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# ON SOME NEW PROPERTIES OF QUATERNION FUNCTIONS

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Abstract. Quaternions discovered by W. R. Hamilton made a great contribution to the progress in noncommutative algebra and vector analysis. However, the analysis of quaternion functions has not been duly developed. The matter is that the notion of a derivative of quaternion functions of a quaternion variable has not been known until recently. The author has succeeded in improving the situation. The present work contains an account of the results obtained by him in this direction. The notion of an $\mathbb{H}$-derivative is introduced for quaternion functions of a quaternion variable. The existence of an $\mathbb{H}$-derivative of elementary functions is established retaining the well-known formulas for the corresponding functions from complex (real) analysis. The rules on the $\mathbb{H}$-differentiation of a sum, a product, and an inverse function are formulated and proved. Necessary and sufficient conditions for the existence of an $\mathbb{H}$-derivative are established. The notions of $\mathbb{C}^2$-differentiation and $\mathbb{C}^2$-holomorphy are introduced for quaternion functions of a quaternion variable. Three equivalent conditions are found, each of them being a necessary and sufficient one for $\mathbb{C}^2$-differentiation. Representations by an integral and a power series are given for $\mathbb{C}^2$-holomorphic functions. It is proved that the harmonicity of functions $f(z)$, $z \cdot f(z)$, and $f(z) \cdot z$ is the necessary and sufficient condition for a function $f$ to be Fueter-regular.

## CONTENTS



## 1. Introduction

The highly important theory of holomorphic (analytic) functions of a complex variable with comprehensive applications to various problems of natural sciences proved to be a serious motive for finding analogous theories for functions of three and more real variables. It turned out that, following

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Frobenius (see [80, p. 117]), such analogous theories did not exist at all for functions of three real variables.

Pursuing the purpose of finding a similar theory for functions of four real variables, W. R. Hamilton (1805–1865)[1] made a truly brilliant discovery by introducing quaternions in science ( [1, 16, 17, 29]). He dedicated the last twenty two years of his life to the construction of the quaternion theory (see [116, p. 212]).

In 1837, J. Bolyai (1802–1860) in Leipzig submitted his remarkable work that forestalled Hamilton's finding to the prize competition, but the jury passed a negative decision. That unfortunate event badly affected the psychological health of Bolyai (see [16, 17]).

**1.1. Fundamental properties of quaternion numbers (see [24, 80]).** The quaternion (constant) units $i_0$, $i_1$, $i_2$, $i_3$ introduced by Hamilton obey the conditions $i_0 = 1$, $i_1^2 = i_2^2 = i_3^2 = i_1 i_2 i_3 = -1$, and the multiplication table

$$i_1 i_2 = -i_2 i_1 = i_3, \quad i_2 i_3 = -i_3 i_2 = i_1, \quad i_3 i_1 = -i_1 i_3 = i_2. \tag{1.1}$$

The diagram below helps memorize this "multiplication table": the product of any two numbers from the set $\{i_1, i_2, i_3\}$ is equal to the third number with sign "+" if the direction of rotation from the first multiplier to the second is clockwise and with sign "−" in the opposite case. We see that the multiplication of quaternions is not commutative: the product depends on the order of factors.

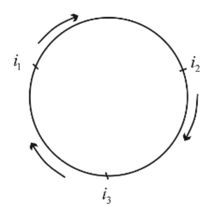


Fig. 1.

Quaternions are numbers of the form

$$a + bi_1 + ci_2 + di_3, \tag{1.2}$$

where $a$, $b$, $c$, and $d$ are real numbers.

The addition and multiplication rules for two quaternions

$$q = a + bi_1 + ci_2 + di_3, \tag{1.3}$$

$$q' = a' + b'i_1 + c'i_2 + d'i_3, \tag{1.4}$$

are respectively

$$q + q' = (a + a') + (b + b')i_1 + (c + c')i_2 + (d + d')i_3 = q' + q, \tag{1.5}$$

$$\begin{aligned} qq' = (aa' - bb' - cc' - dd') + (ab' + ba' + cd' - dc')i_1 \\ + (ac' + ca' + db' - bd')i_2 + (ad' + da' + bc' - cb')i_3. \end{aligned} \tag{1.6}$$

It is easy to verify that the real number $aa' - bb' - cc' - dd'$ is the real part of the products of $qq'$ and $q'q$.

[1]The name William Rowan Hamilton, professor of the Dublin University, President of the Irish Academy of Sciences in 1837–1846, is well known in mechanics, physics, and astronomy not only in connection with quaternions. The great scientist never ceased his tireless search for a higher form of complex numbers. His efforts were crowned with success and on October 16, 1843, while walking to attend the session of the Academy of Sciences in Dublin, he stopped on the Brougham bridge and cut with a knife the multiplication formulas of quaternions on the stone balustrade (see [29, p. 29] and [86]). A few years after 1843, the memorial plaque [71, p. 515] was installed at the place on the bridge across the channel, where Hamilton experienced a sudden inspiration.

Although multiplication of quaternions does not satisfy the commutativity property, it is associative:

$$(q_1 q_2) q_3 = q_1 (q_2 q_3). \tag{1.7}$$

The quaternion

$$\overline{q} = a - bi_1 - ci_2 - di_3 \tag{1.8}$$

is said to be *conjugate* to the quaternion (1.3).

It is obvious that the sum of conjugate quaternions is a real number. The products $q\overline{q}$ and $\overline{q}q$ are also real:

$$q\overline{q} = \overline{q}q = a^2 + b^2 + c^2 + d^2. \tag{1.9}$$

The number $(a^2 + b^2 + c^2 + d^2)^{1/2}$ is called the *modulus* of the quaternion $q$ and is denoted by $|q|$. So,

$$q\overline{q} = \overline{q}q = |q|^2. \tag{1.10}$$

Hence, using Eqs. (1.7) and (1.12), we have

$$|q_1 q_2|^2 = (q_1 \cdot q_2)(\overline{q_1 \cdot q_2}) = q_1 \cdot q_2 \cdot \overline{q}_2 \cdot \overline{q}_1 = q_1 (q_2 \cdot \overline{q}_2) \overline{q}_1 = q_1 |q_2|^2 \overline{q}_1 = |q_2|^2 q_1 \overline{q}_1 = |q_2|^2 |q_1|^2.$$

Therefore

$$|q_1 q_2| = |q_1|\,|q_2|. \tag{1.10$'$}$$

A straightforward calculation shows that the conjugate to the sum is equal to the sum of conjugates,

$$\overline{q_1 + q_2} = \overline{q}_1 + \overline{q}_2. \tag{1.11}$$

while the conjugate to the product is equal to the product of conjugates taken in reverse order:

$$\overline{q_1 q_2} = \overline{q}_2 \cdot \overline{q}_1. \tag{1.12}$$

Since the product of quaternions depends on the order of cofactors, we must separately consider the following two equations:

$$q_2 x = q_1, \tag{1.13}$$

$$x q_2 = q_1; \tag{1.14}$$

here $q_2 \neq 0$. A solution of Eq. (1.13) is called the *left quotient* of $q_1$ and $q_2$ and is denoted by $x_l$, while a solution of Eq. (1.14) is called the right quotient $x_r$. It is easy to obtain the formulas

$$x_l = \frac{1}{|q_2|^2}\, \overline{q}_2 \cdot q_1, \tag{1.15}$$

$$x_r = \frac{1}{|q_2|^2}\, q_1 \cdot \overline{q}_2. \tag{1.16}$$

Taking $q_1 = 1$, we see that each quaternion $q_2 \neq 0$ has the inverse quaternion $\overline{q}_2/|q_2|^2$, which is denoted by $q_2^{-1}$. Therefore,

$$q_2^{-1} = \frac{1}{|q_2|^2}\, \overline{q}_2. \tag{1.17}$$

The uniqueness of solutions of Eqs. (1.13) and (1.14) can also be formulated as follows: if $ab_1 = ab_2$ or $b_1 a = b_2 a$ for $a \neq 0$, then $b_1 = b_2$.

Furthermore, each point $(x_0, x_1, x_2, x_3)$ of the real four-dimensional Euclidean space $\mathbb{R}^4$ is associated with the quaternion $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ with norm $|z| = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{1/2}$. Hence the space $\mathbb{R}^4$ is identified with the quaternion division algebra $\mathbb{H}$.

**1.2. On quaternion functions.** The property of a quaternion function $f(z) = u_0(z) + u_1(z)i_1 + u_2(z)i_2 + u_3(z)i_3$, where $u_k(z)$ are real functions, to possess a finite limit, or to be continuous, or to have a finite partial derivative with respect to a real variable $x_n$ is equivalent to the same properties of all functions $u_m(z)$.

It is natural to construct the theory of quaternion functions using the scheme by which the theory of holomorphic functions of one complex variable is constructed provided that this scheme is realizable.

We can indicate the following three methods of construction of the theory of holomorphic functions of one complex variable.

**1.3. Derivative method.** This method is based on the notion of a derivative as the limit of a ratio of the increment of a function to the increment of an independent variable.

Due to the noncommutativity of multiplication of quaternions, such an approach to a quaternion function $f(z)$ leads to two different notions of the derivative: the *right derivative* $A(z)$ in the form of two equivalent equalities [84]

$$\lim_{h\to 0} \big[f(z+h) - f(z)\big] \cdot h^{-1} = A(z), \quad \Delta f = Ah + \varepsilon_1, \quad \varepsilon_1 \to 0, \quad h \to 0, \tag{1.18}$$

and the *left derivative* $B(z)$, which is expressed by the equivalent equalities

$$\lim_{h\to 0} h^{-1} \cdot \big[f(z+h) - f(z)\big] = B(z), \quad \Delta f = hB + \varepsilon_2, \quad \varepsilon_2 \to 0, \quad h \to 0, \tag{1.19}$$

if the corresponding limits exist.

It turns out that only the functions $\varphi(z) = az + b$ possess the right derivative, only the functions $\psi(z) = za + b$ possess the left derivative, and the functions $\chi(z) = rz + b$ have the unilateral equal derivatives $A = B$, where $a$ and $b$ are any quaternion numbers and $r$ is a real number (see [99]). Subsequently, the same result was also established in works of other authors (see, e.g., [5, 25, 64]).

We also mention the following remarkable formula established in [64] under the assumption $c^2+d^2 > 0$:

$$(a + bi_1 + ci_2 + di_3)^n = a_n + b_n i_1 + c_n i_2 + d_n i_3, \tag{1.20}$$

where the real numbers $a_n$, $b_n$, $c_n$, and $d_n$ are defined by the equalities

$$\begin{aligned} a_n &= \frac{1}{2}\Big[(a - i_1\sqrt{\Delta})^n + (a + i_1\sqrt{\Delta})^n\Big], \\ b_n &= \frac{b}{\sqrt{\Delta}} \cdot \frac{1}{2}\Big[(a - i_1\sqrt{\Delta})^n - (a + i_1\sqrt{\Delta})^n\Big] \cdot i_1, \\ c_n &= \frac{c}{\sqrt{\Delta}} \cdot \frac{1}{2}\Big[(a - i_1\sqrt{\Delta})^n - (a + i_1\sqrt{\Delta})^n\Big] \cdot i_1, \\ d_n &= \frac{d}{\sqrt{\Delta}} \cdot \frac{1}{2}\Big[(a - i_1\sqrt{\Delta})^n - (a + i_1\sqrt{\Delta})^n\Big] \cdot i_1 \end{aligned} \tag{1.21}$$

and

$$\Delta = b^2 + c^2 + d^2.$$

**1.4. Polynomial method.** Consider the polynomial $p(x, y) = \sum\limits_{m,n} A_{m,n} x^m y^n$ of two real variables $x$ and $y$ with complex coefficients $A_{m,n} = \alpha_{m,n} + i\beta_{m,n}$, $i^2 = -1$.

By virtue of the equalities

$$x = \frac{1}{2}\,(\overline{z} + z), \quad y = \frac{1}{2}\, i\,(\overline{z} - z)$$

we obtain the polynomial $p^*(z, \overline{z})$ of the complex variables $z = x + iy$ and $\overline{z} = x - iy$.

The polynomial $p^*(z, \overline{z})$ is a function of only one variable $z$ if and only if the well-known Cauchy–Riemann condition are fulfilled.

However, a similar approach to polynomials of real variables $x_0$, $x_1$, $x_2$, and $x_3$ with quaternion coefficients does not yield a desired result (see [129]). The matter is that the real coordinates $x_k$ of the quaternion $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ are expressed through $z$ by the Hausdorff formulas (see [74]):

$$\begin{aligned} x_0 &= \frac{1}{4}\,(z - i_1 z i_1 - i_2 z i_2 - i_3 z i_3), & x_1 &= \frac{1}{4i_1}\,(z - i_1 z i_1 + i_2 z i_2 + i_3 z i_3), \\ x_2 &= \frac{1}{4i_2}\,(z + i_1 z i_1 - i_2 z i_2 + i_3 z i_3), & x_3 &= \frac{1}{4i_3}\,(z + i_1 z i_1 + i_2 z i_2 - i_3 z i_3) \end{aligned} \tag{1.22}$$

without using the conjugate quaternion $\overline{z} = x_0 - x_1 i_1 - x_2 i_2 - x_3 i_3$.

Note that alongside with these Hausdorff formulas we can also apply, as is easy to verify, the formulas

$$\begin{aligned} x_0 &= \frac{1}{2}\,(\overline{z} + z), & x_1 &= \frac{1}{2}\,(i_1\overline{z} - z i_1), \\ x_2 &= \frac{1}{2}\,(i_2\overline{z} - z i_2), & x_3 &= \frac{1}{2}\,(i_3\overline{z} - z i_3) \end{aligned} \tag{1.23}$$

which, in contrast to the case of complex-valued functions, are not essential.

**1.5. Gradient method.** Looman [91] and Menchoff [100] proved that a continuous function $f(z)$ in a domain $G$ is holomorphic in $G$ if and only if $f(z)$ satisfies in $G$ the Cauchy–Riemann condition

$$f'_x(z) + i f'_y(z) = 0, \quad z = x + iy$$

(see also [117, p. 75]). Later, Tolstov showed that the assertion of Looman and Menshoff remains valid if the continuity of the function $f(z)$ is replaced by its boundedness in $G$ (see [133]). Hence it is clear that the Cauchy–Riemann condition is very important for the function $f(z)$, $z = x + iy$, to be holomorphic in the domain $G$.

Fueter developed a similar method for quaternion functions based on conditions of differential nature that were analogous to the Cauchy–Riemann conditions for functions of one complex variable.

In [54], Fueter introduced the following definition.

A quaternion function $f(z)$ of the quaternion variable $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ that possesses continuous partial derivatives $f'_{x_0}$, $f'_{x_1}$, $f'_{x_2}$, and $f'_{x_3}$ in a domain $G \subset \mathbb{H}$ is said to be *right-regular* in $G$ (notation $f \in F^+(G)$) if the condition

$$\frac{\overline{\partial}_r f}{\partial z} = 0 \tag{1.24}$$

is fulfilled in $G$; similarly, $f(z)$ is said to be *left-regular* if the condition

$$\frac{\overline{\partial}_l f}{\partial z} = 0 \tag{1.25}$$

is fulfilled in $G$; here the quaternion gradient operators $\overline{\partial}_r/\partial z$ and $\overline{\partial}_l/\partial z$ are defined by the equalities

$$\frac{\overline{\partial}_r}{\partial z} = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1}\, i_1 + \frac{\partial}{\partial x_2}\, i_2 + \frac{\partial}{\partial x_3}\, i_3, \tag{1.26}$$

$$\frac{\overline{\partial}_l}{\partial z} = \frac{\partial}{\partial x_0} + i_1\,\frac{\partial}{\partial x_1} + i_2\,\frac{\partial}{\partial x_2} + i_3\,\frac{\partial}{\partial x_3}. \tag{1.27}$$

If $f \in F^+(G) \cap F^-(G)$, then, according to Fueter, the function $f$ is said to be *regular* in $G$ (notation $f \in F(G)$).

Fueter's definitions were subsequently refined by Schuler [119] who weakened the assumption of the continuity of first-order partial derivatives of the function $f(z)$ to the differentiability (according to Stolz) of $f$.

If we introduce into consideration the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \tag{1.28}$$

then we obtain the equality

$$\Delta f = \partial_r(\overline{\partial}_r f) = \partial_l(\overline{\partial}_l f), \tag{1.29}$$

which can be easily verified. Hence it follows that functions that are right- or left-regular in a domain $G \subset \mathbb{H}$ in the Fueter sense are also harmonic, i.e. satisfy the Laplace equation $\Delta f = 0$.

Hence we can conclude that functions $\psi_n(z) = z^n$, which are important for analysis, do not belong to the union $F^+(G) \cup F^-(G)$. For example, $\Delta(z^2) = -4$ since

$$z^2 = (x_0^2 - x_1^2 - x_2^2 - x_3^2) + 2x_0x_1i_1 + 2x_0x_2i_2 + 2x_0x_3i_3.$$

Even the function $\psi_1(z) = z$, although it is harmonic, does not belong to this union.

Thus, the following problem arises: Is it possible to indicate for quaternion functions a differentiation property satisfied by power functions $\psi_n(z) = z^n$ $(n = 0, 1, 2, \ldots)$? The commutativity property of these power functions is obvious.

This problem is solved positively for the basic Hamilton elementary functions

$$z^n, \quad n = 0, 1, 2, \ldots, \tag{1.30}$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \tag{1.31}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots, \tag{1.32}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots, \tag{1.33}$$

and for the logarithmic function $\ln z$.

In the present work, for quaternion functions of a quaternion variable we give the notion of an $\mathbb{H}$-derivative, which exists for all functions mentioned above.

We recall here the opinion of Shilov on the existence of a theory of functions of a quaternion variable (see [124, p. 385]). It concerned the dream of Hamilton to create a theory of a quaternion variable, but the hopes that had been put on quaternions did not come true.

In Secs. 3 and 6 we establish the rule of $\mathbb{H}$-derivation and the necessary and sufficient condition of existence of an $\mathbb{H}$-derivative, which is an analog of the Cauchy–Riemann condition for a complex function of one complex variable.

Furthermore, the notion and the condition of $\mathbb{C}^2$-differentiability and $\mathbb{C}^2$-holomorphy ($\mathbb{C}^2$-analyticity) of quaternion functions with respect to two independent complex variables are given (see Sec. 7). In addition, an integral representation and a representation by a power series are obtained for $\mathbb{C}^2$-holomorphic quaternion functions.

The property of a quaternion function to be right- or left-regular according to Fueter is characterized through the $\mathbb{C}^2$-holomorphy of its two complex components, which are functions with respect to two independent complex variables.

Further, we proved that a function $f$ is Fouter-regular if and only if the functions $f(z)$, $f(z) \cdot z$ and $z \cdot f(z)$ are harmonic. It is established that if $f$ and $f^2$ are harmonic functions, then the equality

$$\sum_{k=0}^{3} (f'_{x_k})^2 = 0$$

is fulfilled. In the complex case, the last equality implies that either of the functions $f$ and $\overline{f}$ is holomorphic.

