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Homology of multiplicative Lie rings

A. Bak[a,*], G. Donadze[b], N. Inassaridze[b], M. Ladra[c]

[a] Department of Mathematics, University of Bielefeld, P. O. Box 100131, 33501 Bielefeld, Germany
[b] A. Razmadze Mathematical Institute, Georgian Academy of Sciences, M. Alexidze St. 1, 0193 Tbilisi, Georgia
[c] Departamento de Álgebra, Facultad de Matemáticas, Universidad de Santiago de Compostela, 15782 Santiago de Compostela, Spain

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Abstract

Two homology theories of multiplicative Lie rings are constructed, studied and compared with the usual homology theories of groups and Lie rings. Central extensions of multiplicative Lie rings are introduced. It is shown that the Steinberg multiplicative Lie ring of a ring is the direct product of the Steinberg group (viewed as a multiplicative Lie ring under the commutator bracket) and the Steinberg Lie ring.

MSC: 18G10; 18G50

1. Introduction

In [2] Ellis introduced multiplicative Lie algebras, which are called multiplicative Lie rings in the body of the current paper, to investigate an interesting combinatorial problem on group commutators. In [10] Point and Wantiez studied algebraic structural properties of multiplicative Lie algebras. In particular, they defined nilpotency and proved several nilpotency results generalizing known ones for groups and Lie algebras.

In the current paper, we investigate further structural properties of multiplicative Lie rings. Then we introduce two homology theories of multiplicative Lie rings and show they satisfy properties analogous to well known ones in the homology theories of groups and Lie rings. Our main results compare our homology groups in degrees 1, 2 and 3 with the corresponding ones for groups and Lie rings. We then introduce the Steinberg multiplicative Lie ring $St^{mlr}(R)$ of a unital ring R and show that its center is isomorphic to the direct product $K_2(R) \times HC_1(R)$ of the algebraic K-theory group $K_2(R)$ and the cyclic homology group $HC_1(R)$. This result will be used in a subsequent paper to study the relationship between a new definition of K-groups of $gl(R)$ (via global actions) and the Lie homology of $gl(R)$.

The rest of the paper is organized as follows. Immediately below, we review some basic notation and conventions. In Section 2, we recall the notion of a multiplicative Lie ring, develop some of its elementary notions of structure, and establish a basic adjointness principle. In Section 3, we introduce two homology theories for multiplicative Lie rings.

* Corresponding author.
E-mail addresses: bak@math.uni-bielefeld.de (A. Bak), donad@rmi.acnet.ge (G. Donadze), inas@rmi.acnet.ge (N. Inassaridze), ladra@usc.es (M. Ladra).

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We verify that they satisfy standard properties analogous to those for the homology theories of groups and Lie rings, and compare our homology groups of degrees 1, 2 and 3 with the corresponding homology groups of groups and Lie rings. In Section 4, we define central extensions of multiplicative Lie rings and show that our second homology group of a perfect multiplicative Lie ring is equal to the kernel of its universal central extension. In Section 5, we relate multiplicative Lie rings to K-theory and cyclic homology. In particular, we define the multiplicative Steinberg Lie ring St^{mlr} via the obvious generators and relations and obtain that it is a universal central extension of the direct product $E_{[,]} \times sl$ where $E_{[,]}$ denotes the multiplicative Lie ring of the elementary group E under commutator bracket and sl the Lie ring generated by all strictly upper and strictly lower triangular matrices.

Notation and conventions. Let *Sets* and $\mathfrak{Gr}$ denote respectively the categories of sets and groups. For elements x, y of a group, let ${}^x y = xyx^{-1}$ and $[x, y] = xyx^{-1}y^{-1}$. For any group G, let $[G, G]$ and $Z(G)$ denote respectively its commutator subgroup and center. By a ring R, we shall always mean an associative ring with identity.

2. Multiplicative Lie rings

We recall the notion of multiplicative Lie ring due to Ellis [2], give some important examples and establish some structural results.

2.1. Definition of multiplicative Lie ring

A *multiplicative Lie ring* consists of a multiplicative (possibly non-abelian) group $\mathfrak{g}$ together with a binary function $\{,\} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which we shall call Lie product, satisfying the following identities:

$$\{x, x\} = 1, \tag{2.1.1}$$

$$\{x, yy'\} = \{x, y\}\, {}^y\{x, y'\}, \tag{2.1.2}$$

$$\{xx', y\} = {}^x\{x', y\}\{x, y\}, \tag{2.1.3}$$

$$\{\{x, y\}, {}^y z\}\{\{y, z\}, {}^z x\}\{\{z, x\}, {}^x y\} = 1, \tag{2.1.4}$$

$${}^z\{x, y\} = \{{}^z x, {}^z y\} \tag{2.1.5}$$

for all $x, x', y, y', z \in \mathfrak{g}$.

