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Generalised Hopf type formulas

The well known Hopf formula for the second integral homology of a group says that for a given group G there is an isomorphism

$$H_2(G) \cong \frac{R \cap [F,F]}{[F,R]} ,$$

where $R \rightarrowtail F \twoheadrightarrow G$ is a free presentation of the group G.

Using topological methods, and in particular the Hurewicz theorem for n-cubes of spaces, [4], which itself is an application of the generalised van Kampen theorem for diagrams of spaces [3], Brown and Ellis in [2] generalised the Hopf formula to higher dimensions.

Theorem BE ([2]). *Let $R_1, \dots, R_n$ be normal subgroups of a group F such that*

$$H_2(F) = 0, \quad H_r\big(F \diagup \prod_{i\in A} R_i\big) = 0, \quad \textit{for } r = |A|+1,\ r = |A|+2,$$

with A a non-empty proper subset of $\langle n\rangle = \{1,\dots,n\}$ (for example, if the groups $F\diagup \prod_{i\in A} R_i$ are free for $A \neq \langle n\rangle$) and $F\diagup \prod_{1\le i\le n} R_i \cong G$. Then there is an isomorphism

$$H_{n+1}(G) \cong \frac{\bigcap_{i=1}^{n} R_i \cap [F,F]}{\prod_{A\subseteq\langle n\rangle} \big[\bigcap_{i\in A} R_i, \bigcap_{i\notin A} R_i\big]} .$$

Later, Ellis using mainly algebraic means, and, in particular, his hyper-relative derived functors, proved the same result, [5].

Trying to derive this result purely algebraically and to obtain Hopf type formulas for more general settings, we suspected that the conditions given above for Theorem BE were not sufficient for getting the generalised Hopf formula for $H_{n+1}(G)$, $n \ge 3$. In fact, we give the following counter-example to Theorem BE:

Let F be a free group with base $\{x_1, x_2\}$, R_1, R_2 and R_3 normal subgroups of the group F generated by the one point sets $\{x_1\}$, $\{x_2\}$ and $\{x_1x_2^{-1}\}$ respectively and $G = 1$. Then we have $F/R_i \cong \mathbb{Z}$, $i = 1,2,3$, $F/R_iR_j = 1$, $i \neq j$ and $[F,F] = [R_1,R_2] = R_i \cap R_j$, $i \neq j$, therefore

$$\frac{\bigcap_{i\in\langle 3\rangle} R_i \cap [F,F]}{\prod_{A\subseteq\langle 3\rangle} \big[\bigcap_{i\in A} R_i, \bigcap_{i\notin A} R_i\big]} \cong \mathbb{Z}$$

whilst $H_3(G) = 1$.

Let the endofunctors $Z_k(G)$ be given by $Z_k(G) = G/\Gamma_k(G)$, $k \ge 2$, where $\{\Gamma_k(G),\ k \ge 1\}$ is the lower central series of a given group G. These Z_k are endofunctors on the category of groups and generalise the abelianization functor, so their non-abelian left derived functors, L_nZ_k, $n \ge 0$, generalise the group homology functors H_n, $n \ge 1$, these

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being the non-abelian derived functors of the group abelianization functor Z_2, cf., for instance, [1].

In [7], a Hopf-like formula is proved for the second Conduché-Ellis homology of precrossed modules using Čech derived functors. The main goal of our work is to develop this method further, and by applying it, to express $L_n Z_k$, $n \geq 1$, $k \geq 2$, by generalised Hopf type formulas. In particular, we will give the stronger conditions needed for Theorem BE. Finally we apply these results to algebraic K-theory.

Some needed notions and notations.