Some relations between the functions $z^n$, $\cos z$, $\sin z$, and $e^z$ are established (see Sec. 13) using the variable imaginary unit quaternion $I_z$ with property $I_z^2 = -1$ (see [71, p. 349]).

The concluding part of the paper (Sec. 14) is dedicated to the applications of quaternions.

## 2. $\mathbb{H}$-Derivative and Its Existence for Elementary Functions

The theory of holomorphic (analytic) functions of one complex variable is based on the notion of a derivative of a complex function with respect to its complex argument. Every elementary function of a complex variable has such a derivative. These derivatives are generalizations of the derivatives of the corresponding functions from real analysis. Quaternions, the appearance of which gave a mighty stimulus to the progress of algebra, are generalizations of complex values $a+bi$ with $i^2=-1$. However, the development of the analysis of quaternion functions was hampered by the lack of the notion of a derivative with good properties. Recently, the author has succeeded in improving the situation by obtaining the results to be discussed below.

**Definition 2.1.** *A quaternion function $f(z)$, where $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, defined in some neighborhood $G \subset \mathbb{H}$ of a point $z^0 = x_0^0 + x_1^0 i_1 + x_2^0 i_2 + x_3^0 i_3$, is said to be $\mathbb{H}$-differentiable at $z^0$ if there exist two sequences of quaternions $A_k(z^0)$ and $B_k(z^0)$ such that $\sum\limits_k A_k(z^0)B_k(z^0)$ is finite and the increment $f(z^0+h) - f(z^0)$ of the function $f(z)$ can be represented as follows:*

$$f(z^0+h) - f(z^0) = \sum_k A_k(z^0) \cdot h \cdot B_k(z^0) + \omega(z^0, h), \tag{2.1}$$

*where*

$$\lim_{h \to 0} \frac{|\omega(z^0,h)|}{|h|} = 0 \tag{2.2}$$

*and $z^0 + h \in G$. In this case, the quaternion $\sum\limits_k A_k(z^0)B_k(z^0)$ is called the $\mathbb{H}$-derivative of the function $f$ at the point $z^0$ and is denoted by $f'(z^0)$. Thus,*

$$f'(z^0) = \sum_k A_k(z^0)B_k(z^0). \tag{2.3}$$

The uniqueness of the $\mathbb{H}$-derivative follows from the fact that the right-hand side of (2.3), if it exists, is just the partial derivative $f'_{x_0}(z^0)$ of $f(z)$ at $z^0$ with respect to its real variable.

In the sequel, the symbol $o(h)$ will denote any function $\omega(z^0,h)$ satisfying (2.2).

**Remark 2.2.** *Note that the same definition still makes good sense for any mapping between Banach algebras. Moreover, all the proofs of our results remain valid (except for Proposition 3.4, which still remains valid if we take $\varphi$ to be invertible in a neighborhood of $z^0$) since in that case only those properties of $\mathbb{H}$ are required, which any Banach algebra has.*

We will show that the basic elementary functions are $\mathbb{H}$-differentiable.

**Proposition 2.3.**

$$(z^n)' = nz^{n-1} \quad \textit{for } n = 0,1,2,\dots \textit{ and for } z \in \mathbb{H}. \tag{2.4}$$

*Proof.* First, we show that the following equality holds for $n = 1, 2, \dots$:

$$(z+h)^n - z^n = z^{n-1}h + z^{n-2}hz + z^{n-3}hz^2 + \dots + zhz^{n-2} + hz^{n-1} + o(h). \tag{2.5}$$

For $n=1$ it is obvious. Assuming now that it is valid for $n=k$, we find

$$\begin{aligned}(z+h)^{k+1} - z^{k+1} &= (z+h)(z+h)^k - z^{k+1} \\ &= (z+h)(z^k + z^{k-1}h + z^{k-2}hz + \dots + zhz^{k-2} + hz^{k-1} + o(h)) - z^{k+1} \\ &= z^{k+1} + z^k h + z^{k-1}hz + \dots + z^2hz^{k-2} + zhz^{k-1} + hz^k + o(h)) - z^{k+1} \\ &= z^k h + z^{k-1}hz + z^{k-2}hz^2 + \dots + z^2hz^{k-2} + zhz^{k-1} + hz^k + o(h).\end{aligned}$$

Then it follows from (2.3) and (2.5) that

$$(z^n)' = z^{n-1} \cdot 1 + z^{n-2} \cdot z + z^{n-3} \cdot z^2 + \dots + z \cdot z^{n-2} + 1 \cdot z^{n-1} = nz^{n-1}. \tag{2.6}$$

Thus, we have proved by induction that $(z^n)' = nz^{n-1}$ for all $n = 0, 1, 2, \ldots$. $\square$

In order to proceed further, we need the following lemma.

**Lemma 2.4.** *The following equalities and estimates are valid for* $|h| < 1$:

$$
\begin{aligned}
\frac{(z+h)^2 - z^2}{2!} &= \frac{zh + hz}{2!} + A_2,\\
\frac{(z+h)^3 - z^3}{3!} &= \frac{z^2h + zhz + hz^2}{3!} + A_3,\\
\frac{(z+h)^4 - z^4}{4!} &= \frac{z^3h + z^2hz + zhz^2 + hz^3}{4!} + A_4,\\
\frac{(z+h)^5 - z^5}{5!} &= \frac{z^4h + z^3hz + z^2hz^2 + zhz^3 + hz^4}{5!} + A_5,
\end{aligned}
$$

*and so on, where*

$$
\begin{aligned}
A_2 &= \frac{1}{2!}\,h^2, \quad |A_2| < |h|^2, \quad A_2 = o(h);\\
A_3 &= \frac{1}{3!}\,(zh^2 + hzh + h^2z + h^3),\\
|A_3| &< \frac{2^3}{3!}\,\big(|z|\,|h|^2 + |h|^3\big)\\
&< \begin{cases} \dfrac{2^3}{3!}\,|h|^2(1+|h|) < \dfrac{2^3}{3!}\,|h|^2\cdot\dfrac{1}{1-|h|} & \textit{for}\quad |z| < 1,\\[2ex] \dfrac{2^3}{3!}\,|z|\,|h|^2(1+|h|) < \dfrac{2^3}{3!}\,|z|\,|h|^2\cdot\dfrac{1}{1-|h|} & \textit{for}\quad |z| \ge 1;\end{cases}\\
A_4 &= \frac{1}{4!}\Big(z^2h^2 + zhzh + zh^2z + zh^3 + hz^2h + hzhz + hzh^2 + h^2z^2 + h^2zh + h^3z + h^4\Big),\\
|A_4| &< \frac{2^4}{4!}\Big(|z|^2\,|h|^2 + |z|\,|h|^3 + |h|^4\Big)\\
&< \begin{cases} \dfrac{2^4}{4!}\,|h|^2(1+|h|+|h|^2) < \dfrac{2^4}{4!}\,|h|^2\cdot\dfrac{1}{1-|h|} & \textit{for}\quad |z| < 1,\\[2ex] \dfrac{2^4}{4!}\,|z|^2\,|h|^2(1+|h|+|h|^2) < \dfrac{2^4}{4!}\,|z|^2\,|h|^2\cdot\dfrac{1}{1-|h|} & \textit{for}\quad |z| \ge 1;\end{cases}\\
A_5 &= \frac{1}{5!}\Big(z^3h^2 + z^2hzh + z^2h^2z + z^2h^3 + zhz^2h + zhzhz + zhzh^2 + zh^2z^2 + zh^2zh\\
&\qquad + zh^3z + zh^4 + hz^3h + hz^2hz + hz^2h^2 + hzhz^2 + hzhzh + hzh^3z +\\
&\qquad + hzh^3 + h^2z^3 + h^2z^2h + h^2zhz + h^2zh^2 + h^3z^2 + h^3zh + h^4z + h^5\Big),\\
|A_5| &< \frac{2^5}{5!}\Big(|z|^3\,|h|^2 + |z|^2\,|h|^3 + |z|\,|h|^4 + |h|^5\Big) <\\
&< \begin{cases} \dfrac{2^5}{5!}\,|h|^2\big(1+|h|+|h|^2+|h|^3\big) < \dfrac{2^5}{5!}\,|h|^2\cdot\dfrac{1}{1-|h|} & \textit{for}\quad |z| < 1,\\[2ex] \dfrac{2^5}{5!}\,|z|^3\,|h|^2\big(1+|h|+|h|^2+|h|^3\big) < \dfrac{2^5}{5!}\,|z|^3\,|h|^2\cdot\dfrac{1}{1-|h|} & \textit{for}\quad |z| \ge 1,\end{cases}
\end{aligned}
$$

*and so on,*

$$|A_n| < \begin{cases} \dfrac{2^n}{n!}\,|h|^2 \cdot \dfrac{1}{1-|h|} & \textit{for} \quad |z| < 1, \\ \dfrac{2^n}{n!}\,|z|^{n-2}|h|^2 \cdot \dfrac{1}{1-|h|} & \textit{for} \quad |z| \geq 1. \end{cases}$$

*Therefore*

$$\sum_{n=3}^{\infty} |A_n| < \begin{cases} |h|^2 \cdot \dfrac{1}{1-|h|} \cdot \displaystyle\sum_{n=3}^{\infty} \dfrac{2^n}{n!} & \textit{for} \quad |z| < 1, \\ |h|^2 \cdot \dfrac{1}{1-|h|} \cdot \displaystyle\sum_{n=3}^{\infty} \dfrac{2^n}{n!}\,|z|^{n-2} & \textit{for} \quad |z| \geq 1 \end{cases}$$

*and the series*

$$\sum_{n=3}^{\infty} \frac{2^n}{n!} \quad \textit{and} \quad \sum_{n=3}^{\infty} \frac{2^n}{n!}\,|z|^{n-2}$$

*converge by the ratio test* (see [98]). *Thus,*

$$\sum_{n=3}^{\infty} |A_n| = o(h)$$

*for any fixed finite quaternion $z$.*

**Proposition 2.5.** *The following equality holds*:

$$(e^z)' = e^z. \tag{2.7}$$

*Proof.* The equality

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

implies that, for any $h \in \mathbb{H}$,

$$e^{z+h} - z^z = h + \frac{(z+h)^2 - z^2}{2!} + \frac{(z+h)^3 - z^3}{3!} + \frac{(z+h)^4 - z^4}{4!} + \cdots\,,$$

and applying Lemma 2.4 to the right-hand side of this equality, we obtain

$$\begin{aligned} &e^{z+h} - e^z \\ &\qquad = h + \frac{1}{2!}\big(zh + hz\big) + \frac{1}{3!}\big(z^2h + zhz + hz^2\big) + \frac{1}{4!}\big(z^3h + z^2hz + zhz^2 + hz^3\big) + \cdots + o(h). \end{aligned}$$

Therefore

$$\begin{aligned} e^{z+h} - e^z = \Big(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\Big)h + \Big(\frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots\Big)hz& \\ + \Big(\frac{1}{3!} + \frac{z}{4!} + \frac{z^2}{5!} + \cdots\Big)hz^2 + \cdots + o(h)& \end{aligned} \tag{2.8}$$

and hence

$$\begin{aligned}(e^z)' &= 1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots \\ &\quad + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots \\ &\quad\quad + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots \\ &\quad\quad\quad + \frac{z^3}{4!} + \cdots \\ &= 1 + 2\,\frac{z}{2!} + 3\,\frac{z^2}{3!} + 4\,\frac{z^3}{4!} + \cdots \\ &= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = e^z. \qquad \square\end{aligned}$$

**Proposition 2.6.** *The following equality holds*:

$$(\sin z)' = \cos z. \tag{2.9}$$

*Proof.*

$$\begin{aligned}\sin(z+h) - \sin z &= (z+h) - \frac{(z+h)^3}{3!} + \frac{(z+h)^5}{5!} - \cdots - z + \frac{z^3}{3!} - \frac{z^5}{5!} + \cdots \\ &= h - \frac{(z+h)^3 - z^3}{3!} + \frac{(z+h)^5 - z^5}{5!} - \cdots \\ &= h - \frac{1}{3!}\,(z^2h + zhz + hz^2) + \frac{1}{5!}\,(z^4h + z^3hz + z^2hz^2 + zhz^3 + hz^4) + \cdots + o(h).\end{aligned}$$

Therefore,

$$\sin(z+h) - \sin z = h + \Big(-\frac{z^2}{3!} + \frac{z^4}{5!}\Big)h + zh\Big(-\frac{z}{3!} + \frac{z^3}{5!}\Big) + h\Big(-\frac{z^2}{3!} + \frac{z^4}{5!}\Big) + \cdots + o(h).$$

Hence

$$\begin{aligned}(\sin z)' &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + z\Big(-\frac{z}{3!} + \frac{z^3}{5!}\Big) - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \\ &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \cos z. \qquad \square\end{aligned}$$

Similarly, we can prove the following assertion.

**Proposition 2.7.** *The following equality holds*:

$$(\cos z)' = -\sin z. \tag{2.10}$$

## 3. Calculation of $\mathbb{H}$-Derivatives

The rules for calculating $\mathbb{H}$-derivatives are identical to those derived in a standard calculus course.

**Proposition 3.1.** *Let $f$ and $\varphi$ be two functions defined in a neighborhood of $z^0 \in \mathbb{H}$. If both functions $f$ and $\varphi$ are $\mathbb{H}$-differentiable at $z^0$, then*

(i) *both $cf$ and $fc$ are $\mathbb{H}$-differentiable at $z^0$ for all $c \in \mathbb{H}$ and*

$$(cf)'(z^0) = cf'(z^0), \quad (fc)'(z^0) = f'(z^0)c;$$

(ii) *$f+\varphi$ is $\mathbb{H}$-differentiable at $z^0$ and*

$$(f+\varphi)'(z^0) = f'(z^0) + \varphi'(z^0);$$

(iii) *$f\varphi$ is $\mathbb{H}$-differentiable at $z^0$ and*

$$(f\varphi)'(z^0) = f'(z^0)\varphi(z^0) + f(z)\varphi'(z^0).$$

*Proof.* The proof of (i) is obvious.

Since $f$ and $\varphi$ are $\mathbb{H}$-differentiable at $z^0$, there are representations

$$f(z^0+h) - f(z^0) = \sum_k A_k h B_k + o(h),$$
$$\varphi(z^0+h) - \varphi(z^0) = \sum_k C_k h D_k + o(h).$$

Then

$$\begin{aligned}(f+\varphi)(z^0+h) - (f+\varphi)(z^0) &= \big[f(z^0+h) - f(z^0)\big] + \big[\varphi(z^0+h) - \varphi(z^0)\big] \\ &= \sum_k A_k h B_k + \sum_k C_k h D_k + o(h),\end{aligned}$$

and hence

$$(f+\varphi)'(z^0) = \sum_k A_k B_k + \sum_k C_k D_k = f'(z^0) + \varphi'(z^0).$$

This proves (ii).

Next, since

$$\begin{aligned}&f(z^0+h)\varphi(z^0+h) - f(z^0)\varphi(z^0) \\ &\quad= \big[f(z^0+h) - f(z^0)\big]\varphi(z^0+h) + f(z^0)\big[\varphi(z^0+h) - \varphi(z^0)\big] \\ &\quad= \Big[\sum_k A_k h B_k + o(h)\Big]\varphi(z^0+h) + f(z^0)\Big[\sum_k C_k h D_k + o(h)\Big] \\ &\quad= \Big[\sum_k A_k h B_k + o(h)\Big]\cdot\Big[\varphi(z^0) + \sum_k C_k h D_k + o(h)\Big] + f(z^0)\Big[\sum_k C_k h D_k + o(h)\Big] \\ &\quad= \Big(\sum_k A_k h B_k\Big)\varphi(z^0) + f(z^0)\sum_k C_k h D_k + o(h),\end{aligned}$$

it follows that

$$(f\varphi)'(z^0) = \Big(\sum_k A_k B_k\Big)\varphi(z^0) + f(z^0)\sum_k C_k D_k = f'(z^0)\varphi(z^0) + f(z^0)\varphi'(z^0);$$

this proves (iii). □

The following two assertions are obtained immediately.

**Corollary 3.2.** *If $f_1, f_2, \dots, f_n$ are $\mathbb{H}$-differentiable functions at a point $z^0$, then their product $f_1 f_2 \cdots f_n$ is also $\mathbb{H}$-differentiable at $z^0$ and we have*

$$\begin{aligned}&(f_1 f_2 \cdots f_n)'(z^0) \\ &\quad= f_1'(z^0) f_2(z^0)\cdots f_n(z^0) + f_1(z^0) f_2'(z^0) f_3(z^0)\cdots f_n(z^0) + \dots + f_1(z^0)\cdots f_{n-1}(z^0) f_n'(z^0).\end{aligned} \tag{3.1}$$

**Corollary 3.3.** *If a function $f$ is $\mathbb{H}$-differentiable at a point $z^0$, then $f^n$ is also $\mathbb{H}$-differentiable at $z^0$ for all $n = 1, 2, \dots$ and we have*

$$(f^n)'(z^0) = f'(z^0) f^{n-1}(z^0) + f(z^0) f'(z^0) f^{n-2}(z^0) + \dots + f^{n-1}(z^0) f'(z^0). \tag{3.2}$$

**Proposition 3.4.** *If a function $\varphi$ is $\mathbb{H}$-differentiable at a point $z^0$ and if $\varphi \neq 0$ in a neighborhood of $z^0$, then*[2] *$1/\varphi$ is also $\mathbb{H}$-differentiable at $z^0$ and*

$$\Big(\frac{1}{\varphi}\Big)'(z^0) = -\frac{1}{\varphi(z^0)} \cdot \varphi'(z^0) \cdot \frac{1}{\varphi(z^0)} . \tag{3.3}$$

*Proof.* First, we prove that the equality

$$q_1^{-1} - q_2^{-1} = q_1^{-1}(q_1 - q_2)q_2^{-1}(q_1 - q_2)q_2^{-1} - q_2^{-1}(q_1 - q_2)q_2^{-1} \tag{3.4}$$

holds for any two nonzero quaternions $q_1$ and $q_2$. Indeed, since $q_1^{-1}$ is the inverse of $q_1$ and $q_2^{-1}$ is the inverse of $q_2$, we obtain

$$\begin{aligned}
&q_1^{-1}(q_1 - q_2)q_2^{-1}(q_1 - q_2)q_2^{-1} - q_2^{-1}(q_1 - q_2)q_2^{-1} \\
&\quad = (1 - q_1^{-1}q_2)q_2^{-1}(q_1q_2^{-1} - 1) - (q_2^{-1}q_1 - 1)q_2^{-1} = (q_2^{-1} - q_1^{-1})(q_1q_2^{-1} - 1) - (q_2^{-1}q_1q_2^{-1} - q_2^{-1}) \\
&\qquad = (q_2^{-1}q_1q_2^{-1} - q_2^{-1} - q_2^{-1} + q_1^{-1}) - (q_2^{-1}q_1q_2^{-1} - q_2^{-1}),
\end{aligned}$$

as desired. Setting $\varphi(z^0 + h)$ and $\varphi(z^0)$ in the equality, we obtain

$$\begin{aligned}
&\frac{1}{\varphi(z^0 + h)} - \frac{1}{\varphi(z^0)} \\
&\quad = \Big\{ -\frac{1}{\varphi(z^0)} + \frac{1}{\varphi(z^0 + h)}\big[\varphi(z^0 + h) - \varphi(z^0)\big]\frac{1}{\varphi(z^0)} \Big\} \cdot \big[\varphi(z^0 + h) - \varphi(z^0)\big]\frac{1}{\varphi(z^0)} .
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{1}{\varphi(z^0 + h)} - \frac{1}{\varphi(z^0)} &= -\frac{1}{\varphi(z^0)}\big[\varphi(z^0 + h) - \varphi(z^0)\big]\frac{1}{\varphi(z^0)} \\
&\quad + \frac{1}{\varphi(z^0 + h)}\big[\varphi(z^0 + h) - \varphi(z^0)\big]\frac{1}{\varphi(z^0)}\big[\varphi(z^0 + h) - \varphi(z^0)\big]\frac{1}{\varphi(z^0)} \\
&= -\frac{1}{\varphi(z^0)}\Big[\sum C_k h D_k + o(h)\Big]\frac{1}{\varphi(z^0)} + o(h) = -\frac{1}{\varphi(z^0)}\Big[\sum C_k h D_k\Big]\frac{1}{\varphi(z^0)} + o(h).
\end{aligned}$$