In [2] the following identities are deduced from (2.1.1) to (2.1.5):

$$\{1, x\} = \{x, 1\} = 1, \tag{2.1.6}$$

$$\{y, x\} = \{x, y\}^{-1}, \tag{2.1.7}$$

$${}^{\{x,y\}}\{x', y'\} = {}^{[x,y]}\{x', y'\}, \tag{2.1.8}$$

$$\{[x, y], x'\} = [\{x, y\}, x'], \tag{2.1.9}$$

$$\{x^{-1}, y\} = {}^{x^{-1}}\{x, y\}^{-1} \quad \text{and} \quad \{x, y^{-1}\} = {}^{y^{-1}}\{x, y\}^{-1} \tag{2.1.10}$$

for all $x, x', y, y' \in \mathfrak{g}$. Due to (2.1.2), (2.1.7) and (2.1.9) we obtain another useful identity

$$\{x, y\}\{x, z\} = \{x, yz\}\{y, [z, x]\}, \quad x, y, z \in \mathfrak{g}. \tag{2.1.11}$$

In fact,

$$\{x, yz\}\{y, [z, x]\} = \{x, y\}\, {}^y\{x, z\}[y, \{z, x\}] = \{x, y\}\{x, z\}.$$

A morphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ of multiplicative Lie rings is a group homomorphism such that $\phi\{x, y\} = \{\phi x, \phi y\}$ for all $x, y \in \mathfrak{g}$. Let

$$\mathcal{LM}$$

denote the category of multiplicative Lie rings and their morphisms.

We recall important examples of multiplicative Lie rings given in [2].

2.2. Examples

(a) Any group G is a multiplicative Lie ring under $\{x, y\} = xyx^{-1}y^{-1}$ for all $x, y \in G$. It is denoted by $G_{[,]}$.

(b) Any group G is also a multiplicative Lie ring under $\{x, y\} = 1$ for all $x, y \in G$. It is called the abelian multiplicative Lie ring of G and is sometimes denoted by $G_\bullet$.

(c) Any ordinary Lie ring L is a multiplicative Lie ring under its Lie product. Moreover, if $\mathfrak{g}$ is a multiplicative Lie ring whose underlying group is abelian then $\mathfrak{g}$ is an ordinary Lie ring.

(d) Let $E \twoheadrightarrow P$ denote a central extension of a group P. Then $x \in P$ acts on $u \in E$ by ${}^x u = \bar{x}u\bar{x}^{-1}$ where $\bar{x} \in E$ is any pre-image of x. The semi-direct product $\mathfrak{g} = E \rtimes P$ of the action of P on E is a multiplicative Lie ring with Lie product $\{,\} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by $\{(u, x), (u', x')\} = ([u\bar{x}, u'\bar{x'}], 1)$ for all $u, u' \in E$ and $x, x' \in P$.

A subgroup $\mathfrak{n}$ of $\mathfrak{g}$ will be *subring* of $\mathfrak{g}$ if $\{x, y\} \in \mathfrak{n}$ for all $x, y \in \mathfrak{n}$. It will be an *ideal* of $\mathfrak{g}$ if it is a normal subgroup and if $\{x, y\} \in \mathfrak{n}$ for all $x \in \mathfrak{n}$ and $y \in \mathfrak{g}$. It follows from (2.1.7) that if $\mathfrak{n}$ is an ideal then $\{y, x\} \in \mathfrak{n}$ for all $x \in \mathfrak{n}$, $y \in \mathfrak{g}$.

By the kernel of a morphism $\phi : \mathfrak{g} \longrightarrow \mathfrak{g}'$ in the category $\mathcal{LM}$ we just mean the kernel of ϕ considered as a group homomorphism. Clearly the kernel $\ker\phi$ is an ideal of $\mathfrak{g}$. Conversely, if $\mathfrak{n}$ is an ideal of $\mathfrak{g}$, the quotient group $\mathfrak{g}/\mathfrak{n}$ inherits the structure of a multiplicative Lie ring. In fact, using (2.1.2) and (2.1.3) we have

$$\begin{aligned}\{x, y\} &= \{x, y'y'^{-1}y\} = \{x, y'\}\,{}^{y'}\{x, y'^{-1}y\} = \{x'x'^{-1}x, y'\}\,{}^{y'}\{x, y'^{-1}y\}\\ &= {}^{x'}\{x'^{-1}x, y'\}\{x', y'\}\,{}^{y'}\{x, y'^{-1}y\}\end{aligned}$$

for all $x, x', y, y' \in \mathfrak{g}$ such that $x'^{-1}x, y'^{-1}y \in \mathfrak{n}$. Thus

$$\{x, y\} = \{x', y'\} \mod \mathfrak{n}.$$

The following notions are taken from [10].

