dots, t_{k-1}, t_k^0, t_{k+1}, \dots, t_n)$, provided that it is assumed that this limit exists and is independent of an arbitrarily chosen collection $c = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ of positive constants.

Since the difference $t_k - t_k^0$ in equality (2.2) is a real number, the necessary and sufficient condition of $\mathbb{R}^n$-differentiability (shortened to differentiability in the sequel) has one and the same form for real, complex and quaternion functions.

Thus, for a quaternion function $f = u_0 + u_1 i_1 + u_2 i_2 + u_3 i_3$ to be differentiable at the point $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ the necessary and sufficient condition is the existence, at z, of finite angular partial derivatives $f'_{\widehat{x}_0}(z)$, $f'_{\widehat{x}_1}(z)$, $f'_{\widehat{x}_2}(z)$, $f'_{\widehat{x}_3}(z)$, where $f'_{\widehat{x}_k} = (u_0)'_{\widehat{x}_k} + i_1 (u_1)'_{\widehat{x}_k} + i_2 (u_2)'_{\widehat{x}_k} + i_3 (u_3)'_{\widehat{x}_k}$, $k = 0, 1, 2, 3$.

Along with this, when the function f is differentiable at a point z, the following equality [2, p. 25; 3, p. 64] is fulfilled for its differential $df(z)$

$$\begin{gathered} df(z) = f'_{\widehat{x}_0}(z)dx_0 + f'_{\widehat{x}_1}(z)dx_1 + f'_{\widehat{x}_2}(z)dx_2 + f'_{\widehat{x}_3}(z)dx_3, \\ df(z) = du_0(z) + i_1 du_1(z) + i_2 du_2(z) + i_3 du_3(z). \end{gathered} \tag{2.3}$$

It can be easily verified that the existence of an angular partial derivative $\frac{\partial f}{\partial \widehat{x}_k}$ of a quaternion function f with respect to a variable x_k is equivalent to the concurrent existence of the angular partial derivatives $\frac{\partial f_1}{\partial \widehat{x}_k}$ and $\frac{\partial f_2}{\partial \widehat{x}_k}$ of the complex functions f_1 and f_2 with respect to the same x_k and the equality

$$\frac{\partial f}{\partial \widehat{x}_k} = \frac{\partial f_1}{\partial \widehat{x}_k} + \frac{\partial f_2}{\partial \widehat{x}_k} i_2, \quad k = 0, 1, 2, 3, \tag{2.4}$$

holds, where

$$\frac{\partial f_1}{\partial \widehat{x}_k} = \frac{\partial u_0}{\partial \widehat{x}_k} + i_1 \frac{\partial u_1}{\partial \widehat{x}_k}, \tag{2.5}$$

$$\frac{\partial f_2}{\partial \widehat{x}_k} = \frac{\partial u_2}{\partial \widehat{x}_k} + i_1 \frac{\partial u_3}{\partial \widehat{x}_k}. \tag{2.6}$$

Moreover, the differentiability of a quaternion function f at a point z is equivalent to the differentiability of the complex functions f_1 and f_2 at z and we have the equality

$$df(z) = df_1(z) + df_2(z) i_2, \tag{2.7}$$

where

$$df_1(z) = du_0(z) + i_1 du_1(z), \quad df_2(z) = du_2(z) + i_1 du_3(z). \tag{2.8}$$

3. $\mathbb{C}^2$-Differentiability of Quaternion Functions

Definition 3.1. A quaternion function $f(z) = f_1(z) + f_2(z) i_2$, $z = (z_1, z_2) = z_1 + z_2 i_2$, is called $\mathbb{C}^2$-differentiable at a point $z^0 = (z_1^0, z_2^0) = z_1^0 + z_2^0 i_2$ if there exist quaternion numbers $d_1 + d'_1 i_2$ and $d_2 + d'_2 i_2$, such that the equality

$$\lim_{z \to z^0} \frac{f(z) - f(z_0) - \sum_{k=1}^{2} (z_k - z_k^0)(d_k + d'_k i_2)}{\|z - z^0\|} = 0 \tag{3.1}$$

is fulfilled.