Hence

$$\Big(\frac{1}{\varphi}\Big)'(z^0) = -\frac{1}{\varphi(z^0)}\Big[\sum C_k D_k\Big]\frac{1}{\varphi(z^0)} = -\frac{1}{\varphi(z^0)} \cdot \varphi'(z^0) \cdot \frac{1}{\varphi(z^0)} . \qquad \square$$

**Corollary 3.5.** *For $z \neq 0$ we have*

$$(z^m)' = mz^{m-1}, \quad m = -1, -2, \dots . \tag{3.5}$$

*Proof.* Setting $n = -m$ and applying Propositions 2.3 and 3.4, we obtain

$$(z^m)' = \Big(\frac{1}{z^n}\Big)' = -\frac{1}{z^n}(z^n)'\frac{1}{z^n} = -\frac{1}{z^n}nz^{n-1}\frac{1}{z^n} = -nz^{-n-1} = mz^{m-1} . \qquad \square$$

**Corollary 3.6.** *For an arbitrary constant $c$, we have*

$$\Big(\frac{1}{c - z}\Big)' = \frac{1}{(c - z)^2}, \quad z \neq c. \tag{3.6}$$

**Corollary 3.7.** *If quaternionic functions $f$ and $\varphi$ are $\mathbb{H}$-differentiable at a point $z^0$ and $\varphi \neq 0$ in a neighborhood of $z^0$, then the functions $f \cdot \dfrac{1}{\varphi}$ and $\dfrac{1}{\varphi} \cdot f$ are also $\mathbb{H}$-differentiable at $z^0$ and we have*

$$\Big(f \cdot \frac{1}{\varphi}\Big)'(z^0) = f'(z^0) \cdot \frac{1}{\varphi(z^0)} - f(z^0)\frac{1}{\varphi(z^0)} \cdot \varphi'(z^0) \cdot \frac{1}{\varphi(z^0)} \tag{3.7}$$

[2] Recall that each nonzero quaternion $q \neq 0$ has a unique inverse inverse determined by the formula (1.17).

*and*

$$\left(\frac{1}{\varphi}\cdot f\right)'(z^0) = -\frac{1}{\varphi(z^0)}\cdot\varphi'(z^0)\cdot\frac{1}{\varphi(z^0)}\,f(z^0) + \frac{1}{\varphi(z^0)}\cdot f'(z^0). \tag{3.8}$$

**Proposition 3.8.** *Let a function $f(z)$ be defined in some neighborhood of a point $z^0 \in \mathbb{H}$ and let a function $F(w)$ be defined in some neighborhood of the point $w^0 = f(z^0)$. Assume that $f$ is $\mathbb{H}$-differentiable at $z^0$ and $F$ is $\mathbb{H}$-differentiable at $w^0$. If $F'(w^0) = \sum\limits_k A_k B_k$, then the composite $Ff$ is $\mathbb{H}$-differentiable at $z^0$ and we have*

$$(Ff)'(z^0) = \sum_k A_k f'(z^0) B_k. \tag{3.9}$$

*Proof.* Let $z$ be in the neighborhood of $z^0$. We set $w = f(z)$. Then

$$F(w) - F(w^0) = \sum_k A_k(w - w^0)B_k + \omega_1(w^0, w),$$
$$f(z) - f(z^0) = \sum_j C_j(z - z^0)D_j + \omega_2(z^0, z),$$

and using these presentations, we calculate

$$\begin{aligned} F(f(z)) - F(f(z^0)) &= \sum_k A_k(f(z) - f(z^0))B_k + \omega_1(f(z^0), f(z)) \\ &= \sum_k A_k\Big(\sum_j C_j(z - z^0)D_j\Big)B_k + o(h) + \omega_1(f(z^0), f(z)) \\ &= \sum_k\sum_j A_k C_j(z - z^0)D_j B_k + o(h) + \omega_1(f(z^0), f(z)). \end{aligned}$$

But since

$$\frac{|\omega_1(f(z^0), f(z))|}{|z - z^0|} = \frac{|\omega_1(f(z^0), f(z))|}{|w - w^0|}\cdot\frac{|w - w^0|}{|z - z^0|} \xrightarrow[z\to z^0]{} 0,$$

we have

$$(Ff)'(z^0) = \sum_k\sum_j A_k C_j D_j B_k = \sum_k A_k\Big(\sum_j C_j D_j\Big)B_k = \sum_k A_k f'(z^0)B_k. \qquad \square$$

For the particular case $F(w) = w^n$, using (2.5), in addition to equality (3.2) we obtain the following assertion.

**Corollary 3.9.** *If a function $f$ is $\mathbb{H}$-differentiable, then*

$$(f^n)' = f^{n-1}\cdot f' + f^{n-2}\cdot f'\cdot f + f^{n-3}\cdot f'\cdot f^2 + \cdots + f'\cdot f^{n-1}. \tag{3.10}$$

## 4. $\mathbb{H}$-Derivative of the Quaternion Logarithm Function

A quaternion $w$ is called the *logarithm* of a finite quaternion $z \neq 0$ if $z = e^w$; in this case we write $w = \ln z$.

In order to define the $\mathbb{H}$-derivative $w' = (\ln z)'$, we first note that the $\mathbb{H}$-derivative of the left-hand side of the identity $z = e^{\ln z}$ exits and is equal to 1 by Proposition 2.3. Applying now Proposition 3.8 to the right-hand side and taking into account (2.8), we obtain

$$\begin{aligned} 1 = \left(1 + \frac{w}{2!} + \frac{w^2}{3!} + \cdots\right)\cdot w' + \left(\frac{1}{2!} + \frac{w}{3!} + \frac{w^2}{4!} + \cdots\right)\cdot w'\cdot w& \\ + \left(\frac{1}{3!} + \frac{w}{4!} + \frac{w^2}{5!} + \cdots\right)\cdot w'\cdot w^2 + \cdots .& \end{aligned} \tag{4.1}$$

Thus, the $\mathbb{H}$-derivative $w' = (\ln z)'$ satisfies Eq. (4.1).

**Remark 4.1.** *If $ww'$ and $w'w$ were equal to each other, then we could write $w \cdot w', w^2 \cdot w', \dots$ instead of $w' \cdot w, w' \cdot w^2, \dots$, and then Eq. (4.1) would take the form*

$$1 = \Big(1 + \frac{w}{2!} + \frac{w^2}{3!} + \cdots\Big) \cdot w' + \Big(\frac{w}{2!} + \frac{w^2}{3!} + \cdots\Big) \cdot w' + \Big(\frac{w^2}{3!} + \frac{w^3}{4!} + \frac{w^4}{5!} + \cdots\Big) \cdot w' + \cdots$$
$$= \Big(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots\Big) \cdot w' = e^w \cdot w' = e^{\ln z} \cdot (\ln z)' = z \cdot (\ln z)'.$$

*So, we can obtain the classical formula*

$$(\ln z)' = \frac{1}{z}, \tag{4.2}$$

*which is well known in the case of a complex variable $z$.*

## 5. Necessary and Sufficient Conditions of Continuity or Differentiability of Functions of Several Real Variables. Criterion of $\mathbb{C}^n$-Differentiability

A quaternion function $f(z) = u_0(z) + u_1(z)i_1 + u_2(z)i_2 + u_3(z)i_3$ of a quaternion variable $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ is continuous or differentiable at a point $z^0 = x_0^0 + x_1^0 i_1 + x_2^0 i_2 + x_3^0 i_3$ (with respect to the set of real variables $(x_0, x_1, x_2, x_3)$) if and only if all real functions $u_k(z)$ possess this property at the point $z^0$. Hence the corresponding results recently obtained by the author will be formulated only for real functions of many real variables. We need these results for our further discussion.

A function of many variables will not have the continuity or differentiability property only because it has the same property with respect to each independent variable.

Functions with this drawback at individual points have been known since the late 19th century, and on the massive set since the 20th century. Namely, the following statement is valid.

**Statement A** (see [134, pp. 432–433])**.** *There exists a function of two variables that is discontinuous at almost every point of the unit square and continuous with respect to every variable at every point of that square.*

These and similar problems were studied, for example, by Z. Piotrowski (see [114]).

Here the problem consists in finding out whether there exists or not any property of a function with respect to an independent real variable and whether the fulfillment of this property for all independent variables will be the necessary and sufficient condition for the continuity or differentiability of the function itself.

In formulating the main results, we use the following notation: $x = (x_1, \dots, x_n)$, $x^0 = (x_1^0, \dots, x_n^0)$, $x(x_k^0) = (x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots, x_n)$.

### 5.1. Continuity conditions.

A function $f$ is said to be *strongly partial continuous* with respect to the variable $x_k$ at a point $x^0$ if the equality

$$\lim_{x \to x^0} \big[f(x) - f(x(x_k^0))\big] = 0 \tag{5.1}$$

is fulfilled, and $f$ is called *separately strongly partial continuous* at the point $x^0$ if $f$ is strongly partial continuous at $x^0$ with respect to every variable, i.e., Eq. (5.1) is fulfilled for all $k = 1, 2, \dots, n$.

**Theorem 5.1** (see [37, 38] and [40, pp. 20–25])**.** *For the continuity of the function $f$ at the point $x^0$, it is necessary and sufficient that it possess separately strong partial continuity at $x^0$.*

The expression

$$f(x) - f(x(x_k^0)) \quad \text{for } |x_j - x_j^0| \le c_j |x_k - x_k^0|,\ j \ne k,$$

depending on the variables $x_1, \dots, x_n$, is called an *angular partial increment* of the function $f$ at the point $x^0$ with respect to the variable $x_k$, corresponding to the collection $c = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ of positive constants.

The *angular partial continuity* of the function $f$ at the point $x^0$ with respect to the variable $x_k$ means the fulfillment of the equality

$$\lim_{\substack{x_k \to x_k^0 \\ |x_j - x_j^0| \le c_j |x_k - x_k^0| \\ j \ne k}} \left[ f(x) - f(x(x_k^0)) \right] = 0 \tag{5.2}$$

for every collection $c = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ of positive constants.

The function $f$ is called *separately angular partial continuous at a point* $x^0$ if with respect to every variable the function $f$ possesses the property of angular partial continuity at the point $x^0$, i.e., if Eq. (5.2) is fulfilled for all $k = 1, \dots, n$ and for every collection $c = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ of positive constants.

**Theorem 5.2** (see [37, 38] and [40, pp. 25–27])**.** *For the continuity of the function $f$ at the point $x^0$, the necessary and sufficient condition is the separately angular partial continuity at $x^0$.*

If in the definition of angular partial continuity we set $c_j = 1$ for all $j \ne k$, then we have the *nonintense angular partial continuity* at the point $x^0$ of the function $f$ with respect to the variable $x_k$.

**Theorem 5.3** (see [40, pp. 27–28])**.** *For the continuity of the function $f$ at the point $x^0$, the necessary and sufficient condition is the separately nonintense angular partial continuity of the function $f$ at the point $x^0$.*

**5.2. Angular partial derivative and an angular gradient.** The existence of all ordinary partial derivatives, i.e., of ordinary gradients of the real function $f$ at the point $x^0$, does not imply the differentiability of $f$ at the point $x^0$. Even the function, possessing a finite gradient at the point $x^0$, may be discontinuous at $x^0$. Such are, for example, most of the functions of two variables at the point $(0, 0)$ as indicated in Piotrowski's work [114].

It is remarkable that this fact can be realized at all points of a set whose plane measure is arbitrarily near to the total measure.

**Statement B** (see [134, Sec. 4])**.** *For every positive number $\mu < 1$, there exists a function $F$ defined on the square*

$$Q = \Big\{ (x, y) \in \mathbb{R}^2;\ 0 \le x \le 1,\ 0 \le y \le 1 \Big\},$$

*possessing finite partial derivatives of all orders at all points of $Q$, and, simultaneously, being discontinuous on a certain set $E \subset Q$ of the plane measure $\mu^2$.*

We say that a function $F$ has an *angular partial derivative* with respect to the variable $x_k$ at the point $x^0$ (notation $f'_{\widehat{x}_k}(x^0)$) if for every collection $c = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ of positive $n - 1$ constants, there exists the independent of $c$ finite limit

$$f'_{\widehat{x}_k}(x^0) = \lim_{\substack{x_k \to x_k^0 \\ |x_j - x_j^0| \le c_j |x_k - x_k^0| \\ j \ne k}} \frac{f(x) - f(x(x_k^0))}{x_k - x_k^0}. \tag{5.3}$$

The existence of $f'_{\widehat{x}_k}(x^0)$ implies the existence of the partial derivative $f'_{x_k}(x^0)$ and the fulfillment of the equality $f'_{x_k}(x^0) = f'_{\widehat{x}_k}(x^0)$. To show this, we must set in (5.3) $x_j = x_j^0$ for all $j \ne k$.

The existence of the angular partial derivative does not, in general, follow from the existence of the ordinary partial derivative. If $f'_{\widehat{x}_k}(x^0)$ is finite, then the function $f$ with respect to the variable $x_k$ has the property of angular partial continuity at the point $x^0$.

If there exist finite $f'_{\widehat{x}_k}(x^0)$, $k=1,\dots,n$, then we say that the function $f$ possesses an *angular gradient* at the point $x^0$ and write

$$\operatorname{anggrad} f(x^0) = \Big(f'_{\widehat{x}_1}(x^0),\dots,f'_{\widehat{x}_n}(x^0)\Big).$$

**Theorem 5.4** (see [37, 39] and [40, pp. 60–64]). *A function $f$ is differentiable at the point $x^0$ if and only if* anggrad $f(x^0)$ *is finite. The total differential $df(x^0)$ of the differentiable at the point $x^0$ function $f$ admits the following representation*:

$$df(x^0) = \sum_{k=1}^{n} f'_{\widehat{x}_k}(x^0)\,dx_k\,.$$

**Theorem 5.5** (see [39] and [40, p. 65]). *A function $f$ is differentiable at the point $x^0$ if and only if the nonintense angular partial derivative*

$$D_{\widehat{x}_k} f(x^0) = \lim_{\substack{x_k\to x_k^0\\ |x_j-x_j^0|\le|x_k-x_k^0|\\ j\ne k}} \frac{f(x)-f(x(x_k^0))}{x_k-x_k^0}$$

*is finite for all $k=1,\dots,n$.*

**Corollary 5.6** (see [39] and [40, p. 65]). *The finiteness of all $D_{\widehat{x}_k} f(x^0)$ implies the finiteness of all $f'_{\widehat{x}_k}(x^0)$ and the fulfillment of the equalities*

$$f'_{\widehat{x}_k}(x^0) = D_{\widehat{x}_k} f(x^0), \quad k=1,\dots,n, \tag{5.4}$$

$$df(x^0) = \sum_{k=1}^{n} D_{\widehat{x}_k} f(x^0)\,dx_k. \tag{5.5}$$

**5.3. Strong partial derivatives and strong gradients.** We say that a function $f$ possesses the *strong partial derivative* with respect to the variable $x_k$ at the point $x^0$ (notation $f'_{[x_k]}(x^0)$) if there exists a finite limit

$$f'_{[x_k]}(x^0) = \lim_{x\to x^0} \frac{f(x)-f(x(x_k^0))}{x_k-x_k^0}\,. \tag{5.6}$$

We say that a function $f$ has the *strong gradient* at the point $x^0$ (notation strgrad $f(x^0)$) if for every $k=1,\dots,n$ there exist finite $f'_{[x_k]}(x^0)$; in this case we write

$$\operatorname{strgrad} f(x^0) = (f'_{[x_k]}(x^0),\dots,f'_{[x_n]}(x^0)). \tag{5.7}$$

If there exists strgrad $f(x^0)$, then there exists anggrad $f(x^0)$, and the equalities strgrad $f(x^0)$ = anggrad $f(x^0)$ = grad $f(x^0)$ hold.

Consequently, we have the following assertion.

**Theorem 5.7** (see [37, 39], and [40, p. 77]). *The existence of the finite* strgrad $f(x^0)$ *implies the existence of the total differential $df(x^0)$ and*

$$\operatorname{strgrad} f(x^0) = \operatorname{anggrad} f(x^0) = \operatorname{grad} f(x^0). \tag{5.8}$$

If grad $f(x)$ is continuous at the point $x^0$, then we have the equality strgrad $f(x^0)$ = grad $f(x^0)$ (see [40, p. 75]).

Moreover, the existence of the finite strgrad $f(x^0)$ does not imply, in general, the continuity of grad $f(x)$ at the point $x^0$. However this fact can be essentially strengthened as follows.

**Theorem 5.8** (see [40, p. 76]). *There exists an absolutely continuous function of two variables that has almost everywhere both a finite strong and a discontinuous gradient.*

**Proposition 5.9** (see [37, 39] and [40, p. 77]). *The finiteness of* anggrad $f(x^0)$ *or, equivalently, the existence of* $df(x^0)$ *does not imply the existence of* strgrad $f(x^0)$.

For example, the function $\lambda(x_1, x_2) = |x_1 \cdot x_2|^{2/3}$ is differentiable at the point $x^0 = (0,0)$, but strgrad $\lambda(x^0)$ does not exist.

The differentiability of the function $\lambda(x_1, x_2) = |x_1 \cdot x_2|^{2/3}$ at the point $x^0 = (0,0)$ is a simple corollary of our next statement.

**Proposition 5.10** (see [39] and [40, p. 66]). *Assume that* $\alpha_j$, $j = 1, \ldots, n$, *are positive numbers. Then the condition*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n > 1 \tag{5.9}$$

*is necessary and sufficient for the everywhere continuous function*

$$\varphi(x_1, \ldots, x_n) = |x_1|^{\alpha_1|} \cdot |x_2|^{\alpha_2} \cdots |x_n|^{\alpha_n} \tag{5.10}$$

*to be differentiable at the point* $x^0 = (0, \ldots, 0)$.

In particular, the function $\gamma(x_1, \ldots, x_n) = |x_1| \cdots |x_n|)^{\alpha}$ is differentiable at the point $x^0$ if and only if $\alpha > 1/n$.