2.3. Some required notions

Let $\mathfrak{g}$ denote a multiplicative Lie ring.

- ◇ Let $\mathfrak{f}$ and $\mathfrak{h}$ be subgroups of $\mathfrak{g}$. The subgroup of $\mathfrak{g}$ generated by all elements $\{x, y\}$, $x \in \mathfrak{f}$, $y \in \mathfrak{h}$ is denoted by $\{\mathfrak{f}, \mathfrak{h}\}$. The subgroup $\{\mathfrak{g}, \mathfrak{g}\}$ is an ideal and is called the *Lie commutator* of the multiplicative Lie ring $\mathfrak{g}$.
- ◇ The set $\mathrm{ZL}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid \text{for all } y \in \mathfrak{g} : \{x, y\} = 1\}$ is an ideal and is called the *Lie center* of the multiplicative Lie ring $\mathfrak{g}$.
- ◇ A multiplicative Lie ring $\mathfrak{g}$ is called *perfect* if $\mathfrak{g} = \{\mathfrak{g}, \mathfrak{g}\}$.

By the above, we know that $\{\mathfrak{g}, \mathfrak{g}\}$ and $\mathrm{ZL}(\mathfrak{g})$ are ideals of $\mathfrak{g}$. Furthermore it is easy to check that the group commutator $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ and the group center $Z(\mathfrak{g})$ of $\mathfrak{g}$ are also ideals of the multiplicative Lie ring $\mathfrak{g}$. In fact, by (2.1.6) and (2.1.9) we get

$$[\{x, y\}, z] = \{[x, y], z\} = 1$$

for all $x \in Z(\mathfrak{g})$, $y, z \in \mathfrak{g}$.

Proposition 2.4. *Let $\mathfrak{g}$ denote a multiplicative Lie ring.*

(a) *If $\mathfrak{g}$ is perfect as a multiplicative Lie ring, then* $\mathrm{ZL}(\mathfrak{g}) \subseteq Z(\mathfrak{g})$ *and* $[\mathfrak{g}, \mathfrak{g}] \subseteq \{\mathfrak{g}, \mathfrak{g}\}$.

(b) *If $\mathfrak{g}$ is perfect as a group, then* $Z(\mathfrak{g}) \subseteq \mathrm{ZL}(\mathfrak{g})$ *and* $\{\mathfrak{g}, \mathfrak{g}\} \subseteq [\mathfrak{g}, \mathfrak{g}]$.

Proof. (a) By assumption any $y \in \mathfrak{g}$ can be written in the form $\prod_{i=1}^n \{x_i, y_i\}$ for some $x_i, y_i \in \mathfrak{g}$. Suppose $x \in \mathfrak{g}$. By (2.1.2) and (2.1.9), we obtain

$$\begin{aligned}[x, y] &= \left[x, \prod_{i=1}^n \{x_i, y_i\}\right] = [x, \{x_1, y_1\}] \cdot {}^{\{x_1,y_1\}}[x, \{x_2, y_2\}] \cdots {}^{\prod_{i=1}^{n-1}\{x_i,y_i\}}[x, \{x_n, y_n\}]\\ &= \{x, [x_1, y_1]\} \cdot {}^{\{x_1,y_1\}}\{x, [x_2, y_2]\} \cdots {}^{\prod_{i=1}^{n-1}\{x_i,y_i\}}\{x, [x_n, y_n]\}.\end{aligned}$$

Suppose now $x \in \mathrm{ZL}(\mathfrak{g})$. Then each $\{x, [x_i, y_i]\} = 1$, $(1 \le i \le n)$. Thus $[x, y] = 1$ and $x \in Z(\mathfrak{g})$. Moreover from equality above it follows directly that $[\mathfrak{g}, \mathfrak{g}] \subseteq \{\mathfrak{g}, \mathfrak{g}\}$.