In that case, we call the sum

$$\sum_{k=1}^{2}(z_k - z_k^0)(d_k + d'_k i_2) \tag{3.2}$$

the $\mathbb{C}^2$-differential of the quaternion function f at the point z^0.

The following statement is true.

Theorem 3.2. *For a quaternion function $f(z) = f_1(z) + f_2(z)i_2$ to be $\mathbb{C}^2$-differentiable at a point z^0 it is necessary and sufficient that one of the following three conditions be fulfilled:*

(i) *The complex functions $f_1(z)$ and $f_2(z)$ are $\mathbb{C}^2$-differentiable at the point z^0;*

(ii) *The equalities*

$$\frac{\partial f}{\partial \widehat{x}_0}(z^0) + i_1 \frac{\partial f}{\partial \widehat{x}_1}(z^0) = 0 \tag{3.3}$$

and

$$\frac{\partial f}{\partial \widehat{x}_2}(z^0) + i_1 \frac{\partial f}{\partial \widehat{x}_3}(z^0) = 0 \tag{3.4}$$

are fulfilled at the point z^0;

(iii) *The equality*

$$df(z^0) = dz_1 \frac{\partial f}{\partial \widehat{z}_1}(z^0) + dz_2 \frac{\partial f}{\partial \widehat{z}_2}(z^0) \tag{3.5}$$

holds, where

$$\frac{\partial f}{\partial \widehat{z}_1} = \frac{\partial f_1}{\partial \widehat{z}_1} + \frac{\partial f_1}{\partial \widehat{z}_1} i_1, \quad \frac{\partial f}{\partial \widehat{z}_2} = \frac{\partial f_1}{\partial \widehat{z}_2} + \frac{\partial f_2}{\partial \widehat{z}_2} i_1 \tag{3.6}$$

and for a complex function $g(z_1, z_2)$ of two complex variables z_1 and z_2 the formal angular partial derivatives $\frac{\partial g}{\partial \widehat{z}_1}$ and $\frac{\partial g}{\partial \widehat{z}_2}$ with respect to z_1 and z_2 are defined by the equality [4]

$$\frac{\partial g}{\partial \widehat{z}_1} = \frac{1}{2}\Big(\frac{\partial g}{\partial \widehat{x}_0} - i_1 \frac{\partial g}{\partial \widehat{x}_1}\Big), \quad \frac{\partial g}{\partial \widehat{z}_2} = \frac{1}{2}\Big(\frac{\partial g}{\partial \widehat{x}_2} - i_1 \frac{\partial g}{\partial \widehat{x}_3}\Big). \tag{3.7}$$

Proof. (i) Equality (3.1) is equivalent to the fulfillment of the following two equalities

$$\lim_{z \to z_0} \frac{f_1(z) - f_1(z^0) - \sum_{k=1}^{2} d_k (z_k - z_k^0)}{\|z - z^0\|} = 0 \tag{3.8}$$

and

$$\lim_{z \to z_0} \frac{f_2(z) - f_2(z^0) - \sum_{k=1}^{2} d'_k (z_k - z_k^0)}{\|z - z^0\|} = 0, \tag{3.9}$$

which are respectively equivalent to the $\mathbb{C}^2$-differentiability of the complex functions $f_1(z)$ and $f_2(z)$ at the point z^0 [4, equality (3.2)].

(ii) According to the statement (i), the $\mathbb{C}^2$-differentiability of a quaternion function $f = f_1+f_2 i_2$ at a point z^0 is equivalent to the $\mathbb{C}^2$-differentiability of the complex functions f_1 and f_2. On the other hand, the $\mathbb{C}^2$-differentiability of the complex function f_1 at the point z^0 is equivalent to the fulfillment of the equalities [4, equality (3.1)]

$$\frac{\partial f_1}{\partial \widehat{x}_0}(z^0) + i_1 \frac{\partial f_1}{\partial \widehat{x}_1}(z^0) = 0, \quad \frac{\partial f_1}{\partial \widehat{x}_2}(z^0) + i_1 \frac{\partial f_1}{\partial \widehat{x}_3}(z^0) = 0. \tag{3.10}$$

Analogously, for the complex function f_2 we have

$$\frac{\partial f_2}{\partial \widehat{x}_0}(z^0) + i_1 \frac{\partial f_2}{\partial \widehat{x}_1}(z^0) = 0, \quad \frac{\partial f_2}{\partial \widehat{x}_2}(z^0) + i_1 \frac{\partial f_2}{\partial \widehat{x}_3}(z^0) = 0. \tag{3.11}$$

If we perform the right multiplication of equalities (3.11) by i_2 and sum the resulting equalities with equalities (3.10), then we will obtain equalities (3.3) and (3.4).