If $\alpha_1 + \alpha_2 + \cdots + \alpha_n \leq 1$, then all $D_{\widehat{x}_k}\varphi(x^0)$ are denied the existence.

This result was later strengthened by G. G. Oniani who proved that the finiteness of a strong gradient is an essentially stronger property than the differentiability.

**Theorem 5.11** (see [109, 110]). *For arbitrary* $n \geq 2$, *there exists a continuous function* $f : [0,1]^n \to \mathbb{R}$ *such that the following conditions hold:*

1. $f$ *is almost everywhere differentiable*;
2. $f$ *is denied having almost everywhere a finite strong gradient.*

The next theorem is an improvement of Theorem 5.11.

**Theorem 5.12** (see [12, Theorem 4]). *For arbitrary* $n \geq 2m$ *there exists a continuous function* $f : [0,1]^n \to \mathbb{R}$, *which is almost everywhere differentiable but is everywhere denied having a finite strong gradient.*

As is known, functions of bounded variation in the Hardy or Arzela sense have the differentiability property almost everywhere, i.e., have a finite angular gradient almost everywhere.

As to the existence of a strong gradient, the functions belonging to the Hardy and Arzela classes behave differently.

**Theorem 5.13** (see [11, 12]). *Every function* $f : [0,1]^n \to \mathbb{R}$ *of bounded variation in the Hardy sense has a finite strong gradient almost everywhere.*

**Theorem 5.14** (see [12, Theorem 3]). *For arbitrary* $n \geq 2$, *there exists a continuous function* $f : [0,1]^n \to \mathbb{R}$ *of bounded variation in the Arzela sense that is denied having a finite strong gradient.*

The results obtained by the author make it possible to classify functions according to the properties of their gradients.

**Theorem 5.15** (see [40, p. 80] and [43, p. 99]). *The class of functions with continuous gradients at a point* $x^0$ *is strictly contained in the class of functions with finite strong gradients at the point* $x^0$, *and the latter class of functions is strictly contained in the class of functions with finite angular gradients at the point* $x^0$. *The latter class coincides with the class of differentiable functions at* $x^0$.

**Remark 5.16.** *The notions of angular and strong gradients were generalized by L. Bantsuri who introduced the notion of a gradient that respect to the basis and in particular established the relationship between the differentiability and the existence of the gradient which he himself had introduced (see [9, 10] and [43, p. 99]).*

### 5.4. $\mathbb{C}^n$-Differentiability criterion.

**Theorem 5.17** (see [41]). *A function $f$ is $\mathbb{C}^n$-differentiable at a point $z \in \mathbb{C}^n$ if and only if the condition*

$$f'_{\widehat{x}_k}(z) + i f'_{\widehat{y}_k}(z) = 0 \tag{5.11}$$

*or, equivalently,*

$$D_{\widehat{x}_k} f(z) + i D_{\widehat{y}_k} f(z) = 0 \tag{5.12}$$

*holds for all $k = 1, \dots, n$, where $z = (z_1, \dots, z_n)$ and $z_k = x_k + y_k$.*

This theorem implies the following classical assertion.

**Hartog's Main Theorem (see [41, p. 17]).** *A function $f$, holomorphic (analytic) with respect to each variable in an open set $G \subset \mathbb{C}^n$, is $\mathbb{C}^n$-holomorphic ($\mathbb{C}^n$-analytic) in $G$.*

## 6. Necessary and Sufficient Conditions for the $\mathbb{H}$-Differentiability of Quaternion Functions

Now let us examine how the $\mathbb{H}$-differentiability of a quaternion function $f(z) = u_0(z) + u_1(z)i_1 + u_2(z)i_2 + u_3(z)i_3$ of a quaternion variable $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ is related to the existence of a differential $df(z)$ (with respect to real variables $x_0$, $x_1$, $x_2$, and $x_3$).

Since partial angular derivatives are derivatives with respect to real variables (see Eq. (5.3)), the conditions of differentiability for real, complex, and quaternion functions are written in one and the same form.

It then follows that for the differentiability of a quaternion function $f$ at a point $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, the necessary and sufficient condition is the existence of the finite partial angular derivatives

$$f'_{\widehat{x}_k} = (u_0)'_{\widehat{x}_k} + i_1 (u_2)'_{\widehat{x}_k} + i_2 (u_2)'_{\widehat{x}_k} + i_3 (u_3)'_{\widehat{x}_k}, \quad k = 0, 1, 2, 3.$$

Moreover, if $f$ is differentiable at $z$, the following equalities hold for its differential $df(z)$:

$$\begin{aligned} df(z) &= f'_{\widehat{x}_0}(z)\, dx_0 + f'_{\widehat{x}_1}(z)\, dx_1 + f'_{\widehat{x}_2}(z)\, dx_2 + f'_{\widehat{x}_3}(z)\, dx_3, \\ df(z) &= du_0(z) + i_1\, du_1(z) + i_2\, du_2(z) + i_3\, du_3(z). \end{aligned} \tag{6.1}$$

**Theorem 6.1** (see [44]). *If a quaternion function $f$ is $\mathbb{H}$-differentiable at a point $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, then $f$ is differentiable at the same point $z$ and its partial angular derivatives $f'_{\widehat{x}_0}(z)$, $f'_{\widehat{x}_1}(z)$, $f'_{\widehat{x}_2}(z)$, and $f'_{\widehat{x}_3}(z)$ can be expressed in terms of the $\mathbb{H}$-derivative $f'(z) = \sum\limits_k A_k(z) B_k(z)$ as follows*:

$$f'_{\widehat{x}_0}(z) = \sum_k A_k(z) B_k(z) = f'(z), \tag{6.2}$$

$$f'_{\widehat{x}_1}(z) = \sum_k A_k(z) i_1 B_k(z), \tag{6.3}$$

$$f'_{\widehat{x}_2}(z) = \sum_k A_k(z) i_2 B_k(z), \tag{6.4}$$

$$f'_{\widehat{x}_3}(z) = \sum_k A_k(z) i_3 B_k(z). \tag{6.5}$$

*Moreover, we have*

$$df(z) = \sum_k A_k(z)\, dz\, B_k(z). \tag{6.6}$$

*Proof.* Since $dz = dx_0 + i_1 dx_1 + i_2 dx_2 + i_3 dx_3$ and there exists the $\mathbb{H}$-derivative $f'(z)$, the increment $f(z+dz) - f(z)$ can be represented as in Eq. (2.1). Thus we have

$$\begin{aligned} f(x_0 + dx_0, x_1 + dx_1, x_2 + dx_2, x_3 + dx_3) - f(x_0, x_1, x_2, x_3) \\ = dx_0 \sum_k A_k(z) B_k(z) + dx_1 \sum_k A_k(z) i_1 B_k(z) \\ + dx_2 \sum_k A_k(z) i_2 B_k(z) + dx_3 \sum_k A_k(z) i_3 B_k(z) + o(dz). \end{aligned}$$

It follows that $f$ is differentiable at the point $z$ and the following equality holds:

$$\begin{aligned} df(z) = dx_0 \sum_k A_k(z) B_k(z) + dx_1 \sum_k A_k(z) i_1 B_k(z) \\ + dx_2 \sum_k A_k(z) i_2 B_k(z) + dx_3 \sum_k A_k(z) i_3 B_k(z). \end{aligned} \tag{6.7}$$

Thus, Eq. (6.6) is fulfilled. Now, by virtue of Eq. (6.1), we obtain Eq. (6.2)–(6.5) from (6.7). □

**Remark 6.2.** *Equality* (6.7) *can be interpreted as follows. As in the classical case, the differential $df(z)$ of the $\mathbb{H}$-differentiable function $f$ is linear with respect to the differential $dz$ of the independent variable $z$.*

**Theorem 6.3.** *If a quaternion function $f$ is differentiable at a point $z$ and its partial angular derivatives $f'_{\widehat{x}_0}(z)$, $f'_{\widehat{x}_1}(z)$, $f'_{\widehat{x}_2}(z)$ and $f'_{\widehat{x}_3}(z)$ can be expressed in the forms* (6.2)–(6.5) *for some quaternions $A_k(z)$ and $B_k(z)$, then $f$ is $\mathbb{H}$-differentiable at the point $z$ and*

$$f'(z) = \sum_k A_k(z) B_k(z). \tag{6.8}$$

*Proof.* By the differentiability of $f$ at $z$, we have

$$f(z+dz) - f(z) = df(z) + o(dz),$$

which, when compared with (6.1), yields

$$f(z+dz) - f(z) = f'_{\widehat{x}_0}(z)\, dx_0 + f'_{\widehat{x}_1}(z)\, dx_1 + f'_{\widehat{x}_2}(z)\, dx_2 + f'_{\widehat{x}_3}(z)\, dx_3 + o(dz).$$

Now multiplying both sides of Eqs. (6.2)–(6.5) by the real numbers $dx_0$, $dx_1$, $dx_2$, and $dx_3$, respectively, and adding the resulting equalities, we obtain

$$f(z+dz) - f(z) = \sum_k A_k(z) dz B_k(z) + o(dz),$$

which means that the $\mathbb{H}$-derivative $f'(z)$ exists and Eq. (6.8) holds. □

Combining Theorems 6.1 and 6.3, we obtain the following assertion.

**Theorem 6.4.** *The existence of the differential $df(z)$ of a quaternion function $f$ and its representation in the form*

$$df(z) = \sum_k A_k(z) dz B_k(z) \tag{6.9}$$

*are equivalent to the existence of the derivative $f'(z)$ and its representation in the form*

$$f'(z) = \sum_k A_k(z) B_k(z), \tag{6.10}$$

*where $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ and $dz = dx_0 + i_1 dx_1 + i_2 dx_2 + i_3 dx_3$.*

*Proof.* Since the existence of the differential $df(z)$ is equivalent to the existence of partial angular derivatives $f'_{\widehat{x}_m}(z)$ and its representability as in Eq. (6.1), one concludes from (6.9) that

$$f'_{\widehat{x}_m}(z) = \sum_k A_k(z) i_m B_k(z), \quad i_0 = 1, \quad m = 0, 1, 2, 3,$$

and

$$f(z + dz) - f(z) = \sum_k A_k(z) dz B_k(z) + \omega(z, dz), \tag{6.11}$$

where

$$\lim_{dz \to 0} \frac{|\omega(z, dz)|}{|dz|} = 0. \tag{6.12}$$

Now (6.10) follows immediately from (6.11) and (6.12).

Conversely, if $f'(z)$ exists and (6.10) holds, then Eqs. (6.11) and (6.12) also hold, from which we conclude that $df(z)$ exists and Eq. (6.9) is fulfilled. □

**Corollary 6.5** (see [44])**.** *When $x_2 = 0 = x_3$ and $u_2 = 0 = u_3$, one has a complex function $f(z) = u(z) + iv(z)$ of a complex variable $z = x + iy$. In this case, Eq. (6.9) has the form*

$$df(z) = c(z)\, dz = c(z)\, dx + ic(z) dy,$$

*where $c(z) = \sum_k A_k(z) B_k(z)$, from which we obtain the equalities*

$$f'_{\widehat{x}}(z) = c(z), \quad f'_{\widehat{y}}(z) = ic(z).$$

*Thus we have*

$$f'_{\widehat{x}}(z) + i f'_{\widehat{y}}(z) = 0. \tag{6.13}$$

Note that Eq. (6.13) is a necessary and sufficient condition for the complex function $f$ to be $\mathbb{C}^1$-differentiable at the point $z$ (see [1, pp. 85 and 65] and [41, p. 15] when $n = 1$). Moreover, we obtain the well known equalities

$$f'(z) = f'_{\widehat{x}}(z), \quad f'(z) = -i f'_{\widehat{y}}(z)$$

for the derivative $f'(z)$.

**Corollary 6.6** (see [44])**.** *For the quaternion $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, we have*

$$dz^n = z^{n-1}\, dz + z^{n-2}\, dz \cdot z + z^{n-3}\, dz \cdot z^2 + \cdots + z\, dz \cdot z^{n-2} + dz \cdot z^{n-1}$$

*for all $n = 0, 1, 2, \dots$.*

*Proof.* According to Eq. (6.2) we have

$$(z^n)' = z^{n-1} \cdot 1 + z^{n-2} \cdot z + z^{n-3} \cdot z^2 + \cdots + z \cdot z^{n-2} + 1 \cdot z^{n-1} = n z^{n-1} \tag{6.14}$$

for all $n = 0, 1, 2, \dots$. Combining this with (6.6) gives the desired result. □

**Corollary 6.7** (see [44])**.** *For the partial derivatives of the functions $f_n(z) = z^n$, $n = 0, 1, 2, \dots$, with respect to real variables $x_k$, $k = 0, 1, 2, 3$, we have*

$$(z^n)'_{x_k} = z^{n-1} \cdot i_k + z^{n-2} \cdot i_k \cdot z + \cdots + z \cdot i_k \cdot z^{n-2} + i_k \cdot z^{n-1}. \tag{6.15}$$

*Proof.* It suffices to apply (6.2)–(6.5) to (6.14) and take into account that the existence of partial angular derivatives implies that partial derivatives with respect to one and the same variable exist and are equal to one another. □

## 7. $\mathbb{C}^2$-Differentiability of Quaternion Functions and Their Representation by Integrals and Series

Introduce the complex variables $z_1 = x_0 + x_1 i_1$ and $z_2 = x_2 + x_3 i_1$; then the quaternion $z$ can be written in the form

$$z = z_1 + z_2 i_2 \tag{7.1}$$

or, briefly, $z = (z_1, z_2)$. Hence the four-dimensional real Euclidean space $\mathbb{R}^4$ is identified with the two-dimensional complex space $\mathbb{C}^2$ having points $z = (z_1, z_2)$.

The conjugate quaternion $\overline{z} = x_0 - x_1 i_1 - x_2 i_2 - x_3 i_3$ has the form $\overline{z} = \overline{z}_1 - z_2 i_2$, where $\overline{z}_1 = x_0 - x_1 i_1$. We also have the equality

$$z_2 i_2 = i_2 \overline{z}_2. \tag{7.2}$$

Therefore, $\overline{z_1 + z_2 i_2} = \overline{z}_1 - i_2 \overline{z}_2$. The equality $z = 0$ is equivalent to two equalities $z_1 = 0$ and $z_2 = 0$.

The product $zw$ of two quaternions $z = z_1 + z_2 i_2$ and $w = w_1 + w_2 i_2$ is defined by the formula

$$zw = (z_1 w_1 - \overline{w}_2 z_2) + (w_2 z_1 + z_2 \overline{w}_1) i_2$$

(see [80, p. 37]). In particular, for the complex variables $z_1$ and $z_2$ we have

$$z_1 z_2 = z_2 z_1, \quad z_1 \in \mathbb{C}^1, \quad z_2 \in \mathbb{C}^1. \tag{7.3}$$

The set of all points $z = (z_1, z_2) \in \mathbb{C}^2$ with the property

$$\|z - z^0\| < \delta,$$

where

$$\|z\| = \|z_1\| + \|z_2\|, \quad \|z_1\| = |x_0| + |x_1|, \quad \|z_2\| = |x_2| + |x_3|,$$

is called the $\delta$-neighborhood of a point $z^0 = (z_1^0, z_2^0) \in \mathbb{C}^2$; we denote it by $U(z^0, \delta)$. We also denote by the symbol $U(z^0)$ the neighborhood of a point $z^0$ in general.

Similarly to Eq. (7.1), the function $u = f(z)$ takes the form

$$f = f_1 + f_2 i_2, \tag{7.4}$$

where

$$f_1(z_1, z_2) = u_0(z_1, z_2) + i_1 u_1(z_1, z_2),$$
$$f_2(z_1, z_2) = u_2(z_1, z_2) + i_1 u_3(z_1, z_2).$$

It can be easily verified that the existence of an angular partial derivative $\partial f/\partial \widehat{x}_k$ of a quaternion function $f$ with respect to a variable $x_k$ is equivalent to the concurrent existence of the angular partial derivatives $\partial f_1/\partial \widehat{x}_k$ and $\partial f_2/\partial \widehat{x}_k$ of the complex functions $f_1$ and $f_2$ with respect to the same $x_k$, and the equality

$$\frac{\partial f}{\partial \widehat{x}_k} = \frac{\partial f_1}{\partial \widehat{x}_k} + \frac{\partial f_2}{\partial \widehat{x}_k}\, i_2, \quad k = 0, 1, 2, 3, \tag{7.5}$$

holds, where

$$\frac{\partial f_1}{\partial \widehat{x}_k} = \frac{\partial u_0}{\partial \widehat{x}_k} + i_1 \frac{\partial u_1}{\partial \widehat{x}_k}, \tag{7.6}$$

$$\frac{\partial f_2}{\partial \widehat{x}_k} = \frac{\partial u_2}{\partial \widehat{x}_k} + i_1 \frac{\partial u_3}{\partial \widehat{x}_k}. \tag{7.7}$$

Moreover, the differentiability of a quaternion function $f$ at a point $z$ is equivalent to the differentiability of the complex functions $f_1$ and $f_2$ at $z$, and we have the equality

$$df(z) = df_1(z) + df_2(z) i_2, \tag{7.8}$$

where

$$df_1(z) = du_0(z) + i_1 du_1(z), \quad df_2(z) = du_2(z) + i_1 du_3(z). \tag{7.9}$$

### 7.1. $\mathbb{C}^2$-Differentiability of quaternion functions.

**Definition 7.1** (see [45]). *A quaternion function $f(z) = f_1(z) + f_2(z)i_2$, $z = (z_1, z_2) = z_1 + z_2 i_2$, is said to be $\mathbb{C}^2$-differentiable at a point $z^0 = (z_1^0, z_2^0) = z_1^0 + z_2^0 i_2$ if there exist quaternions $d_1 + d_1' i_2$ and $d_2 + d_2' i_2$, such that the equality*

$$\lim_{z\to z^0} \frac{f(z) - f(z_0) - \sum\limits_{k=1}^{2} (z_k - z_k^0)(d_k + d_k' i_2)}{\|z - z^0\|} = 0 \tag{7.10}$$

*is fulfilled. In this case, we the sum*

$$\sum_{k=1}^{2} (z_k - z_k^0)(d_k + d_k' i_2) \tag{7.11}$$

*is called the $\mathbb{C}^2$-differential of the quaternion function $f$ at the point $z^0$.*

**Theorem 7.2** (see [45]). *A quaternion function $f(z) = f_1(z) + f_2(z)i_2$ is $\mathbb{C}^2$-differentiable at a point $z^0$ if and only if one of the following three conditions holds*:

(i) *the complex functions $f_1(z)$ and $f_2(z)$ are $\mathbb{C}^2$-differentiable at the point $z^0$*;

(ii) *the equalities*

$$\frac{\partial f}{\partial \widehat{x}_0}(z^0) + i_1 \frac{\partial f}{\partial \widehat{x}_1}(z^0) = 0, \tag{7.12}$$

$$\frac{\partial f}{\partial \widehat{x}_2}(z^0) + i_1 \frac{\partial f}{\partial \widehat{x}_3}(z^0) = 0 \tag{7.13}$$

*hold*;

(iii) *the equality*

$$df(z^0) = dz_1 \frac{\partial f}{\partial \widehat{z}_1}(z^0) + dz_2 \frac{\partial f}{\partial \widehat{z}_2}(z^0) \tag{7.14}$$

*holds, where*

$$\frac{\partial f}{\partial \widehat{z}_1} = \frac{\partial f_1}{\partial \widehat{z}_1} + \frac{\partial f_2}{\partial \widehat{z}_1} i_1, \qquad \frac{\partial f}{\partial \widehat{z}_2} = \frac{\partial f_1}{\partial \widehat{z}_2} + \frac{\partial f_2}{\partial \widehat{z}_2} i_1 \tag{7.15}$$

*and for the complex function $g(z_1, z_2)$ of two complex variables $z_1$ and $z_2$ the formal angular partial derivatives $\partial g/\partial \widehat{z}_1$ and $\partial g/\partial \widehat{z}_2$ with respect to $z_1$ and $z_2$ are defined by the equality*

$$\frac{\partial g}{\partial \widehat{z}_1} = \frac{1}{2}\Big(\frac{\partial g}{\partial \widehat{x}_0} - i_1 \frac{\partial g}{\partial \widehat{x}_1}\Big), \qquad \frac{\partial g}{\partial \widehat{z}_2} = \frac{1}{2}\Big(\frac{\partial g}{\partial \widehat{x}_2} - i_1 \frac{\partial g}{\partial \widehat{x}_3}\Big). \tag{7.16}$$

(see [41]).