(b) is proved similarly to (a), replacing (2.1.2) by the corresponding rule for group commutators. □

Proposition 2.5. *Let* $\mathfrak{g}$ *be a perfect multiplicative Lie ring. Then the multiplicative Lie ring* $\mathfrak{g}/Z(\mathfrak{g})$ *has trivial center, i.e.*

$$\mathrm{ZL}(\mathfrak{g}/Z(\mathfrak{g})) = 1.$$

Proof. Let $x \in \mathfrak{g}$. If suffices to show that if $\{x, y\} \in Z(\mathfrak{g})$ for all $y \in \mathfrak{g}$ then $[x, y] = 1$ for all $y \in \mathfrak{g}$. As in the proof of 2.4(a), we can write

$$[x, y] = \left[x, \prod_{i=1}^{n}\{x_i, y_i\}\right] = [x, \{x_1, y_1\}] \cdot {}^{\{x_1,y_1\}}[x, \{x_2, y_2\}] \cdots {}^{\prod_{i=1}^{n-1}\{x_i,y_i\}}[x, \{x_n, y_n\}] .$$

But by (2.1.9) and (2.1.2), we also have

$$[x, \{x_i, y_i\}] = \{x, [x_i, y_i]\} = \{x, x_i\} \cdot {}^{x_i}\{x, y_i\} \cdot {}^{x_i y_i}\{x, x_i{}^{-1}\} \cdot {}^{x_i y_i x_i{}^{-1}}\{x, y_i{}^{-1}\} = 1.$$

It follows that $[x, y] = 1$. □

The construction $G_\bullet$ in 2.2(b) defines a functor

$$\mathfrak{Gr} \longrightarrow \mathcal{LM}, \quad G \mapsto G_\bullet,$$

which embeds $\mathfrak{Gr}$ onto full subcategory of group objects of $\mathcal{LM}$. We shall use this functor to identify $\mathfrak{Gr}$ with its image in $\mathcal{LM}$. Define the *abelianization functor*

$$\mathfrak{Ab} : \mathcal{LM} \longrightarrow \mathfrak{Gr}, \quad \mathfrak{g} \mapsto \frac{\mathfrak{g}}{\{\mathfrak{g}, \mathfrak{g}\}}.$$

Remark. The underlying group of $\mathfrak{Ab}(\mathfrak{g})$ is in general non-abelian, as in the case $G_\bullet$ above.

The following proposition is easy and left to the reader.

Proposition 2.6. *The abelianization functor* $\mathfrak{Ab} : \mathcal{LM} \longrightarrow \mathfrak{Gr}$ *is left adjoint to the inclusion functor* $\mathfrak{Gr} \hookrightarrow \mathcal{LM}$.

3. Homologies of multiplicative Lie rings

The rest of the paper is devoted to the investigation of multiplicative Lie rings from the homological point of view. Two homology theories of multiplicative Lie rings are constructed as non-abelian derived functors of the abelianization functor with respect to two different projective classes in the category $\mathcal{LM}$. Standard properties analogous to those for the homology theories of groups and Lie rings are established and our homology groups of degrees 1, 2, 3 are compared with the corresponding homology groups of groups and Lie rings.

3.1. *Free multiplicative Lie rings*

We shall need the notions of free multiplicative Lie rings over a given set and over a given group.

Recall that a *magma* is a binary operation on a set M, i.e. a function $M \times M \longrightarrow M$. Let X denote a set and $M(X)$ the *free magma* over X. $M(X)$ is constructed as follows. Let $X_1 = X$. Let $n > 1$ and assume the set X_p has been defined for all $1 \le p < n$. Define $X_n = \coprod_{p+q=n} X_p \times X_q$ (=disjoint union). Then $M(X) = \coprod_{n \ge 1} X_n$ with the obvious binary operation. Let $F(M(X))$ denote the free group over the set $M(X)$. Let $\{,\}$ denote the binary operation on $F(M(X))$. The formal rules expressed by (2.1.2), (2.1.3) and (2.1.10) provide a straightforward procedure for extending $\{,\}$ to a binary operation $\{,\}$ on $F(M(X))$. For any subset $S \subseteq F(M(X))$, let $N(S)$ denote the normal subgroup of $F(M(X))$ generated by S and $\{\{a, s\} | a \in F(M(X)), s \in S\}$. Let $\mathcal{N}(S) = \cup_{i \geq 1} N^i(S)$ where for $i > 1$, $N^i(S) = N(N^{i-1}(S))$. Let $relF(M(X))$ denote the set of all elements

$$\{a, a\}, \quad \{a, bb'\}\,{}^b\{a, b'\}^{-1}\{a, b\}^{-1}, \quad \{aa', b\}\{a, b\}^{-1}\,{}^a\{a', b\}^{-1},$$
$$\{\{b, a\}, {}^a c\}\{\{a, c\}, {}^c b\}\{\{c, b\}, {}^b a\}, \quad {}^c\{a, b\}\{{}^c a, {}^c b\}^{-1}$$

of $F(M(X))$ such that $a, b, c \in F(M(X))$. Let $Q(M(X)) = \mathcal{N}(relF(M(X)))$. It is easy to check that $\mathfrak{F}(X) = F(M(X))/Q(M(X))$ is a multiplicative Lie ring, called the *free multiplicative Lie ring over the set* X, and that the construction $X \mapsto \mathfrak{F}(X)$ defines a covariant functor

$$\mathfrak{F} : Sets \longrightarrow \mathcal{LM}, \quad X \mapsto \mathfrak{F}(X).$$