(iii) Again, by virtue of statement (i), the $\mathbb{C}^2$-differentiability of the quaternion function f is equivalent to the $\mathbb{C}^2$-differentiability of the complex functions f_1 and f_2. But the complex function f_1 is $\mathbb{C}^2$-differentiable at the point z^0 if and only if the equality [4, equality (3.7)]

$$df_1(z^0) = \sum_{k=1}^{2} \frac{\partial f_1}{\partial \widehat{z}_k}(z^0) dz_k \tag{3.12}$$

is fulfilled.

Analogously, for the complex function f_2 to be $\mathbb{C}^2$-differentiable at a point z^0 it is necessary and sufficient that the equality

$$df_2(z^0) = \sum_{k=1}^{2} \frac{\partial f_2}{\partial \widehat{z}_k}(z^0) dz_k \tag{3.13}$$

be fulfilled.

Using (1.3) we can rewrite equalities (3.12) and (3.13) in the form

$$df_1 = dz_1 \frac{\partial f_1}{\partial \widehat{z}_1} + dz_2 \frac{\partial f_1}{\partial \widehat{z}_2}, \quad df_2 = dz_1 \frac{\partial f_2}{\partial \widehat{z}_1} + dz_2 \frac{\partial f_2}{\partial \widehat{z}_2}. \tag{3.14}$$

Hence we obtain the equality

$$df_1 + df_i i_2 = dz_1 \frac{\partial (f_1 + f_2 i_2)}{\partial \widehat{z}_1} + dz_2 \frac{\partial (f_1 + f_2 i_2)}{\partial \widehat{z}_2},$$

from which by virtue of (2.7) we obtain equality (3.5). □

Remark 3.3. The equivalence of the $\mathbb{C}^2$-differentiability of a quaternion function $f = f_1+f_2 i_2$ with the concurrent $\mathbb{C}^2$-differentiability of its complex components f_1 and f_2 (see statement (i) from Theorem 3.2) has no analogue for the $\mathbb{C}^1$-differentiability in the domain. That this is so follows from the fact that a $\mathbb{C}^1$-differentiable real function in a domain is necessarily constant in this domain.

Theorem 3.4. *The $\mathbb{C}^2$-differential of a quaternion function f is equal to the differential of this function.*

Proof. For the coefficients d_k and d'_k figuring in equalities (3.8) and (3.9) we know the equalities [5, p. 31]

$$d_k = \frac{\partial f_1}{\partial z_k}(z^0), \quad d'_k = \frac{\partial f_2}{\partial z_k}(z^0).$$

But for a $\mathbb{C}^2$-differentiable complex function the partial derivative with respect to the variable z_k is equal to its angular partial derivative with respect to the same z_k [4, equality (2.1)]. Therefore the $\mathbb{C}^2$-differential of the function $f = f_1 + f_2 i_2$ defined by equality (3.2) at the point z^0 is written as

$$\sum_{k=1}^{2} dz_k \frac{\partial f}{\partial \widehat{z}_k}(z^0).$$

But the latter expression is equal by virtue of equality (3.5) to $df(z^0)$. □

4. $\mathbb{C}^2$-Holomorphy of Quaternion Functions

Definition 4.1. A quaternion function $f(z) = f_1(z) + f_2(z)i_2$ will be called $\mathbb{C}^2$-holomorphic at a point z^0 or in a domain $D \subset \mathbb{C}^2$ if f is $\mathbb{C}^2$-differentiable in the neighborhood of z^0 or at every point of the domain D.

The following statement holds true.