*Proof.* (i) Equality (7.10) is equivalent to the fulfillment of the following two equalities:

$$\lim_{z\to z_0} \frac{f_1(z) - f_1(z^0) - \sum\limits_{k=1}^{2} d_k (z_k - z_k^0)}{\|z - z^0\|} = 0, \tag{7.17}$$

$$\lim_{z\to z_0} \frac{f_2(z) - f_2(z^0) - \sum\limits_{k=1}^{2} d_k' (z_k - z_k^0)}{\|z - z^0\|} = 0, \tag{7.18}$$

which are equivalent to the $\mathbb{C}^2$-differentiability of the complex functions $f_1(z)$ and $f_2(z)$, respectively, at the point $z^0$ (see [41, Eq. (3.2)]).

(ii) According to the statement (i), the $\mathbb{C}^2$-differentiability of a quaternion function $f = f_1 + f_2 i_2$ at a point $z^0$ is equivalent to the $\mathbb{C}^2$-differentiability of the complex functions $f_1$ and $f_2$. On the other hand, the $\mathbb{C}^2$-differentiability of the complex function $f_1$ at the point $z^0$ is equivalent to the fulfillment of the equalities (see [41, Eq. (3.1) or Theorem 5.10])

$$\frac{\partial f_1}{\partial \widehat{x}_0}(z^0) + i_1 \frac{\partial f_1}{\partial \widehat{x}_1}(z^0) = 0, \quad \frac{\partial f_1}{\partial \widehat{x}_2}(z^0) + i_1 \frac{\partial f_1}{\partial \widehat{x}_3}(z^0) = 0. \tag{7.19}$$

Similarly, for the complex function $f_2$ we have

$$\frac{\partial f_2}{\partial \widehat{x}_0}(z^0) + i_1 \frac{\partial f_2}{\partial \widehat{x}_1}(z^0) = 0, \quad \frac{\partial f_2}{\partial \widehat{x}_2}(z^0) + i_1 \frac{\partial f_2}{\partial \widehat{x}_3}(z^0) = 0. \tag{7.20}$$

If we premultiply Eqs. (7.20) by $i_2$ and add the resulting equalities with Eqs. (7.19), then we obtain Eqs. (7.12) and (7.13).

(iii) Again, by virtue of the statement (i), the $\mathbb{C}^2$-differentiability of the quaternion function $f$ is equivalent to the $\mathbb{C}^2$-differentiability of the complex functions $f_1$ and $f_2$. But the complex function $f_1$ is $\mathbb{C}^2$-differentiable at a point $z^0$ if and only if the following equality holds:

$$df_1(z^0) = \sum_{k=1}^{2} \frac{\partial f_1}{\partial \widehat{z}_k}(z^0)\, dz_k \tag{7.21}$$

(see [41, Eq. (3.7)]).

Similarly, the complex function $f_2$ is $\mathbb{C}^2$-differentiable at a point $z^0$ if and only if the equality

$$df_2(z^0) = \sum_{k=1}^{2} \frac{\partial f_2}{\partial \widehat{z}_k}(z^0)\, dz_k \tag{7.22}$$

is valid.

Using (7.3) we can rewrite Eqs. (7.21) and (7.22) as follows:

$$df_1 = dz_1 \frac{\partial f_1}{\partial \widehat{z}_1} + dz_2 \frac{\partial f_1}{\partial \widehat{z}_2}, \quad df_2 = dz_1 \frac{\partial f_2}{\partial \widehat{z}_1} + dz_2 \frac{\partial f_2}{\partial \widehat{z}_2}. \tag{7.23}$$

Hence we obtain the equality

$$df_1 + df_2 i_2 = dz_1 \frac{\partial(f_1 + f_2 i_2)}{\partial \widehat{z}_1} + dz_2 \frac{\partial(f_1 + f_2 i_2)}{\partial \widehat{z}_2},$$

from which by virtue of (7.8) we obtain Eq. (7.14). □

**Remark 7.3.** *The equivalence of the $\mathbb{C}^2$-differentiability of a quaternion function $f = f_1 + f_2 i_2$ to the concurrent $\mathbb{C}^2$-differentiability of its complex components $f_1$ and $f_2$ (see the statement (i) of Theorem 7.2) has no analogs for the $\mathbb{C}^1$-differentiability in a domain. That fact follows from the fact that a $\mathbb{C}^1$-differentiable real function in a domain is necessarily constant in this domain.*

**Theorem 7.4.** *The $\mathbb{C}^2$-differential of a quaternion function $f$ is equal to the differential of this function.*

*Proof.* For the coefficients $d_k$ and $d'_k$ that are involved in Eqs. (7.17) and (7.18), the following equalities hold (see [123, p. 31]):

$$d_k = \frac{\partial f_1}{\partial z_k}(z^0), \quad d'_k = \frac{\partial f_2}{\partial z_k}(z^0).$$

But for a $\mathbb{C}^2$-differentiable complex function, the partial derivative with respect to the variable $z_k$ is equal to its angular partial derivative with respect to the same $z_k$ (see [41, Eq. (2.1)]). Therefore, the $\mathbb{C}^2$-differential of the function $f = f_1 + f_2 i_2$ defined by Eq. (7.11) at the point $z^0$ is written as

$$\sum_{k=1}^{2} dz_k \frac{\partial f}{\partial \widehat{z}_k}(z^0).$$

By virtue of Eq. (7.14), the last expression is equal to $df(z^0)$. □

### 7.2. $\mathbb{C}^2$-Holomorphy of quaternion functions.

**Definition 7.5** (see [45]). *A quaternion function $f(z) = f_1(z) + f_2(z)i_2$ is said to be $\mathbb{C}^2$-holomorphic at a point $z^0$ or in a domain $D \subset \mathbb{C}^2$ if $f$ is $\mathbb{C}^2$-differentiable in the neighborhood of $z^0$ or at every point of the domain $D$.*

**Proposition 7.6.** *A quaternion function $f(z)$ is $\mathbb{C}^2$-holomorphic at a point $z^0$ or in a domain $D \subset \mathbb{C}^2$ if and only if one of the conditions* (i)–(iii) *of Theorem 7.2 holds in a neighborhood of $z^0$ or at every point of the domain $D$.*

In particular, we have the following assertion.

**Proposition 7.7.** *The $\mathbb{C}^2$-holomorphy at a point or in a domain of a quaternion function $f(z) = f_1(z) + f_2(z)i_2$ is equivalent to the concurrent $\mathbb{C}^2$-holomorphy at the same point or in the same domain of the complex functions $f_1(z)$ and $f_2(z)$.*

### 7.3. Integral representations of $\mathbb{C}^2$-holomorphic quaternion functions.

**Theorem 7.8** (see [45]). *Let a quaternion function $f(z) = f_1(z) + f_2(z)i_2$ be $\mathbb{C}^2$-holomorphic in a domain $D \subset \mathbb{C}^2$, which is the Cartesian product of simply connected domains $D_1 \subset \mathbb{C}^1$ and $D_2 \subset \mathbb{C}^1$. Then at any point $z = (z_1, z_2)$ the representation*

$$f(z_1, z_2) = -\frac{1}{4\pi^2} \int\limits_{\Gamma_1} \int\limits_{\Gamma_2} \frac{dt_1\, dt_2}{(t_1 - z_1)(t_2 - z_2)}\, f(t_1, t_2) \tag{7.24}$$

*is fulfilled, where $\Gamma_1$ and $\Gamma_2$ are any closed paths in $D_1$ and $D_2$, respectively, which envelop the points $z_1$ and $z_2$.*

*Proof.* By Proposition 7.7, we have the following equalities (see [123, p. 28]):

$$f_1(z_1, z_2) = -\frac{1}{4\pi^2} \int\limits_{\Gamma_1} \int\limits_{\Gamma_2} \frac{f_1(t_1, t_2)}{(t_1 - z_1)(t_2 - z_2)}\, dt_1\, dt_2, \tag{7.25}$$

$$f_2(z_1, z_2) = -\frac{1}{4\pi^2} \int\limits_{\Gamma_1} \int\limits_{\Gamma_2} \frac{f_2(t_1, t_2)}{(t_1 - z_1)(t_2 - z_2)}\, dt_1\, dt_2. \tag{7.26}$$

By virtue of Eq. (7.3) we can write

$$f_1(t_1, t_2)\, dt_1\, dt_2 = dt_1\, dt_2\, f_1(t_1, t_2), \quad f_2(t_1, t_2)\, dt_1\, dt_2 = dt_1\, dt_2\, f_2(t_1, t_2).$$

Hence, from Eqs. (7.25) and (7.26) we obtain the equality

$$f_1(z_1, z_2) + f_2(z_1, z_2)i_2 = -\frac{1}{4\pi^2} \int\limits_{\Gamma_2\Gamma_1} \frac{dt_1\, dt_2}{(t_1 - z_1)(t_2 - z_2)} \left[f_1(t_1, t_2) + f_2(t_1, t_2)i_2\right],$$

which is equivalent to Eq. (7.24). □

**Theorem 7.9** (see [45]). *If a quaternion function $f(z_1, z_2) = f_1(z_1, z_2) + f_2(z_1, z_2)i_2$ is $\mathbb{C}^2$-holomorphic in the Cartesian product $D_1 \times D_2$ of simply connected domains $D_1 \subset \mathbb{C}^1$ and $D_2 \subset \mathbb{C}^1$, then its partial derivatives $f'_{z_1}$ and $f'_{z_2}$ are also $\mathbb{C}^2$-holomorphic quaternion functions in $D_1 \times D_2 \subset \mathbb{C}^2$.*

*Proof.* According to Proposition 7.7, the $\mathbb{C}^2$-holomorphy of a quaternion function $f$ implies the $\mathbb{C}^2$-holomorphy of the complex functions $f_1$ and $f_2$ defined by Eqs. (7.25) and (7.26). Therefore, their partial derivatives

$$\frac{df_1}{\partial z_1}, \quad \frac{df_1}{\partial z_2}, \quad \frac{df_2}{\partial z_1}, \quad \frac{df_2}{\partial z_2}$$

are $\mathbb{C}^2$-holomorphic complex functions in $D_1 \times D_2$. Thus, Eqs. (7.19) and (7.20), which hold for the functions $f_1$ and $f_2$, are also valid for the partial derivatives

$$\frac{df_1}{\partial z_1}, \quad \frac{df_1}{\partial z_2}, \quad \frac{df_2}{\partial z_1}, \quad \frac{df_2}{\partial z_2}.$$

Hence it follows that, as was shown in the proof of Theorem 7.2, these partial derivatives satisfy Eqs. (7.12) and (7.13), i.e., are $\mathbb{C}^2$-holomorphic quaternion functions by virtue of the statement (ii) of Theorem 7.2. □

### 7.4. Representation of $\mathbb{C}^2$-holomorhic functions by power series.

**Theorem 7.10** (see [45]). *Let a quaternion function $f(z) = f_1(z) + f_2(z)i_2$ be $\mathbb{C}^2$-holomorphic in a domain $D \subset \mathbb{C}^2$, which is the Cartesian product of simply connected domains $D_1 \subset \mathbb{C}^1$ and $D_2 \subset \mathbb{C}^1$. Then at any point $z = (z_1, z_2) \in D$ from the neighborhood of $z^0 = (z_1^0, z_2^0) \in D$ the representation of $f$ by the power series*

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} (z_1 - z_1^0)^m (z_2 - z_2^0)^n c_{mn} \tag{7.27}$$

*is fulfilled, where the quaternion coefficients $c_{mn}$ of the function $f$ are defined by the equalities*

$$c_{mn} = -\frac{1}{4\pi^2} \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} \frac{dt_1\, dt_2}{(t_1 - z_1^0)^{m+1}(t_2 - z_2^0)^{n+1}}\, f(t_1, t_2), \tag{7.28}$$

$$m!n!c_{mn} = \Big(\frac{\partial^{m+n} f(z_1, z_2)}{\partial z_1^m \partial z_2^n}\Big)_{\substack{z_1 = z_1^0 \\ z_2 = z_2^0}}. \tag{7.29}$$

*Proof.* By Proposition 7.7, the complex functions $f_1$ and $f_2$ are $\mathbb{C}^2$-holomorphic or, equivalently, $\mathbb{C}^2$-analytic in the domain $D$. Hence we have the equalities

$$f_1(z_1, z_2) = \sum_{m,n=0}^{\infty} {}^1c_{mn} (z_1 - z_1^0)^m (z_2 - z_2^0)^n, \tag{7.30}$$

$$f_2(z_1, z_2) = \sum_{m,n=0}^{\infty} {}^2c_{mn} (z_1 - z_1^0)^m (z_2 - z_2^0)^n, \tag{7.31}$$

where the complex coefficients of the functions $f_1$ and $f_2$ are given by the formulas (see [123, p. 49])

$${}^1c_{mn} = -\frac{1}{4\pi^2} \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} \frac{f_1(t_1, t_2)}{(t_1 - z_1^0)^{m+1}(t_2 - z_2^0)^{n+1}}\, dt_1\, dt_2, \tag{7.32}$$

$${}^2c_{mn} = -\frac{1}{4\pi^2} \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} \frac{f_2(t_1, t_2)}{(t_1 - z_1^0)^{m+1}(t_2 - z_2^0)^{n+1}}\, dt_1\, dt_2. \tag{7.33}$$

Using (7.3) and the equality $f_1 + f_2 i_2 = f$, from (7.30)–(7.31) and (7.32)–(7.33) we obtain respectively Eqs. (7.27) and (7.28).

As to Eq. (7.29), it is obtained from the well-known formulas (see [123, p. 31])

$$m!n!{}^1c_{mn} = \Big(\frac{\partial^{m+n} f_1(t_1, t_2)}{\partial t_1^m \partial t_2^n}\Big)_{\substack{t_1 = z_1^0 \\ t_2 = z_2^0}},$$

$$m!n!{}^2c_{mn} = \Big(\frac{\partial^{m+n} f_2(t_1, t_2)}{\partial t_1^m \partial t_2^n}\Big)_{\substack{t_1 = z_1^0 \\ t_2 = z_2^0}},$$

taking into account the equalities

$$\frac{df}{dz_1} = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_1}\, i_2, \quad \frac{df}{dz_2} = \frac{\partial f_1}{\partial z_2} + \frac{\partial f_2}{\partial z_2}\, i_2. \qquad \square$$

## 8. Properties of Right- and Left-Regular Functions of Two Complex Variables

**8.1.** As was mentioned above, a differentiable (with respect to a set of real variables $(x_0, x_1, x_2, x_3)$) quaternion function $f(z)$, where $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, is right-regular (see (1.24)) in a domain $G \subset \mathbb{H}$ if it satisfies the condition

$$\frac{\overline{\partial}_r f}{\partial z} = 0$$

and is left-regular (see (1.25)) in $G$ if it satisfies the condition

$$\frac{\overline{\partial}_l f}{\partial z} = 0.$$

But we know (see Sec. 5) that the differentiability of $f$ is equivalent to the existence of its angular partial derivatives

$$\frac{\partial f}{\partial \widehat{x}_0}(z), \quad \frac{\partial f}{\partial \widehat{x}_1}(z), \quad \frac{\partial f}{\partial \widehat{x}_2}(z), \quad \frac{\partial f}{\partial \widehat{x}_3}(z), \quad z \in G.$$

Therefore, a quaternion function $f(z)$ is right-regular in a domain $G \subset \mathbb{H}$ if and only if it satisfies in $G$ the condition

$$\frac{\overline{\partial}_r f}{\partial \widehat{z}} = 0 \tag{8.1}$$

and left-regular if and only if it satisfies in $G$ the condition

$$\frac{\overline{\partial}_l f}{\partial \widehat{z}} = 0, \tag{8.2}$$

where we assume that

$$\frac{\overline{\partial}_r}{\partial \widehat{z}} = \frac{\partial}{\partial \widehat{x}_0} + \frac{\partial}{\partial \widehat{x}_1}\, i_1 + \frac{\partial}{\partial \widehat{x}_2}\, i_2 + \frac{\partial}{\partial \widehat{x}_3}\, i_3, \tag{8.3}$$

$$\frac{\overline{\partial}_l}{\partial \widehat{z}} = \frac{\partial}{\partial \widehat{x}_0} + i_1 \frac{\partial}{\partial \widehat{x}_1} + i_2 \frac{\partial}{\partial \widehat{x}_2} + i_3 \frac{\partial}{\partial \widehat{x}_3}. \tag{8.4}$$

Now we can write the operators $\overline{\partial}_r/\partial\widehat{z}$ and $\overline{\partial}_l/\partial\widehat{z}$ using the function $f_1$ and $f_2$, where $f_1(z_1, z_2) + f_2(z_1, z_2) i_2 = f(z_1, z_2)$ and $z_1 = x_0 + i_1 x_1$, $z_2 = x_2 + i_1 x_3$. Equality (8.3) takes the form

$$\begin{aligned}
\frac{\overline{\partial}_r f}{\partial \widehat{z}} &= \frac{\partial f_1}{\partial \widehat{x}_0} + \frac{\partial f_2}{\partial \widehat{x}_0}\, i_2 + \Big(\frac{\partial f_1}{\partial \widehat{x}_1} + \frac{\partial f_2}{\partial \widehat{x}_1}\, i_2\Big) i_1 + \Big(\frac{\partial f_1}{\partial \widehat{x}_2} + \frac{\partial f_2}{\partial \widehat{x}_2}\, i_2\Big) i_2 + \Big(\frac{\partial f_1}{\partial \widehat{x}_3} + \frac{\partial f_2}{\partial \widehat{x}_3}\, i_2\Big) i_3 \\
&= \frac{\partial f_1}{\partial \widehat{x}_0} + \frac{\partial f_1}{\partial \widehat{x}_1}\, i_1 + \frac{\partial f_1}{\partial \widehat{x}_2}\, i_2 + \frac{\partial f_1}{\partial \widehat{x}_3}\, i_3 + \frac{\partial f_2}{\partial \widehat{x}_0}\, i_2 - \frac{\partial f_2}{\partial \widehat{x}_2}\, i_1 i_2 - \frac{\partial f_2}{\partial \widehat{x}_2} + \frac{\partial f_2}{\partial \widehat{x}_3}\, i_1 \\
&= \frac{\partial f_1}{\partial \widehat{x}_0} + \frac{\partial f_1}{\partial \widehat{x}_1}\, i_1 + \Big(\frac{\partial f_1}{\partial \widehat{x}_2} + \frac{\partial f_1}{\partial \widehat{x}_3}\, i_1\Big) i_2 + \Big(\frac{\partial f_2}{\partial \widehat{x}_0} - \frac{\partial f_2}{\partial \widehat{x}_1}\, i_1\Big) i_2 - \Big(\frac{\partial f_2}{\partial \widehat{x}_2} - \frac{\partial f_2}{\partial \widehat{x}_3}\, i_1\Big).
\end{aligned}$$