Now let G denote a group and let $\mathfrak{F}(G)$ denote the free multiplicative Lie ring over the underlying set G. Let $\mathfrak{Q}(G)$ denote the ideal of the multiplicative Lie ring $\mathfrak{F}(G)$ generated by the elements

$$|g||h||gh|^{-1} \quad \text{for all } g, h \in G,$$

where $|g|$ denotes the element of $\mathfrak{F}(G)$ defined by the element $g \in G$. The quotient multiplicative Lie ring $\mathfrak{F}(G)/\mathfrak{Q}(G)$ will be called the *free multiplicative Lie ring over the group* G and will be denoted by $\mathfrak{L}(G)$. This construction clearly defines a covariant functor

$$\mathfrak{L} : \mathfrak{Gr} \longrightarrow \mathcal{LM}, \quad G \mapsto \mathfrak{L}(G).$$

Let

$$\mathcal{U} : \mathcal{LM} \longrightarrow Sets$$

denote the forgetful functor assigning to each multiplicative Lie ring its underlying set and let

$$\mathcal{V} : \mathcal{LM} \longrightarrow \mathfrak{Gr}$$

denote the forgetful functor associating to each multiplicative Lie ring its underlying group.

The following proposition is straightforward and left to the reader.

Proposition 3.1.1. *The functors $\mathcal{U}$ and $\mathcal{V}$ are right adjoints to the functors $\mathfrak{F}$ and $\mathfrak{L}$, respectively.*

We assume the reader is familiar with cotriples and projective classes. See, for example, [3, Chapter 2] or [13, Section 8.6]. It is well known that every adjoint pair of functors induces a cotriple. Let $(\mathfrak{F}\mathcal{U}, \tau, \delta)$ (resp. $(\mathfrak{L}\mathcal{V}, \tau', \delta')$) denote the cotriple in $\mathcal{LM}$ defined by the adjoint pair $(\mathfrak{F}, \mathcal{U})$ (resp. $(\mathfrak{L}, \mathcal{V})$) in 3.1.1. Let $\mathcal{P}$ and $\mathcal{Q}$ denote the projective classes in the category $\mathcal{LM}$ induced by the cotriples $(\mathfrak{F}\mathcal{U}, \tau, \delta)$ and $(\mathfrak{L}\mathcal{V}, \tau', \delta')$, respectively. The following lemma describes these projective classes. It is easy to prove and will be very useful in the following.

Lemma 3.1.2. (i) *A morphism $\mathfrak{g} \xrightarrow{\phi} \mathfrak{g}'$ in the category $\mathcal{LM}$ is a $\mathcal{P}$-epimorphism (resp. $\mathcal{Q}$-epimorphism) if and only if ϕ admits a set-theoretic (resp. group-theoretic) section.*

(ii) *A free multiplicative Lie ring over a set (resp. over a group) belongs to the projective class $\mathcal{P}$ (resp. $\mathcal{Q}$).*

It is easy to see that the category $\mathcal{LM}$ has finite limits. This implies, cf. [3, Definition 2.22], the existence of non-abelian left derived functors $\mathcal{L}_n^{\mathcal{P}}\mathfrak{Ab} : \mathcal{LM} \longrightarrow \mathfrak{Gr}, n \geq 0$, and $\mathcal{L}_n^{\mathcal{Q}}\mathfrak{Ab} : \mathcal{LM} \longrightarrow \mathfrak{Gr}, n \geq 0$, of the abelianization functor $\mathfrak{Ab} : \mathcal{LM} \longrightarrow \mathfrak{Gr}$ relative to the projective classes $\mathcal{P}$ and $\mathcal{Q}$, respectively. More explicitly, the functors $\mathcal{L}_n^{\mathcal{P}}\mathfrak{Ab}, n \geq 0$, and $\mathcal{L}_n^{\mathcal{Q}}\mathfrak{Ab}, n \geq 0$, are defined as n-th homotopy groups of the (pseudo)simplicial groups $(\mathfrak{Ab}(\mathfrak{f}_*), \mathfrak{Ab}(d_0^0), \mathfrak{Ab}(\mathfrak{g}))$ and $(\mathfrak{Ab}(\mathfrak{h}_*), \mathfrak{Ab}(d_0^0), \mathfrak{Ab}(\mathfrak{g}))$ given by applying the functor $\mathfrak{Ab}$ dimension-wise to a $\mathcal{P}$-projective (pseudo)simplicial resolution $(\mathfrak{f}_*, d_0^0, \mathfrak{g})$ and to a $\mathcal{Q}$-projective (pseudo)simplicial resolution $(\mathfrak{h}_*, d_0^0, \mathfrak{g})$, respectively.