Proposition 4.1. *For a quaternion function $f(z)$ to $\mathbb{C}^2$-holomorphic at a point z^0 or in any domain $D \subset \mathbb{C}^2$ it is necessary and sufficient that one of conditions* (i)–(iii) *from Theorem* 3.2 *be fulfilled in the neighbothood of x^0 or at every point of the domain D.*

In particular, we have

Proposition 4.2. *The $\mathbb{C}^2$-holomorphy at a point or in a domain of a quartenion function $f(z) = f_1(z) + f_2(z)i_2$ is equivalent to the concurrent $\mathbb{C}^2$-holomorphy at the same point or in the same domain of the complex functions $f_1(z)$ and $f_2(z)$.*

5. Integral Representations of $\mathbb{C}^2$-Holomorphic Quaternion Functions

Theorem 5.1. *Let a quaternion function $f(z) = f_1(z) + f_2(z)i_2$ be $\mathbb{C}^2$-holomorphic in a domain $D \subset \mathbb{C}^2$ which is the Cartesian product of simply connected domains $D_1 \subset \mathbb{C}^1$ and $D_2 \subset \mathbb{C}^1$. Then at any point $z = (z_1, z_2)$ the representation*

$$f(z_1, z_2) = -\frac{1}{4\pi^2} \int\limits_{\Gamma_1} \int\limits_{\Gamma_2} \frac{dt_1 dt_2}{(t_1 - z_1)(t_2 - z_2)} f(t_1, t_2), \tag{5.1}$$

is fulfilled, where Γ_1 *and* Γ_2 *are any closed paths in* D_1 *and* D_2*, respectively, which envelop the points* z_1 *and* z_2*.*

Proof. By Proposition 4.2. we have the equalities [5, p. 28]

$$f_1(z_1, z_2) = -\frac{1}{4\pi^2} \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} \frac{f_1(t_1, t_2)}{(t_1 - z_1)(t_2 - z_2)} \, dt_1 dt_2, \tag{5.2}$$

$$f_2(z_1, z_2) = -\frac{1}{4\pi^2} \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} \frac{f_2(t_1, t_2)}{(t_1 - z_1)(t_2 - z_2)} \, dt_1 dt_2. \tag{5.3}$$

By virtue of equality (1.3) we can write $f_1(t_1, t_2)dt_1dt_2 = dt_1dt_2f_1(t_1, t_2)$ and $f_2(t_1, t_2)dt_1dt_2 = dt_1dt_2f_2(t_1, t_2)$. Hence, from equalities (5.2) and (5.3) we obtain the equality

$$f_1(z_1, z_2) + f_2(z_1, z_2)i_2 =$$
$$= -\frac{1}{4\pi^2} \int\limits_{\Gamma_2\Gamma_1} \frac{dt_1 dt_2}{(t_1 - z_1)(t_2 - z_2)} [f_1(t_1, t_2) + f_2(t_1, t_2)i_2],$$

which is equivalent to equality (5.1) □

Theorem 5.2. *If a quaternion function* $f(z_1, z_2) = f_1(z_1, z_2) + f_2(z_1, z_2)i_2$ *is* $\mathbb{C}^2$*-holomorphic in the Cartesian product* $D_1 \times D_2$ *of simply connected domains* $D_1 \subset \mathbb{C}^1$ *and* $D_2 \subset \mathbb{C}^1$*, then its partial derivatives* f'_{z_1} *and* f'_{z_2} *are also* $\mathbb{C}^2$*-holomorphic quaternion functions in* $D_1 \times D_2 \subset \mathbb{C}^2$*.*

Proof. According to Proposition 4.2, the $\mathbb{C}^2$-holomorphy of a quaternion function f imples the $\mathbb{C}^2$-holomorphy of the complex functions f_1 and f_2 given by equalities (5.2) and (5.3). Therefore their partial derivatives $\frac{df_1}{\partial z_1}$, $\frac{df_1}{\partial z_2}$, $\frac{df_2}{\partial z_1}$ and $\frac{df_2}{\partial z_2}$ are $\mathbb{C}^2$-holomorphic complex functions in $D_1 \times D_2$. Thus equalities (3.10) and (3.11) which are fulfilled for the functions f_1 and f_2 will also be fulfilled for their partial derivatives $\frac{df_1}{\partial z_1}$, $\frac{df_1}{\partial z_2}$, $\frac{df_2}{\partial z_1}$, $\frac{df_2}{\partial z_2}$. Hence it follows that, as was shown when proving Theorem 3.2, these partial derivatives satisfy equalities (3.3) and (3.4), i.e. are $\mathbb{C}^2$-holomorphic quaternion functions by virtue of statement (ii) from Theorem 3.2. □