Taking into consideration the equalities $\overline{z}_1 = x_0 - i_1 x_2$ and $\overline{z}_2 = x_2 - i_1 x_3$ and introducing the notation

$$\frac{\partial f_1}{\partial \widehat{x}_0} + \frac{\partial f_1}{\partial \widehat{x}_1}\, i_1 = 2\,\frac{\partial f_1}{\partial \widehat{\overline{z}}_1}, \quad \frac{\partial f_1}{\partial \widehat{x}_2} + \frac{\partial f_1}{\partial \widehat{x}_3}\, i_1 = 2\,\frac{\partial f_1}{\partial \widehat{\overline{z}}_2},$$

$$\frac{\partial f_2}{\partial \widehat{x}_0} - \frac{\partial f_2}{\partial \widehat{x}_1}\, i_1 = 2\,\frac{\partial f_2}{\partial \widehat{z}_1}, \quad \frac{\partial f_2}{\partial \widehat{x}_2} - \frac{\partial f_2}{\partial \widehat{x}_3}\, i_1 = 2\,\frac{\partial f_2}{\partial \widehat{z}_2},$$

we have

$$\frac{\overline{\partial}_r f}{\partial \widehat{z}} = 2\Big[\frac{\partial f_1}{\partial \widehat{\overline{z}}_1} + \frac{\partial f_1}{\partial \widehat{\overline{z}}_2}\, i_2 + \frac{\partial f_2}{\partial \widehat{z}_1}\, i_2 - \frac{\partial f_2}{\partial \widehat{z}_2}\Big] = 2\Big[\Big(\frac{\partial f_1}{\partial \widehat{\overline{z}}_1} - \frac{\partial f_2}{\partial \widehat{z}_2}\Big) + \Big(\frac{\partial f_1}{\partial \widehat{\overline{z}}_2} + \frac{\partial f_2}{\partial \widehat{z}_1}\Big) i_2\Big]. \tag{8.5}$$

Therefore, Eq. (8.1) is equivalent to the system of equalities (see Sec. 7)

$$\frac{\partial f_1}{\partial \widehat{\overline{z}}_1} = \frac{\partial f_2}{\partial \widehat{z}_2}, \quad \frac{\partial f_1}{\partial \widehat{\overline{z}}_2} = -\frac{\partial f_2}{\partial \widehat{z}_1}. \tag{8.6}$$

**Proposition 8.1.** *If the second component $f_2(z_1,z_2)$ of a quaternion function $f(z_1,z_2) = f_1(z_1,z_2) + f_2(z_1,z_2)i_2$ is a complex function equal to zero in a domain $G \subset \mathbb{H}$, then the right-regularity of the complex function $f(z_1,z_2) = f_1(z_1,z_2)$ is the property of its $\mathbb{C}^2$-differentiability in $G$.*

*Proof.* Since $f_2(z_1,z_2) = 0$ for all $(z_1,z_2) \in G$, the system (8.6) implies that the system

$$\frac{\partial f_1}{\partial \widehat{\overline{z}}_1} = 0, \quad \frac{\partial f_1}{\partial \widehat{\overline{z}}_2} = 0 \tag{8.7}$$

is fulfilled in $G$. But the complex function $f_1(z_1,z_2)$ is $\mathbb{C}^2$-differentiable in $G$ if and only if the conditions (8.7) are fulfilled (see [41]). ☐

**Corollary 8.2.** *The fulfillment of the conditions*

$$f_2(z_1,z_2) = 0, \quad \frac{\overline{\partial}_r f_1}{\partial \widehat{z}} = 0 \tag{8.8}$$

*in $G$ implies the fulfillment of either of the following two conditions*:

(1) *the function $f_1(z_1,z_2)$ is $\mathbb{C}^2$-differentiable in $G$* (see [41]);

(2) *the equality*

$$df_1 = \frac{\partial f_1}{\partial \widehat{z}_1}\, dz_1 + \frac{\partial f_1}{\partial \widehat{z}_2}\, dz_2 \tag{8.9}$$

*is fulfilled in $G$* (see [41]).

Furthermore, the system (8.6) implies the following assertions.

**Proposition 8.3.** *If the complex functions $f_1(z_1,z_2)$ and $\overline{f}_2(z_1,z_2)$ are $\mathbb{C}^2$-holomorphic in $G$ with respect to $(z_1,z_2)$, then the quaternion function $f = f_1 + f_2 i_2$ is right-regular in $G$.*

**Proposition 8.4.** *If a quaternion function $f(z_1,z_2) = f_1(z_1,z_2) + f_2(z_1,z_2)i_2$ is right-regular in the domain $G$, then the complex functions $f_1(z_1,z_2)$ and $\overline{f}_2(z_1,z_2)$ simultaneously are or are not $\mathbb{C}^2$-holomorphic in $G$ with respect to $(z_1,z_2)$.*

**8.2.** The assertions given below clearly show that the properties of right- and left-regular functions are, generally speaking, different.

For the operator $\overline{\partial}_l f/\partial\widehat{z}$ we have

$$\begin{aligned}\frac{\overline{\partial}_l f}{\partial \widehat{z}} &= \Big(\frac{\partial f_1}{\partial \widehat{x}_0} + \frac{\partial f_2}{\partial \widehat{x}_0}\, i_2\Big) + i_1\Big(\frac{\partial f_1}{\partial \widehat{x}_1} + \frac{\partial f_2}{\partial \widehat{x}_2}\, i_2\Big) + i_2\Big(\frac{\partial f_1}{\partial \widehat{x}_2} + \frac{\partial f_2}{\partial \widehat{x}_2}\, i_2\Big) + i_3\Big(\frac{\partial f_1}{\partial \widehat{x}_3} + \frac{\partial f_2}{\partial \widehat{x}_3}\, i_2\Big)\\ &= \Big(\frac{\partial f_1}{\partial \widehat{x}_0} + i_1 \frac{\partial f_1}{\partial \widehat{x}_1}\Big) + \Big(i_2 \frac{\partial f_1}{\partial \widehat{x}_2} + i_3 \frac{\partial f_1}{\partial \widehat{x}_3}\Big) + \Big(\frac{\partial f_2}{\partial \widehat{x}_0}\, i_2 + i_1 \frac{\partial f_2}{\partial \widehat{x}_1}\, i_2\Big) + i_2 \frac{\partial f_2}{\partial \widehat{x}_2}\, i_2 + i_3 \frac{\partial f_2}{\partial \widehat{x}_3}\, i_2.\end{aligned}$$

If we perform the substitution

$$i_3 = -i_2 i_1, \quad \frac{\partial f_2}{\partial \widehat{x}_0}\, i_2 = i_2\overline{\Big(\frac{\partial f_2}{\partial \widehat{x}_0}\Big)}, \quad i_1 \frac{\partial f_2}{\partial \widehat{x}_1} = -i_2 i_1 \overline{\Big(\frac{\partial f_2}{\partial \widehat{x}_1}\Big)}$$

and take into account the fact that the conjugate to the derivative with respect to a real variable of a complex function is equal to the derivative of its conjugate function, then we obtain

$$\begin{aligned}\frac{\overline{\partial}_l f}{\partial \widehat{z}} &= \Big(\frac{\partial f_1}{\partial \widehat{x}_0} + i_1 \frac{\partial f_1}{\partial \widehat{x}_1}\Big) + i_2\Big(\frac{\partial f_1}{\partial \widehat{x}_2} - i_1 \frac{\partial f_1}{\partial \widehat{x}_3}\Big) + i_2\Big(\frac{\partial \overline{f}_2}{\partial \widehat{x}_0} - i_1 \frac{\partial \overline{f}_2}{\partial \widehat{x}_1}\Big) + \Big(-\frac{\partial \overline{f}_2}{\partial \widehat{x}_2} - i_1 \frac{\partial \overline{f}_2}{\partial \widehat{x}_3}\Big)\\ &= 2\Big[\Big(\frac{\partial f_1}{\partial \widehat{\overline{z}}_1} - \frac{\partial \overline{f}_2}{\partial \widehat{\overline{z}}_2}\Big) + i_2\Big(\frac{\partial f_1}{\partial \widehat{z}_2} + \frac{\partial \overline{f}_2}{\partial \widehat{z}_1}\Big)\Big]\end{aligned}$$

(note that at the end of the procedure we have used the equality $t_1t_2 = t_2t_1$, $t_1 \in \mathbb{C}$, $t_2 \in \mathbb{C}$). Thus, the equality $\overline{\partial}_l f/\partial\widehat{z} = 0$ is equivalent to the system

$$\frac{\partial f_1}{\partial \widehat{\overline{z}}_1} = \frac{\partial \overline{f}_2}{\partial \widehat{\overline{z}}_2}, \quad \frac{\partial f_1}{\partial \widehat{z}_2} = -\frac{\partial \overline{f}_2}{\partial \widehat{z}_1}. \tag{8.10}$$

This system gives rise to the following assertions.

**Proposition 8.5.** *If a quaternion function $f = f_1 + f_2 i_2$ is left-regular in the domain $G$ and the complex function $f_1$ is $\mathbb{C}^2$-holomorphic with respect to $(z_1, \overline{z}_2)$, then this complex function $f_2$ is $\mathbb{C}^2$-holomorphic with respect to $(z_1, \overline{z}_2)$.*

**Proposition 8.6.** *If complex functions $f_1$ and $f_2$ are $\mathbb{C}^2$-holomorphic in $G$ with respect to $(z_1, \overline{z}_2)$, then the quaternion function $f = f_1 + f_2 i_2$ is left-regular in $G$.*

**Proposition 8.7.** *If a quaternion function $f = f_1 + f_2 i_2$ is left-regular in the domain $G$, then the complex functions $f_1$ and $f_2$ simultaneously are or are not $\mathbb{C}^2$-holomorphic in $G$ with respect to $(z_1, \overline{z}_2)$.*

## 9. On Harmonic and Regular Functions

**9.1.** We already know (see Sec. 1) that if a quaternion function $f(z)$ of a quaternion variable $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ is right- or left-regular in a domain $G \subset \mathbb{H}$, then it is a harmonic function in this domain $G$, i.e., a function $f$ is a solution in $G$ of the Laplace equation

$$\Delta f = 0, \tag{9.1}$$

where the Laplace operator $\Delta$ is defined by the equality

$$\Delta = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\,. \tag{9.2}$$

Since (9.2) contains partial derivatives with respect to the real variables $x_0$, $x_1$, $x_2$, and $x_3$, Eq. (9.1) is equivalent to the following system of equalities in $G$:

$$\Delta u_0(z) = 0, \quad \Delta u_1(z) = 0, \quad \Delta u_2(z) = 0, \quad \Delta u_3(z) = 0, \tag{9.3}$$

where

$$f(z) = u_0(z) + u_1(z)i_1 + u_2(z)i_2 + u_3(z)i_3, \quad z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3. \tag{9.4}$$

Therefore, if the quaternion function (9.4) is right- or left-regular in a domain $G \subset \mathbb{H}$, then its components (i.e., the real functions $u_0(z)$, $u_1(z)$, $u_2(z)$, and $u_3(z)$) are harmonic functions in $G$.

The following assertion is easy to verify.

**Proposition 9.1.** *If a real function $u(z)$ is harmonic in a domain $G \subset \mathbb{H}$, then the function*

$$F = \frac{\partial u}{\partial x_0} - \frac{\partial u}{\partial x_1}\, i_1 - \frac{\partial u}{\partial x_2}\, i_2 - \frac{\partial u}{\partial x_3}\, i_3 \tag{9.5}$$

*is regular in $G$.*

The following assertion is remarkable.

**Theorem 9.2** (see [54]). *If a real function $u_0(z)$ is harmonic in a simply connected domain $G \subset \mathbb{H}$, then there exist harmonic in $G$ real functions $u_1(z)$, $u_2(z)$ and $u_3(z)$ such that the quaternion function*

$$f(z) = u_0(z) + u_1(z)i_1 + u_2(z)i_2 + u_3(z)i_3 \tag{9.6}$$

*is regular in $G$.*

As we already know (see Sec. 1), the power functions $\psi_n(z) = z^n$ are not harmonic functions, but the functions $\Delta z^n$ are such. Namely, the following assertion is valid.

**Theorem 9.3** (see [54]). *Functions $\Delta z^n$, $n = 0, 1, 2, \ldots$, are regular and, therefore, harmonic in any finite domain $G \subset \mathbb{H}$.*

**9.2.** If a quaternion function $f(z)$ is different from zero in a domain $G \subset \mathbb{H}$, then the equality

$$\frac{1}{f} \cdot f = 1 \tag{9.7}$$

is fulfilled in $G$. Hence, using the equality

$$(\varphi \cdot \psi)'_{x_k} = \varphi'_{x_k} \cdot \psi + \varphi \cdot \psi'_{x_k} \tag{9.8}$$

we have

$$\Big(\frac{1}{f}\Big)'_{x_k} \cdot f + \frac{1}{f} \cdot f'_{x_k} = 0,$$

$$\Big(\frac{1}{f}\Big)'_{x_k} \cdot f \cdot i_k + \frac{1}{f} \cdot f'_{x_k} \cdot i_k = 0.$$

Therefore,

$$\sum_{k=0}^{3} \Big(\frac{1}{f}\Big)'_{x_k} \cdot f \cdot i_k + \frac{1}{f}\Big(\sum_{k=0}^{3} f'_{x_k} \cdot i_k\Big) = 0. \tag{9.9}$$

**Proposition 9.4.** *The right regularity of a quaternion function $f$ that differs from zero in a domain $G \subset \mathbb{H}$ is equivalent to the fulfillment of the equality*

$$\sum_{k=0}^{3} \Big(\frac{1}{f}\Big)'_{x_k} \cdot f \cdot i_k = 0 \tag{9.10}$$

*in $G$.*

Similarly, when $f(z) \neq 0$, $z \in G$, from the equality $f \cdot \frac{1}{f} = 1$ we have

$$f'_{x_k} \cdot \frac{1}{f} + f \cdot \Big(\frac{1}{f}\Big)'_{x_k} = 0, \tag{9.11}$$

$$\Big(\sum_{k=0}^{3} i_k f'_{x_k}\Big) \cdot \frac{1}{f} + \sum_{k=0}^{3} i_k f \Big(\frac{1}{f}\Big)'_{x_k} = 0. \tag{9.12}$$

**Proposition 9.5.** *The left regularity of a quaternion function $f$ that differs from zero in a domain $G$ is equivalent to the fulfillment of the equality*

$$\sum_{k=0}^{3} i_k \cdot f \cdot \Big(\frac{1}{f}\Big)'_{x_k} = 0 \tag{9.13}$$

*in $G$.*

**9.3.** From Eq. (9.9) we obtain

$$\Big(\frac{1}{f}\Big)''_{x_k x_k} \cdot f + \Big(\frac{1}{f}\Big)'_{x_k} \cdot f'_{x_k} + \Big(\frac{1}{f}\Big)'_{x_k} \cdot f'_{x_k} + \frac{1}{f} \cdot f''_{x_k x_k} = 0,$$

i.e.,

$$\Big(\frac{1}{f}\Big)''_{x_k x_k} \cdot f + 2\Big(\frac{1}{f}\Big)'_{x_k} \cdot f'_{x_k} + \frac{1}{f} \cdot f''_{x_k x_k} = 0.$$

Therefore,

$$\Big[\sum_{k=0}^{3} \Big(\frac{1}{f}\Big)''_{x_k x_k}\Big] \cdot f + 2\sum_{k=0}^{3} \Big(\frac{1}{f}\Big)'_{x_k} \cdot f'_{x_k} + \frac{1}{f} \sum_{k=0}^{3} f''_{x_k x_k} = 0. \tag{9.14}$$

In addition, we have

$$\Big(\frac{1}{f}\Big)'_{x_k} = -\frac{1}{f} \cdot f'_{x_k} \cdot \frac{1}{f}. \tag{9.15}$$

Now Eq. (9.14) takes the form

$$\left[\sum_{k=0}^{3}\left(\frac{1}{f}\right)''_{x_kx_k}\right]\cdot f-\frac{2}{f}\sum_{k=0}^{3}f'_{x_k}\cdot\frac{1}{f}\cdot f'_{x_k}+\frac{1}{f}\sum_{k=0}^{3}f''_{x_kx_k}=0. \tag{9.16}$$

Furthermore, Eq. (9.11) implies

$$\begin{gathered}f''_{x_kx_k}\cdot\frac{1}{f}+f'_{x_k}\cdot\left(\frac{1}{f}\right)'_{x_k}+f'_{x_k}\cdot\left(\frac{1}{f}\right)'_{x_k}+f\cdot\left(\frac{1}{f}\right)''_{x_kx_k}=0,\\ \left(\sum_{k=0}^{3}f''_{x_kx_k}\right)\cdot\frac{1}{f}+2\sum_{k=0}^{3}f'_{x_k}\cdot\left(\frac{1}{f}\right)'_{x_k}+f\sum_{k=0}^{3}\left(\frac{1}{f}\right)''_{x_kx_k}=0.\end{gathered} \tag{9.17}$$

Equalities (9.14), (9.16), and (9.17) give rise to the following assertion.

**Theorem 9.6.** *If a quaternion function $f$ differs from zero in a domain $G\subset\mathbb{H}$ and both its inverse function $f^{-1}$ and $f$ itself are harmonic in $G$, then the equalities*

$$\sum_{k=0}^{3}\left(\frac{1}{f}\right)'_{x_k}\cdot f'_{x_k}=0, \tag{9.18}$$

$$\sum_{k=0}^{3}f'_{x_k}\cdot\frac{1}{f}\cdot f'_{x_k}=0, \tag{9.19}$$

$$\sum_{k=0}^{3}f'_{x_k}\cdot\left(\frac{1}{f}\right)'_{x_k}=0 \tag{9.20}$$

*are fulfilled in $G$.*

**9.4.** One can easily prove the following equalities:

$$\Delta\bigl(z\cdot f(z)\bigr)=2\sum_{k=0}^{3}i_k\cdot f'_{x_k}+z\cdot\Delta(f), \tag{9.21}$$

$$\Delta\bigl(f(z)\cdot z\bigr)=2\sum_{k=0}^{3}f'_{x_k}\cdot i_k+\Delta(f)\cdot z. \tag{9.22}$$

From Eq. (9.21) we obtain the following two assertions.