A concise treatment of the special case of abelian derived functors is found in [13, Section 8.7] and examples of non-abelian derived functors are given there also. It is known (see [3, Corollary 2.37]) that the derived functors relative to the projective class induced by a cotriple are isomorphic to the derived functors relative to this cotriple.

It is easy to check that $\mathfrak{Ab}$ is a right exact functor, i.e. it sends a short exact sequence of multiplicative Lie rings $1 \to \mathfrak{f} \to \mathfrak{g} \to \mathfrak{h} \to 1$ to an exact sequence of groups $\mathfrak{Ab}(\mathfrak{f}) \to \mathfrak{Ab}(\mathfrak{g}) \to \mathfrak{Ab}(\mathfrak{h}) \to 1$. Hence by [3, Proposition 2.26],

$$\mathcal{L}_0^{\mathcal{P}}\mathfrak{Ab}(\mathfrak{g}) \cong \mathfrak{Ab}(\mathfrak{g}) \cong \mathcal{L}_0^{\mathcal{Q}}\mathfrak{Ab}(\mathfrak{g}).$$

Definition 3.2. Let $\mathfrak{g}$ denote a multiplicative Lie ring and $n \geq 1$. Define the n-th *strong homology* group of $\mathfrak{g}$ by

$$HS_n^{mlr}(\mathfrak{g}) = \mathcal{L}_{n-1}^{\mathcal{P}}\mathfrak{Ab}(\mathfrak{g}).$$

Define the n-th *weak homology* group of $\mathfrak{g}$ by

$$HW_n^{mlr}(\mathfrak{g}) = \mathcal{L}_{n-1}^{\mathcal{Q}}\mathfrak{Ab}(\mathfrak{g}).$$

By the isomorphisms just preceding 3.2, we have

$$HS_1^{mlr}(\mathfrak{g}) = \mathcal{L}_0^{\mathcal{P}}\mathfrak{Ab}(\mathfrak{g}) \cong \mathfrak{Ab}(\mathfrak{g}) \cong \mathcal{L}_0^{\mathcal{Q}}\mathfrak{Ab}(\mathfrak{g}) = HW_1^{mlr}(\mathfrak{g}).$$

Let G denote an abelian group. Clearly $\mathfrak{L}(G)$ is the ordinary free Lie ring over G. Therefore for a given Lie ring $\mathfrak{g}$, the classical Chevalley–Eilenberg homology groups and our weak homology groups are isomorphic.

3.3. Hopf formula

For future calculations, we prove now the *Hopf formula* for HS_2^{mlr} and HW_2^{mlr}.

Call a short exact sequence $\mathfrak{r} \rightarrowtail \mathfrak{f} \overset{\phi}{\twoheadrightarrow} \mathfrak{g}$ of multiplicative Lie rings a presentation of $\mathfrak{g}$ in the projective class $\mathcal{P}$ (resp. $\mathcal{Q}$) if $\mathfrak{f} \in \mathcal{P}$ (resp. $\mathfrak{f} \in \mathcal{Q}$) and ϕ is a $\mathcal{P}$-epimorphism (resp. $\mathcal{Q}$-epimorphism).

Theorem 3.3.1. *Let $\mathfrak{g}$ denote a multiplicative Lie ring and $\mathfrak{r} \rightarrowtail \mathfrak{f} \overset{\phi}{\twoheadrightarrow} \mathfrak{g}$ (resp. $\mathfrak{s} \rightarrowtail \mathfrak{h} \overset{\psi}{\twoheadrightarrow} \mathfrak{g}$) be a presentation of $\mathfrak{g}$ in the projective class $\mathcal{P}$ (resp. $\mathcal{Q}$). Then there are isomorphisms of groups*

$$HS_2^{mlr}(\mathfrak{g}) \cong \frac{\mathfrak{r} \cap \{\mathfrak{f}, \mathfrak{f}\}}{\{\mathfrak{f}, \mathfrak{r}\}}, \quad HW_2^{mlr}(\mathfrak{g}) \cong \frac{\mathfrak{s} \cap \{\mathfrak{h}, \mathfrak{h}\}}{\{\mathfrak{h}, \mathfrak{s}\}}.$$

We shall prove only the first isomorphism. The proof of the second is similar. We need the following lemma.