6. Representation of $\mathbb{C}^2$-Holomorhic Functions by Power Series

Theorem 6.1. *Let a quaternion function* $f(z) = f_1(z) + f_2(z)i_2$ *be* $\mathbb{C}^2$*-holomorphic in a domain* $D \subset \mathbb{C}^2$ *which is the Cartesian product of simply connected domains* $D_1 \subset \mathbb{C}^1$ *and* $D_2 \subset \mathbb{C}^1$*. Then at any point* $z =$

$(z_1,z_2)\in D$ from the neighborhood of $z^0=(z_1^0,z_2^0)\in D$ the representation of f by the power series

$$f(z_1,z_2)=\sum_{m,n=0}^{\infty}(z_1-z_1^0)^m(z_2-z_2^0)^n c_{mn}, \tag{6.1}$$

is fulfilled, where the quaternion coefficients c_{mn} of the function f are defined by the equalities

$$c_{mn}=-\frac{1}{4\pi^2}\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}\frac{dt_1dt_2}{(t_1-z_1^0)^{m+1}(t_2-z_2^0)^{n+1}}\,f(t_1,t_2), \tag{6.2}$$

$$m!n!c_{mn}=\Big(\frac{\partial^{m+n}f(z_1,z_2)}{\partial z_1^m\partial z_2^n}\Big)_{\substack{z_1=z_1^0\\ z_2=z_2^0}}. \tag{6.3}$$

Proof. By virtue of Proposition 4.2, the complex functions f_1 and f_2 are $\mathbb{C}^2$-holomorphic or, which is the same, $\mathbb{C}^2$-analytic in the domain D. Hence we have the equalities

$$f_1(z_1,z_2)=\sum_{m,n=0}^{\infty}{}^1c_{mn}(z_1-z_1^0)^m(z_2-z_2^0)^n, \tag{6.4}$$

$$f_2(z_1,z_2)=\sum_{m,n=0}^{\infty}{}^2c_{mn}(z_1-z_1^0)^m(z_2-z_2^0)^n, \tag{6.5}$$

where the complex coefficients of the functions f_1 and f_2 are given by the formulas

$${}^1c_{mn}=-\frac{1}{4\pi^2}\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}\frac{f_1(t_1,t_2)}{(t_1-z_1^0)^{m+1}(t_2-z_2^0)^{n+1}}\,dt_1dt_2, \tag{6.6}$$

$${}^2c_{mn}=-\frac{1}{4\pi^2}\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}\frac{f_2(t_1,t_2)}{(t_1-z_1^0)^{m+1}(t_2-z_2^0)^{n+1}}\,dt_1dt_2. \tag{6.7}$$

Using (1.3) and the equality $f_1+f_2i_2=f$, from (6.4), (6.5) and (6.6), (6.7) we obtain respectively equalities (6.1) and (6.2). As to equality (6.3), it is obtained from the well known formulas [5, p. 31]

$$m!n!{}^1c_{mn}=\Big(\frac{\partial^{m+n}f_1(t_1,t_2)}{\partial t_1^m\partial t_2^n}\Big)_{\substack{t_1=z_1^0\\ t_2=z_2^0}},$$

$$m!n!{}^2c_{mn}=\Big(\frac{\partial^{m+n}f_2(t_1,t_2)}{\partial t_1^m\partial t_2^n}\Big)_{\substack{t_1=z_1^0\\ t_2=z_2^0}},$$

taking into account the equalities

$$\frac{df}{dz_1}=\frac{\partial f_1}{\partial z_1}+\frac{\partial f_2}{\partial z_1}i_2,\quad \frac{df}{dz_2}=\frac{\partial f_1}{\partial z_2}+\frac{\partial f_2}{\partial z_2}i_2.$$ □

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