**Proposition 9.7.** *If a quaternion function $f(z)$ defined in a domain $G\subset\mathbb{H}$ is left-regular in $G$, then the function $z\cdot f(z)$ is harmonic in $G$.*

**Proposition 9.8.** *If quaternion functions $f(z)$ and $z\cdot f(z)$ defined in a domain $G\subset\mathbb{H}$ are harmonic in $G$, then the function $f(z)$ is left-regular in the domain $G$.*

Similarly, from Eq. (9.22) we also obtain two assertions.

**Proposition 9.9.** *If a quaternion function $f(z)$ defined in a domain $G\subset\mathbb{H}$ is right-regular in $G$, then the function $f(z)\cdot z$ is harmonic in $G$.*

**Proposition 9.10.** *If quaternion functions $f(z)$ and $f(z)\cdot z$ defined in a domain $G\subset\mathbb{H}$ are harmonic in $G$, then the function $f(z)$ is right-regular in the domain $G$.*

Therefore, the function $\psi_2(z)=z^2$ is not harmonic since the function $\psi_1(z)=z$ does not belong to the union $F^+(G)\cup F^-(G_)$ (see Sec. 1).

**Remark 9.11.** *Propositions 9.8 and 9.10 generalize to the case of quaternion functions the well-known assertion from the theory of of complex functions of one complex variable $z=x+iy$ given by the equality*

$$\Delta(zh(z))=2(h'_x+ih'_y)+z\Delta(h). \tag{9.23}$$

**9.5.** For a quaternion function $f$ given in a domain $G \subset \mathbb{H}$ we consider the function $\varphi = f^2 = f \cdot f$. Then we have the relations

$$\varphi'_{x_k} = f'_{x_k} \cdot f + f \cdot f'_{x_k}, \quad \varphi''_{x_k x_k} = f''_{x_k x_k} \cdot f + f'_{x_k} \cdot f'_{x_k} + f'_{x_k} \cdot f'_{x_k} + f \cdot f''_{x_k x_k}$$

and

$$\Delta(f^2) = \Delta(f) \cdot f + f \cdot \Delta(f) + 2 \sum_{k=0}^{3} (f'_{x_k})^2, \tag{9.24}$$

from which we obtain the following assertion.

**Proposition 9.12.** *If quaternion functions $f$ and $f^2$ defined in a domain $G \subset \mathbb{H}$ are harmonic in $G$, then the equality*

$$\sum_{k=0}^{3} (f'_{x_k})^2 = 0 \tag{9.25}$$

*is fulfilled in $G$, i.e., the sum of squares of partial derivatives of the function $f$ is equal to zero.*

**Remark 9.13.** *From the theory of complex functions of one complex variable $z = x + iy$ we know that the fulfillment of the equality $(f'_x)^2 + (f'_y)^2 = 0$ in the domain $D$ implies that either of the two functions $f$ and $\overline{f}$ is holomorphic in $D$.*

**9.6.** If $f(z) = u_0(z) + u_1(z)i_1 + u_2(z)i_2 + u_3(z)i_3$, then

$$f^2(z) = -|f(z)|^2 + 2u_0(z)f(z). \tag{9.26}$$

Indeed,

$$\begin{aligned} f^2 &= (u_0^2 - u_1^2 - u_2^2 - u_3^2) + 2u_0u_1i_1 + 2u_0u_2i_2 + 2u_0u_3i_3 \\ &= (u_0^2 - u_1^2 - u_2^2 - u_3^2) + 2u_0(u_1i_1 + u_2i_2 + u_3i_3) \\ &= (u_0^2 - u_1^2 - u_2^2 - u_3^2) + 2u_0(f - u_0) = -(u_0^2 + u_1^2 + u_2^2 + u_3^2) + 2u_0 f = -|f|^2 + 2u_0 f. \end{aligned}$$

**9.7.** Equalities (9.21) and (9.22) imply the following theorem.

**Theorem 9.14.** *A function $f$ is Fueter-regular if and only if the functions $f(z)$, $f(z) \cdot z$, and $z \cdot f(z)$ are harmonic.*

## 10. Integral Properties of Regular Functions

The following integral properties of right- and left-regular functions were established by Fueter.

**Theorem 10.1** (see [54])**.** *Let quaternion functions $f(z)$ and $\psi(z)$ be respectively right- and left-regular in a domain $G \subset \mathbb{H}$. If $G$ is a closed and smooth surface in $G$ that bounds a simply connected domain, then the following equalities are fulfilled:*

$$\int_\sigma f(z)\, dZ = 0, \tag{10.1}$$

$$\int_\sigma dZ\, \psi(z) = 0, \tag{10.2}$$

$$\int_\sigma f(z)\, dZ\, \psi(z) = 0, \tag{10.3}$$

*where*

$$dZ = (\cos\alpha_0 + i_1 \cos\alpha_1 + i_2 \cos\alpha_2 + i_3 \cos\alpha_3)\, ds$$

*and $\alpha_m$ is the angle formed by the positively directed $Ox_m$-axis and the outward normal drawn at a point $z \in \sigma$ and $ds$ is a surface element of the corresponding unit sphere in $\mathbb{H}$.*

**Theorem 10.2** (see [54]). *Let a quaternion function $f(z)$ be right-regular in a domain $G \subset \mathbb{H}$. Then at every point $z^0 \in G$ the equality*

$$f(z^0) = \frac{1}{8\pi^2} \int\limits_{\sigma} f(z)\, dZ\, \Delta(z - z^0)^{-1} \tag{10.4}$$

*is fulfilled for any closed and smooth surface $\sigma$ which is homothetic to the point $z^0$.*

## 11. Quaternions and Vector Algebra

The discovery of quaternions motivated various studies in mathematics and physics. Owing to quaternions, a fruitful trend appeared in mathematics, namely, vector algebra. Below we give a brief account of the relationship between the product of two vector quaternions and the scalar and vector products of two respective vectors from a three-dimensional vector space (see [80]).

Every quaternion $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ is the formal sum of a real number $x_0$ and a vector $x_1 i_1 + x_2 i_2 + x_3 i_3$ (in a three-dimensional space we have the Cartesian coordinate systems and $i_1$, $i_2$, and $i_3$ are unit vectors starting from the origin and directed along the coordinate axes) which is directed from the origin to the point $(x_1, x_2, x_3)$. Recall that it was Hamilton who used the term *vector* for the first time in 1845 (see [116, p. 276]).

The number $x_0$ is called the *scalar* (or real) part, while the expression $x_1 i_1 + x_2 i_2 + x_3 i_3$ is called the *vector* (or imaginary) part of the quaternion $z$.

Let us consider two vector quaternions

$$z' = x_1' i_1 + x_2' i_2 + x_3' i_3, \quad z'' = x_1'' i_1 + x_2'' i_2 + x_3'' i_3. \tag{11.1}$$

From the rule of multiplication of quaternions we obtain the equality

$$z'z'' = -(x_1'x_1'' + x_2'x_2'' + x_3'x_3'') + (x_2'x_3'' - x_3'x_2'')i_1 + (x_3'x_1'' - x_1'x_3'')i_2 + (x_1'x_2'' - x_2'x_1'')i_3. \tag{11.2}$$

Hence it is clear that the scalar part with the minus sign of the product $z'z''$ is the scalar product of the vectors $z' = (x_1', x_2', x_3')$ and $z'' = (x_1'', x_2'', x_3'')$, which is denoted by $(z', z'')$ , i.e.,

$$(z', z'') = x_1'x_1'' + x_2'x_2'' + x_3'x_3''. \tag{11.3}$$

The vector part of the product $z'z''$ is the vector product of the vectors $z'$ and $z''$, which is denoted by $[z', z'']$. Thus,

$$[z', z''] = (x_2'x_3'' - x_3'x_2'')i_1 + (x_3'x_1'' - x_1'x_3'')i_2 + (x_1'x_2'' - x_2'x_1'')i_3. \tag{11.4}$$

Therefore, for the product of the vector quaternions $z'$ and $z''$ we have the equality

$$z'z'' = -(z', z'') + [z', z'']. \tag{11.5}$$

Thus, the scalar and vector products of three-dimensional vectors are parts of the product of the corresponding two vector quaternions. This shows the importance of quaternions especially in view of the fact that the operations of scalar and vector multiplication underlie such an important area of mathematics as vector algebra, having various applications both in mathematics and in physics.

Here we draw attention to the following fact. Given the right-hand side of Eq. (11.3) and anyone of the vectors from its left-hand side (say $z''$), there exists an infinite set of vectors $z'$ that satisfy Eq. (11.3).

Similarly, given the right-hand side of Eq. (11.4) and a vector $z''$, there exists an infinite set of vectors $z'$ that satisfy Eq. (11.4).

Along with this, Eq. (11.5) shows that for its given right part and given $z''$, the unknown vector $z'$ is the right quotient obtained by dividing $[z', z''] - (z', z'')$ by $z''$, which is equal to

$$z' = \frac{1}{|z''|^2} \cdot \big( -(z', z'') + [z', z''] \big) \cdot \overline{z''}. \tag{11.6}$$

Finally, it should be said that any rotation of a three-dimensional space about the origin can be given by means of some quaternion $Q$ with norm 1. The rotation corresponding to $Q$ transforms the vector $z' = (x_1', x_2', x_3')$ to the vector $Qz'Q^{-1}$ (see [34]). Here $Q^{-1}$ denotes as usual the inverse quaternion to $Q$, i.e., such that $QQ^{-1} = 1$.

Various applications of quaternions will be discussed in the concluding section of this work.

## 12. Information on Other Properties of Quaternions

We will use the notation $z' = x_1 i_1 + x_2 i_2 + x_3 i_3$ for the vector part of the quaternion $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$.

**1⁰.** The equality

$$(z')^2 = -|z'|^2 \tag{12.1}$$

is valid. Indeed, $z'\overline{z'} = z'(-z') = -(z')^2$.

**2⁰.** For the quaternions

$$z_1 = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3, \quad z_2 = y_0 + y_1 i_1 + y_2 i_2 + y_3 i_3 \tag{12.2}$$

the inequality

$$|z_1 + z_2| \le |z_1| + |z_2| \tag{12.3}$$

holds.

*Proof.* We have the relations

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) \\ &= z_1\overline{z}_1 + z_1\overline{z}_2 + z_2\overline{z}_1 + z_2\overline{z}_2 = |z_1|^2 + |z_2|^2 + z_1\overline{z}_2 + z_2\overline{z}_1. \end{aligned}$$

But

$$z_1\overline{z}_2 + z_2\overline{z}_1 = 2(x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3) = 2\sum_{k=0}^{3} x_k y_k \tag{12.4}$$

and

$$\left|z_1\overline{z}_2 + z_2\overline{z}_1\right| \le 2\sum_{k=0}^{3} |x_k|\,|y_k| \le 2\Big(\sum_{k=0}^{3} x_k^2\Big)^{1/2} \cdot \Big(\sum_{k=0}^{3} y_k^2\Big)^{1/2} = 2|z_1|\,|z_2|.$$

Therefore,

$$|z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1|\,|z_2| = \big(|z_1| + |z_2|\big)^2. \qquad \square$$

**3⁰.** The equality

$$\overline{z_1' z_2'} = z_2' z_1' \tag{12.5}$$

is fulfilled for the vector parts $z_1'$ and $z_2'$ of the quaternions (12.2).

*Proof.* By the formula (1.12) we have

$$\overline{z_1' z_2'} = \overline{z_2'} \cdot \overline{z_1'} = (-z_2')(-z_1') = z_2' z_1'. \qquad \square$$

**$4^0$.** The equality
$$(z_1z_2)^{-1} = z_2^{-1} \cdot z_1^{-1}, \quad z_1 \neq 0, \quad z_2 \neq 0, \tag{12.6}$$
or, equivalently,
$$\frac{1}{z_1z_2} = \frac{1}{z_2} \cdot \frac{1}{z_1}, \quad z_1 \neq 0, \quad z_2 \neq 0, \tag{12.7}$$
is fulfilled.

*Proof.* By virtue of the formulas (1.17) and (1.12) we have
$$\frac{1}{z_1z_2} = \frac{1}{|z_1z_2|^2}\,\overline{z_1z_2} = \frac{1}{|z_1z_2|^2} \cdot \overline{z}_2 \cdot \overline{z}_1 = \frac{1}{|z_2|^2} \cdot \overline{z}_2 \cdot \frac{1}{|z_1|^2}\overline{z}_1 = z_2^{-1} \cdot z_1^{-1}. \qquad \square$$

**$5^0$.** The equality
$$\frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_1}\,(z_2 - z_1)\,\frac{1}{z_2}, \quad z_1 \neq 0, \quad z_2 \neq 0, \tag{12.8}$$
holds.

*Proof.* By the formula (1.17) we have
$$\begin{aligned}\frac{1}{z_1}\,(z_2 - z_1)\,\frac{1}{z_2} &= \frac{1}{|z_1|^2}\,\overline{z}_1(z_2 - z_1)\,\frac{1}{|z_2|^2}\,\overline{z}_2 \\ &= \frac{1}{|z_1|^2}\,\overline{z}_1 z_2\,\frac{1}{|z_2|^2}\,\overline{z}_2 - \frac{1}{|z_1|^2}\,\overline{z}_1 z_1\,\frac{1}{|z_2|^2}\,\overline{z}_2 = \frac{1}{|z_1|^2}\,\overline{z}_1 - \frac{1}{|z_2|^2}\,\overline{z}_2 = \frac{1}{z_1} - \frac{1}{z_2}. \qquad \square\end{aligned}$$

**$6^0$.** The equality
$$\frac{1}{z_1}(z_1 - z_2)\,\frac{1}{z_2} = \frac{1}{z_2}\,(z_1 - z_2)\,\frac{1}{z_1} \tag{12.9}$$
holds.

*Proof.* Multiplying Eq. (12.8) by $(-1)$ we obtain
$$\frac{1}{z_2} - \frac{1}{z_1} = \frac{1}{z_1}\,(z_1 - z_2)\,\frac{1}{z_2}\,. \tag{12.10}$$
By virtue of (12.8) the left-hand side of this equality is equal to
$$\frac{1}{z_2}\,(z_1 - z_2)\,\frac{1}{z_1}\,.$$
Thus, Eq. (12.9) is fulfilled. $\square$

**Remark 12.1.** *Equality* (12.9) *implies that we cannot reduce both of its parts by $z_1 - z_2$. Otherwise we would obtain the equality $z_1z_2 = z_2z_1$ which is not fulfilled at all.*

**$7^0$.** The equality
$$z^2 = -|z|^2 + 2x_0z \tag{12.11}$$
is fulfilled for the quaternion $z = x_0 + x_1i_1 + x_2i_2 + x_3i_3$.

*Proof.* We have the relations
$$\begin{aligned}z^2 &= (x_0^2 - x_1^2 - x_2^2 - x_3^2) + 2x_0x_1i_1 + 2x_0x_2i_2 + 2x_0x_3i_3 \\ &= (x_0^2 - x_1^2 - x_2^2 - x_3^2) + 2x_0(x_1i_1 + x_2i_2 + x_3i_3) \\ &= (x_0^2 - x_1^2 - x_2^2 - x_3^2) + 2x_0(z - x_0) \\ &= -(x_0^2 + x_1^2 + x_2^2 + x_3^2) + 2x_0z = -|z|^2 + 2x_0z. \qquad \square\end{aligned}$$

**8$^0$.** The equality

$$1+z+z^2+\cdots+z^{n-1}=\begin{cases}\dfrac{1-z^n}{1-z} & \text{for } z\neq 1,\\ n & \text{for } z=1\end{cases} \qquad (12.12)$$

holds.

*Proof.* For $z \neq 1$ we have

$$(1-z)(1+z+z^2+\cdots+z^{n-1}) = 1+z+\cdots+z^{n-1}-z-z^2-\cdots-z^{n-1}-z^n = 1-z^n$$

and

$$(1+z+z^2+\cdots+z^{n-1})(1-z) = 1+z+\cdots+z^{n-1}-z-z^2-\cdots-z^{n-1}-z^n = 1-z^n.$$

Therefore, the first row in Eq. (12.12) is valid, while the validity of the second row from above is obvious. □

**9$^0$.** If $|z| < 1$, then we have the equality

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \qquad (12.13)$$

which is a consequence of Eq. (12.12).

**10$^0$.** Let us define all quaternions which will commute with the given quaternion $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$. Assume that the sought quaternion is

$$L = y_0 + y_1 i_1 + y_2 i_2 + y_3 i_3.$$

Then from the equality $zL = Lz$ we obtain the equalities $x_2 y_1 = x_1 y_2$, $x_3 y_1 = x_1 y_3$, and $x_3 y_2 = x_2 y_3$, i.e.

$$\frac{y_1}{x_1} = \frac{y_2}{x_2} = \frac{y_3}{x_3}.$$

If we denote the common value of these equalities by $\lambda$, which is a real number, then we have $y_1 = \lambda x_1$, $y_2 = \lambda x_2$, $y_3 = \lambda x_3$. Therefore, $L = y_0 + \lambda(z - x_0) = (y_0 - x_0\lambda) + \lambda z$. Hence

$$L = r_1 + r_2 z, \qquad (12.14)$$

where $r_1$ and $r_2$ are any real numbers.

**11$^0$.** We want to know for which $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ the equality

$$i_1 z i_1 = z \qquad (12.15)$$

is fulfilled. We have

$$i_1 z i_1 = i_1(x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3) i_1 = -x_0 - x_1 i_1 + x_2 i_2 + x_3 i_3.$$

The equality

$$-x_0 - x_1 i_1 + x_2 i_2 + x_3 i_3 = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$$

will be fulfilled for $x_0 = 0$, $x_1 = 0$, and any real $x_2$ and $x_3$. Therefore, the equality

$$i_1(x_2 i_2 + x_3 i_3) i_1 = x_2 i_2 + x_3 i_3 \qquad (12.16)$$

is valid for any real $x_2$ and $x_3$.

**12$^0$.** For the vector parts $z_1'$ and $z_2'$ of quaternions (12.2) we have the equality

$$z_1' z_2' = -z_2' z_1 \qquad (12.17)$$

if the scalar part of the product $z_1' z_2'$ is equal to zero.

This statement follows from Eq. (11.2) if we replace there $z'$ by $z_1'$ and $z''$ by $z_2'$.

$\mathbf{13}^0$ (A. Hurwitz, see [77]). The equality

$$z(ab)z^{-1} = zaz^{-1} \cdot zbz^{-1} \tag{12.18}$$

holds.