(1) Given a group F and n normal subgroups, $R_1, \ldots, R_n$, then $(F; R_1, \ldots, R_n)$ will be called a *normal* $(n+1)$*-ad of groups.* For a given $(n+1)$-ad of groups $(F; R_1, \ldots, R_n)$, $A \subseteq \langle n \rangle$ and for $k \geq 1$ denote by $D_k(F; A)$ the following normal subgroup of the group F

$$\prod_{A_1 \cup A_2 \cup \cdots \cup A_k = A} [\bigcap_{i \in A_1} R_i, [\bigcap_{i \in A_2} R_i, \ \ldots \ , [\bigcap_{i \in A_{k-1}} R_i, \bigcap_{i \in A_k} R_i] \ldots]] \, .$$

(2) Let us consider the set $\langle n \rangle = \{1, \ldots, n\}$. The subsets of $\langle n \rangle$ are ordered by inclusion. This ordered set determines in the usual way a category $\underline{C_n}$. An *n-cube of groups* is a functor $\mathfrak{F} : \underline{C_n} \to \mathfrak{Gr}$, $A \mapsto \mathfrak{F}_A$, $\rho_B^A \mapsto \alpha_B^A$. It is easy to see that there exists a natural homomorphism $\mathfrak{F}_A \xrightarrow{\alpha_A} \lim\limits_{B \supset A} \mathfrak{F}_B$ for any $A \subseteq \langle n \rangle$, $A \neq \langle n \rangle$.

(3) Let G be a group. An n-cube of groups $\mathfrak{F}$ will be called an *n-presentation of the group* G if $\mathfrak{F}_{\langle n \rangle} = G$. An n-presentation $\mathfrak{F}$ of G is called *free* and *exact* if the group $\mathfrak{F}_A$ is free and the homomorphism α_A is surjective for all $A \neq \langle n \rangle$.

(4) Define the functor $Z_\infty : \mathfrak{Gr} \to \mathfrak{Gr}$ as follows: for a given group G, $Z_\infty(G) = \varprojlim_j Z_j(G)$; for a given group homomorphism $\lambda : G \to H$, $Z_\infty(\lambda)$ is the group homomorphism induced by the $Z_j(\lambda)$.

Main Theorem. *Let G be a group, $\mathfrak{F}$ be a free exact n-presentation of G and $k \geq 2$. Then*

$$L_n Z_k(G) \cong \frac{\bigcap_{i \in \langle n \rangle} R_i \cap \Gamma_k(F)}{D_k(F; R_1, \ldots, R_n)} \, , \quad n \geq 1 \, ,$$

where $(F; R_1, \ldots, R_n)$ is the normal $(n+1)$-ad of groups induced by $\mathfrak{F}$.

The application of our Main Theorem, but in particular case, to algebraic K-theory will be given virtue to the following

Theorem. *Let $\mathfrak{R}$ be a ring with unit and $(F_*, d_0^0, GL(\mathfrak{R}))$ be a free pseudosimplicial resolution of the general linear group $GL(\mathfrak{R})$. Then there is an exact sequence of abelian groups*

$$0 \longrightarrow \varprojlim_j{}^{(1)} \left(\frac{\Big(\bigcap_{i \in \langle n+1 \rangle} \operatorname{Ker} d_{i-1}^n \Big) \cap \Gamma_j(F_n)}{D_j(F_n; \operatorname{Ker} d_0^n, \ldots, \operatorname{Ker} d_n^n)} \right) \longrightarrow K_{n+1}(\mathfrak{R}) \longrightarrow$$

$$\longrightarrow \varprojlim_j \left(\frac{\Big(\bigcap_{i \in \langle n \rangle} \operatorname{Ker} d_{i-1}^{n-1} \Big) \cap \Gamma_j(F_{n-1})}{D_j(F_{n-1}; \operatorname{Ker} d_0^{n-1}, \ldots, \operatorname{Ker} d_{n-1}^{n-1})} \right) \longrightarrow 0$$

for $n \geq 1$, where $\varprojlim_j{}^{(1)}$ is the first right derived functor of the functor $\varprojlim_j$ (for the definition see e.g. [6]*).*

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