Lemma 3.3.2. *Let $\mathfrak{g}_*$ denote a simplicial multiplicative Lie ring. Then*

$$\{\mathfrak{g}_1, \ker d_0^1\} = \ker d_0^1 \cap \{\mathfrak{g}_1, \mathfrak{g}_1\}.$$

Proof. It suffices to show that

$$\ker d_0^1 \cap \{\mathfrak{g}_1, \mathfrak{g}_1\} \subseteq \{\mathfrak{g}_1, \ker d_0^1\}. \tag{3.3.3}$$

We begin by showing that

$$\{\mathfrak{g}_1, \mathfrak{g}_1\} = \{\mathfrak{g}_1, \ker d_0^1\} \cdot (\{\mathfrak{g}_1, \mathfrak{g}_1\} \cap s_0\mathfrak{g}_0). \tag{3.3.4}$$

It suffices to show that the left hand side is contained in the right hand side. Let $v, w \in \ker d_0^1$. Let $x = s_0d_0^1(x)v$ and $y = s_0d_0^1(y)w$. By (2.1.2) and (2.1.3),

$$\begin{aligned}\{x, y\} &= \{x, s_0d_0^1(y)w\} = \{x, s_0d_0^1(y)\} \cdot {}^{s_0d_0^1(y)}\{x, w\} = \{s_0d_0^1(x)v, s_0d_0^1(y)\} \cdot {}^{s_0d_0^1(y)}\{x, w\} \\ &= {}^{s_0d_0^1(x)}\{v, s_0d_0^1(y)\} \cdot \{s_0d_0^1(x), s_0d_0^1(y)\} \cdot {}^{s_0d_0^1(y)}\{x, w\}\,.\end{aligned}$$

But by (2.1.5) ${}^{s_0d_0^1(x)}\{v, s_0d_0^1(y)\}$, ${}^{s_0d_0^1(y)}\{x, w\} \in \{\mathfrak{g}_1, \ker d_0^1\}$ and $\{s_0d_0^1(x), s_0d_0^1(y)\} \in \{\mathfrak{g}_1, \mathfrak{g}_1\} \cap s_0\mathfrak{g}_0$. Thus

$$\{\mathfrak{g}_1, \mathfrak{g}_1\} \subseteq \{\mathfrak{g}_1, \ker d_0^1\} \cdot (\{\mathfrak{g}_1, \mathfrak{g}_1\} \cap s_0\mathfrak{g}_0).$$

Now we are back to (3.3.3). By (3.3.4), we can write any element $x \in \ker d_0^1 \cap \{\mathfrak{g}_1, \mathfrak{g}_1\}$ as a product

$$x = x'y,$$

where $x' \in \{\mathfrak{g}_1, \ker d_0\}$ any $y \in \{\mathfrak{g}_1, \mathfrak{g}_1\} \cap s_0\mathfrak{g}_0$. It follows that $y = x'^{-1}x$ belongs to $\ker d_0^1 \cap s_0\mathfrak{g}_0 = 1$. Whence $x = x'$ and (3.3.3) is proved. □

Proof of 3.3.1. Let $(\mathfrak{f}_*, d_0^0, \mathfrak{g})$ be a $\mathcal{P}$-projective simplicial resolution (cf. [3]) of the multiplicative Lie ring $\mathfrak{g}$ induced by the free presentation $\mathfrak{f} \twoheadrightarrow \mathfrak{g}$. The long exact homotopy sequence induced by the short exact sequence of augmented simplicial groups

$$1 \longrightarrow \{\mathfrak{f}_*, \mathfrak{f}_*\} \longrightarrow \mathfrak{f}_* \longrightarrow \mathfrak{Ab}(\mathfrak{f}_*) \longrightarrow 1,$$

gives by 3.2 the isomorphism of groups

$$HS_2^{mlr}(\mathfrak{g}) \cong \frac{\ker \tilde{d}_0^0}{\tilde{d}_1^1(\ker \tilde{d}_0^1)}.$$

Since $\tilde{d}_i^n$ is a restriction of d_i^n to $\{\mathfrak{f}_n, \mathfrak{f}_n\}$, it follows that $\ker \tilde{d}_0^0 = \ker d_0^0 \cap \{\mathfrak{f}_0, \mathfrak{f}_0\} = \mathfrak{r} \cap \{\mathfrak{f}, \mathfrak{f}\}$ and $\ker \tilde{d}_0^1 = \ker d_0^1 \cap \{\mathfrak{f}_1, \mathfrak{f}_1\}$.