Indeed, by the formula (1.16) we have

$$zaz^{-1} \cdot zbz^{-1} = za\,\frac{1}{|z|^2}\,\overline{z} \cdot zbz^{-1} = z(ab)z^{-1}.$$

**Remark 12.2.** *Now it becomes possible to prove the formula (see [42])*

$$\Big(\frac{1}{\varphi}\Big)'(z^0) = -\frac{1}{\varphi(z^0)} \cdot \varphi'(z_0) \cdot \frac{1}{\varphi(z^0)} \tag{12.19}$$

*only after the author proved (see [42]) the following equality (see the formula* (12.9) *above):*

$$\frac{1}{z_1}\,(z_1 - z_2)\,\frac{1}{z_2} = \frac{1}{z_2}\,(z_1 - z_2)\,\frac{1}{z_1}\,. \tag{12.20}$$

*Thus the inequality*

$$\frac{1}{z_1} \cdot \frac{1}{z_2} \neq \frac{1}{z_2} \cdot \frac{1}{z_1}$$

*"transforms to the equality" as a result of "intrusion" of the difference $z_2 - z_1$ in the role of a multiplier between $1/z_2$ and $1/z_1$ in both parts of this inequality. In this context, a question arises as to the physical nature of this statement—in the author's opinion this topic is interesting to investigate.*

## 13. Interrelations between the Functions $z^n$, $\cos z$, $\sin z$, and $e^z$

Let us consider a nonzero quaternion $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, $z \neq 0$. Since

$$z = |z|\Big(\frac{x_0}{|z|} + \frac{x_1}{|z|}\,i_1 + \frac{x_2}{|z|}\,i_2 + \frac{x_3}{|z|}\,i_3\Big), \quad |z| = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{1/2} > 0,$$

and

$$\Big(\frac{x_0}{|z|}\Big)^2 + \Big|\frac{x_1}{|z|}\,i_1 + \frac{x_2}{|z|}\,i_2 + \frac{x_3}{|z|}\,i_3\Big|^2 = 1,$$

there exists a real number $\theta$ such that $0 \leq \theta \leq \pi$ and

$$\frac{x_0}{|z|} = \cos\theta \tag{13.1}$$

and

$$\Big|\frac{x_1}{|z|}\,i_1 + \frac{x_2}{|z|}\,i_2 + \frac{x_3}{|z|}\,i_3\Big|^2 = \sin\theta. \tag{13.2}$$

We rewrite the last equality in the form

$$|x_1 i_1 + x_2 i_2 + x_3 i_3| = |z|\sin\theta \tag{13.3}$$

or, equivalently,

$$\sqrt{x_1^2 + x_2^2 + x_3^2} = |z|\sin\theta. \tag{13.4}$$

On the other hand, we have

$$\begin{aligned} x_1 i_1 + x_2 i_2 + x_3 i_3 &= \sqrt{x_1^2 + x_2^2 + x_3^2}\,\frac{x_1 i_1 + x_2 i_2 + x_3 i_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\ &= \sqrt{x_1^2 + x_2^2 + x_3^2}\,\Big(\frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\,i_1 + \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\,i_2 + \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\,i_3\Big). \end{aligned} \tag{13.5}$$

Assuming that the quaternion $z$ is not a real number, we introduce the notation (see [71, p. 349])

$$I_z = \frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\,i_1 + \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\,i_2 + \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\,i_3. \tag{13.6}$$

Then we have the equalities

$$z = x_0 + I_z\sqrt{x_1^2 + x_2^2 + x_3^2}\,, \tag{13.7}$$

$$z = |z|(\cos\theta + I_z \sin\theta). \tag{13.8}$$

It is obvious that

$$|I_z| = 1 \tag{13.9}$$

and by the formula (9.26) we have

$$I_z^2 = -1. \tag{13.10}$$

Therefore, $I_z$ is a variable quaternion collective imaginary unit (as different from Hamiltonian constant imaginary units).

It is obvious that

$$\overline{I}_z = -I_z = I_{\overline{z}}. \tag{13.11}$$

The equalities

$$I_z^3 = -I_z, \quad I_z^4 = 1, \quad I_z^{4n+m} = I_z^m, \qquad 1 \le m \le 4, \quad n = 1, 2, \dots, \tag{13.12}$$

are obvious. Similarly, we obtain the equalities

$$\overline{z} = x_0 - I_z\sqrt{x_1^2 + x_2^2 + x_3^2}\,, \tag{13.13}$$

$$\overline{z} = |z|(\cos\theta - I_z \sin\theta), \tag{13.14}$$

$$z^{-1} = |z|^{-2}\big(x_0 - I_z\sqrt{x_1^2 + x_2^2 + x_3^2}\big), \tag{13.15}$$

$$z^{-1} = |z|^{-1}(\cos\theta - I_z \sin\theta). \tag{13.16}$$

From Eq. (13.7) we obtain

$$z \cdot I_z = -\sqrt{x_1^2 + x_2^2 + x_3^2} + x_0 I_z, \tag{13.17}$$

$$I_z \cdot z = -\sqrt{x_1^2 + x_2^2 + x_3^2} + x_0 I_z \tag{13.18}$$

and, therefore,

$$z \cdot I_z = I_z \cdot z. \tag{13.19}$$

Furthermore,

$$(z \cdot I_z)^2 = z \cdot I_z \cdot z \cdot I_z = z \cdot I_z \cdot I_z \cdot z = -z^2, \tag{13.20}$$

$$(I_z \cdot z)^2 = I_z \cdot z \cdot I_z \cdot z = z \cdot I_z \cdot I_z \cdot z = -z^2. \tag{13.21}$$

The following equalities are fulfilled by virtue of Eq. (13.19):

$$(z + I_z)^2 = z^2 + 2I_z \cdot z + I_z^2 = z^2 + 2z \cdot I_z - 1, \tag{13.22}$$

$$(z + I_z)(z - I_z) = z^2 - I_z^2 = z^2 - 1. \tag{13.23}$$

**Remark 13.1.** *The well-known formula*

$$(t_1 + t_2)^2 = t_1^2 + 2t_1 t_2 + t_2^2$$

*from complex analysis for the square of a sum does not hold in general for quaternions. Indeed,*

$$(i_1 + i_2)^2 = (i_1 + i_2)(i_1 + i_2) = i_1^2 + i_1 i_2 + i_2 i_1 + i_2^2 = i_1^2 + i_2^2 = -2,$$

$$i_1^2 + 2i_1 i_2 + i_2^2 = -2 + 2i_3 = 2(i_3 - 1).$$

If $a$ and $b$ are real functions of some variables, then the equality

$$(a + I_z b)^2 = a^2 + 2abI_z - b^2 \tag{13.24}$$

holds. Indeed,

$$(a + I_z b)^2 = (a + I_z b)(a + I_z b) = a^2 + abI_z + abI_z + I_z^2 b^2 = a^2 + 2aI_z - b^2.$$

Furthermore,

$$|a + I_z b| = \sqrt{a^2 + b^2}. \tag{13.25}$$

As a matter of fact, by the formulas (1.10), (1.11), and (13.11) we have

$$|a + I_z b|^2 = (a + I_z b)(\overline{a + I_z b}) = (a + I_z b)(a - I_z b) = a^2 - abI_z + abI_z - b^2 I_z^2 = a^2 + b^2.$$

For the functions $z^n$, $e^z$, $\cos z$, and $\sin z$ defined by Eqs. (1.30)–(1.33) we formulate the following propositions, in which $r$ is a real function of some variables.

**Proposition 13.2** (A. Moivre's formula (see [24, p. 21])).

$$(\cos r + I_z \sin r)^n = \cos nr + I_z \sin nr \tag{13.26}$$

This formula can be proved by induction.

**Proposition 13.3.** *The equality*

$$e^{r \cdot I_z} = \cos r + I_z \sin r \tag{13.27}$$

*holds.*

*Proof.* Using Eqs. (13.10) and (13.12), from Eqs. (1.31)–(1.33) we obtain

$$\begin{aligned} e^{r \cdot I_z} &= 1 + rI_z - \frac{r^2}{2!} - \frac{r^3}{3!} I_z + \frac{r^4}{4!} - \cdots \\ &= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) + \left(r - \frac{r^3}{3!} + \frac{r^5}{5!} - \cdots\right) I_z = \cos r + I_z \sin r. \end{aligned}$$

□

**Proposition 13.4.** *The equality*

$$e^{z \cdot I_z} = \cos z + I_z \sin z \tag{13.28}$$

*is fulfilled for any quaternion $z$ which is not a real number.*

*Proof.* We have

$$\begin{aligned} e^{z \cdot I_z} &= 1 + zI_z - \frac{z^2}{2!} - I_z \frac{z^3}{3!} + \frac{z^4}{4!} + I_z \frac{z^5}{5!} - \cdots \\ &= \left(1 - \frac{r^2}{2!} + \frac{r^4}{4!} - \cdots\right) + I_z \left(r - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = \cos z + I_z \sin z. \end{aligned}$$

□

**Proposition 13.5.** *The equalities*

$$e^{-z \cdot I_z} = \cos z - I_z \sin z \tag{13.29}$$

*are fulfilled under the conditions of Proposition* 13.4.

*Proof.* From Eq. (1.31) we have

$$\begin{aligned} e^{-zI_z} &= 1 - zI_z + \frac{(-zI_z)^2}{2!} + \frac{1}{3!}(-zI_z)^3 + \frac{1}{4!}(-zI_z)^4 + \cdots \\ &= 1 - I_z \cdot z + \frac{1}{2!}(zI_z)^2 - \frac{1}{3!}(zI_z)^2(zI_z) + \frac{1}{4!}(zI_z)^2(zI_z)^2 + \cdots . \end{aligned}$$

Now, using Eq. (13.20) we obtain

$$\begin{aligned}e^{-zI_z} &= 1 - zI_z - \frac{z^2}{2!} + \frac{1}{3!}\,z^2 zI_z + \frac{1}{4!}\,(-z^2)(-z^2) + \cdots \\ &= \Big(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\Big) - I_z\Big(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\Big) \\ &= \cos z - I_z \sin z.\end{aligned}$$

□

**Proposition 13.6.** *The equalities*

$$\cos z = \frac{1}{2}\,(e^{-z\cdot I_z} + e^{z\cdot I_z}), \tag{13.30}$$

$$\sin z = \frac{1}{2}\,I_z(e^{-z\cdot I_z} - e^{z\cdot I_z}) \tag{13.31}$$

*are fulfilled under the conditions of Proposition* 13.4.

*Proof.* Equality (13.30) is obtained by adding Eqs. (13.28) and (13.29). Next, by subtracting Eq. (13.29) from (13.28) we obtain

$$e^{z\cdot I_z} - e^{-z\cdot I_z} = 2I_z \sin z.$$

From this, taking into account (13.10), we have

$$I_z(e^{z\cdot I_z} - e^{-z\cdot I_z}) = -2\sin z$$

and

$$\frac{1}{2}\,I_z(e^{-z\cdot I_z} - e^{z\cdot I_z}) = \sin z.$$

□

**Proposition 13.7.** *The equality*

$$z = |z|e^{\theta\cdot I_z}, \quad 0 \le \theta \le \pi, \tag{13.32}$$

*holds for any nonreal quaternion* $z$.

The proof of Proposition 13.7 is based on Eqs. (13.8) and (13.27).

**Proposition 13.8.** *The equalities*

$$e^{\pi I_z} = -1, \quad e^{2\pi I_z} = 1 \tag{13.33}$$

*obtained from* (13.27) *are fulfilled under the conditions of Proposition* 13.7.

**Proposition 13.9.** *The equality*

$$(e^{r\cdot I_z})^n = e^{nr\cdot I_z} \tag{13.34}$$

*is fulfilled under the conditions of Proposition* 13.7.

*Proof.* From Eqs. (13.27) and (13.26) we obtain

$$(e^{r\cdot I_z})^n = (cosr + I_z \sin r)^n = \cos nr + I_z \sin nr. \tag{13.35}$$

On the other hand, from Eq. (1.31) we obtain

$$\begin{aligned}e^{nr\cdot I_z} &= 1 + nr\cdot I_z - \frac{1}{2!}\,n^2r^2 - \frac{1}{3!}\,n^3r^3\cdot I_z + \cdots \\ &= \Big(1 - \frac{1}{2!}\,(nr)^2 + \frac{1}{4!}\,(nr)^4 - \cdots\Big) + I_z\Big(nr - \frac{1}{3!}\,(nr)^3 + \cdots\Big) \\ &= \cos nr + I_z \sin nr.\end{aligned}$$

Thus,

$$\cos nr + I_z \sin nr = e^{nr\cdot I_z}. \tag{13.36}$$

Now Eq. (13.34) follows from (13.35) and (13.36). □

**Proposition 13.10.** *The equality*

$$z^n = |z|^n e^{n\theta \cdot I_z} \tag{13.37}$$

*is fulfilled under the conditions of Proposition* 13.7.

The proof of Proposition 13.10 is based on Eqs. (13.32) and (13.34).

**Proposition 13.11.** *For real numbers $\theta$ and $\psi$ we have*

$$e^{(\theta+\psi)I_z} = e^{\theta \cdot I_z} \cdot e^{\psi \cdot I_z} = e^{\psi \cdot I_z} \cdot e^{\theta \cdot I_z}. \tag{13.38}$$

*Proof.* Using the formula (13.27) we have

$$e^{(\theta+\psi)I_z} = \cos(\theta+\psi) + I_z \sin(\theta+\psi) = \cos\theta\cos\psi - \sin\theta\sin\psi + I_z \sin\theta\cos\psi + I_z\cos\theta\sin\psi.$$

On the other hand,

$$(\cos\theta + I_z\sin\theta)(\cos\psi + I_z\sin\psi) = \cos\theta\cos\psi - \sin\theta\sin\psi + I_z\sin\theta\cos\psi + I_z\cos\theta\sin\psi.$$

Therefore,

$$e^{(\theta+\psi)I_z} = e^{\theta \cdot I_z} \cdot e^{\psi \cdot I_z} = e^{\psi \cdot I_z} \cdot e^{\theta \cdot I_z}.$$

□

**Corollary 13.12.** *For any real $\psi$ we have*

$$e^{\psi \cdot I_z} \cdot e^{-\psi \cdot I_z} = e^{-\psi \cdot I_z} \cdot e^{\psi \cdot I_z} = 1. \tag{13.39}$$

**Remark 13.13.** *In the complex analysis of one complex variable, the equality $e^{t_1+t_2} = e^{t_1} \cdot e^{t_2}$ is valid for any complex numbers $t_1$ and $t_2$. A similar equality does not hold in quaternion analysis. Indeed, if in the formula* (13.27) *we assume that $x_0 = 0$, $x_1 = 1$, $x_2 = 0 = x_3$, and $r = 1$, then $I_z = i_1$ and*

$$e^{i_1} = \cos 1 + i_1 \sin 1.$$

*Similarly,*

$$e^{i_2} = \cos 1 + i_2 \sin 1.$$

*If $x_0 = 0$, $x_1 = 1$, $x_2 = 1$, $x_3 = 0$, and $r = \sqrt{2}$, then*

$$I_z = \frac{1}{\sqrt{2}} i_1 + \frac{1}{\sqrt{2}} i_2$$

*and*

$$e^{i_1+i_2} = e^{\sqrt{2}\, I_z} = \cos\sqrt{2} + \frac{i_1+i_2}{\sqrt{2}} \sin\sqrt{2} \neq e^{i_1} \cdot e^{i_2}.$$

## 14. Applications of Quaternions

**14.1. Vector analysis.** Historically, the first important application of vector quaternions is given by Eq. (11.5), according to which a three-dimensional vector can be uniquely defined if we know its scalar and vector products on another given three-dimensional vector (a vector quaternion). In this case, the unknown vector is the left or the right quotient obtained by dividing the four-dimensional vector by another vector (see Eq. (11.6)).

The case of two-dimensional vectors is exceptional! For such vectors, the operations of multiplication and division are introduced as multiplication and division of appropriate complex numbers of the form $a + bi$, where $a$ and $b$ are real numbers and $i^2 = -1$!

The condition $i^2 = -1$ allows one to solve any first-order equation, i.e. an equation, where the coefficient of the unknown is nonzero. This result is achieved by multiplying the equation by a conjugate constant value with respect to the coefficient of the unknown, this coefficient being obviously not equal to zero.

Possibilities of generalizing an algebra of vectors to dimensions higher than two are few in number. The matter is that according to the Frobenius theorem the system of complex numbers $a+bi$ with $i^2 =$

$-1$ is the only possible extension of the field of real numbers with all the addition and multiplication laws remaining preserved.

But if we reject the commutative property of multipliers, then there will be one more possibility to consider four-dimensional vectors (the system of Hamilton quaternions) having the associative property $(q_1 q_2)q_3 = q_1(q_2 q_3)$, but $q_1 q_2 \neq q_2 q_1$.

If we also give up the associative property, then we will have eight-dimensional vectors (Cayley octaves) with weak associativity variants $(uv)v = u(vv)$ and $v(vu) = (vv)u$.

Therefore, complex numbers and quaternions are an ultimate possible extension of the notion of a real number with the basic properties of the field of real numbers preserved.

Thus, W. R. Hamilton made a truly outstanding contribution to the elaboration of the fundamental principles of modern operational calculus. For Hamilton's bibliography, see the book [115, pp. 262–269].

Physicists, mechanical experts, and technologists have widely used the techniques provided by quaternions, placing special emphasis on the ideas of vector and vector and scalar multiplication. In the applied branches of science, quaternions were replaced by ordinary vector calculus.

The books [65, 66] were the outcome of Hamilton's creative research efforts. Hamilton's writings are collected in the books [67–71].

The application of quaternions in vector analysis is described in [127, pp. 45–50 and 259–263].

As to the applications in other areas, we will briefly say about them below.

**14.2. Space flight mechanics.** The book [24] summarizes the results obtained earlier in [15, 22, 23, 30, 101, 120, 135]. In [24], the idea is realized concerning the potential application of quaternion methods to the solution of general theoretical problems, as well as to practical motion control problems of solid bodies and flying aircraft.

The application of quaternions makes it possible to create quite a convenient and obvious formalism employing the Rodrigues–Hamilton parameters which, as different from Euler angles, do not degenerate for any position of a solid body.

It is noteworthy here that in describing the helical motion of a solid body, use is made of biquaternions which are not mentioned in the book [24] and which do not possess the remarkable property of quaternions consisting in that the equality of a quaternion module to zero is equivalent to the equality of all quaternion components to zero.

**14.3. Physics and mechanics.** In these areas, the most important results were obtained by W. P. Hamilton himself (see [67–71]). A detailed account of Hamilton's results is given by L. S. Polak in [116].

For other results in this direction a reference should be made to the works [15, 22–24, 27, 30, 35, 51, 52, 59, 61, 78, 83, 101, 103, 120, 126, 135–137].

**14.4. Number theory.** The use of quaternions in the solution of various problems from the theory of numbers yielded a lot of serious results. The works [7, 8, 32, 33, 47–49, 72, 73, 77, 87–90, 111, 112, 122, 132, 138–141] are dedicated to this large class of questions.

**14.5. Equation theory.** As different from the complex case, big difficulties arise when solving equations in quaternions. For EXAMPLE, each point of the unit three-dimensional sphere $x_1^2 + x_2^2 + x_3^2 = 1$ is a solution of the equation (see Eq. (13.10))

$$I_z^2 + 1 = 0. \tag{14.1}$$

The works [21, 107, 108] are worth mentioning in this direction.

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