Using 3.3.2, we obtain

$$\tilde{d}_1^1(\ker \tilde{d}_0^1) = \tilde{d}_1^1(\ker d_0^1 \cap \{\mathfrak{f}_1, \mathfrak{f}_1\}) = \tilde{d}_1^1\{\mathfrak{f}_1, \ker d_0^1\} = \{\mathfrak{f}, \mathfrak{r}\}. \quad □$$

We continue the comparing of the strong and the weak homology groups of multiplicative Lie rings.

Let G denote a group. For a normal subgroup $S \subseteq G$, define the subgroups $\Gamma_n(S, G)$ of $\mathfrak{L}(G)$ inductively by

$$\Gamma_1(S, G) = S \quad \text{and} \quad \Gamma_n(S, G) = \{\Gamma_{n-1}(S, G), G\}, \quad n \geq 2.$$

We shall write $\Gamma_n(G)$ instead of $\Gamma_n(G, G)$.

Lemma 3.4. *Let $\tau : G \to H$ be a group epimorphism. Then there is an exact sequence*

$$0 \longrightarrow \prod_{n\geq 2} \Gamma_n(S, G) \longrightarrow \{\mathfrak{L}(G), \mathfrak{L}(G)\} \longrightarrow \{\mathfrak{L}(H), \mathfrak{L}(H)\} \longrightarrow 0,$$

where $S = \ker \tau$.

Proof. By [2, Lemma 15] there is a short exact sequence

$$0 \longrightarrow \prod_{n\geq 1} \Gamma_n(S, G) \longrightarrow \mathfrak{L}(G) \xrightarrow{\mathfrak{L}(\tau)} \mathfrak{L}(H) \longrightarrow 0.$$

Obviously the subsequence

$$0 \longrightarrow \left(\prod_{n\geq 1} \Gamma_n(S, G)\right) \cap \{\mathfrak{L}(G), \mathfrak{L}(G)\} \longrightarrow \{\mathfrak{L}(G), \mathfrak{L}(G)\} \xrightarrow{\mathfrak{L}(\tau)} \{\mathfrak{L}(H), \mathfrak{L}(H)\} \longrightarrow 0$$

is exact.

Since $(\prod_{n\geq 2} \Gamma_n(S, G)) \subset \{\mathfrak{L}(G), \mathfrak{L}(G)\}$ and $\Gamma_1(S, G) \cap \{\mathfrak{L}(G), \mathfrak{L}(G)\} = 1$, we obtain

$$\left(\prod_{n\geq 1} \Gamma_n(S, G)\right) \cap \{\mathfrak{L}(G), \mathfrak{L}(G)\} = \prod_{n\geq 2} \Gamma_n(S, G). \quad □$$

Proposition 3.5. *Let G denote a group, $\mathfrak{h}$ a multiplicative Lie ring and $\tau : G \longrightarrow \mathfrak{h}$ an epimorphism of groups. Then there is a natural isomorphism*

$$\{\mathfrak{L}(G), \mathfrak{L}(G)\}/\{\mathfrak{L}(G), \mathfrak{q}\} \cong \{\mathfrak{L}(\mathfrak{h}), \mathfrak{L}(\mathfrak{h})\}/\{\mathfrak{L}(\mathfrak{h}), \mathfrak{r}\},$$

where $\mathfrak{q} = \ker(\mathfrak{L}(G) \xrightarrow{\alpha} \mathfrak{h})$, $\mathfrak{r} = \ker(\mathfrak{L}(\mathfrak{h}) \xrightarrow{\beta} \mathfrak{h})$, *and α and β are induced respectively by τ and $1_{\mathfrak{h}}$.*

Proof. Clearly $\mathfrak{L}(\tau)$ induces an epimorphism $\{\mathfrak{L}(G), \mathfrak{q}\} \longrightarrow \{\mathfrak{L}(\mathfrak{h}), \mathfrak{r}\}$. Moreover $\Gamma_n(\ker\tau, G) \subseteq \{\mathfrak{L}(G), \mathfrak{q}\}$ for $n \geq 2$. Therefore by 3.4 we get the short exact sequence

$$0 \longrightarrow \prod_{n\geq 2} \Gamma_n(S, G) \longrightarrow \{\mathfrak{L}(G), \mathfrak{q}\} \xrightarrow{\mathfrak{L}(\tau)} \{\mathfrak{L}(\mathfrak{h}), \mathfrak{r}\} \longrightarrow 0.$$

The isomorphism in the proposition follows immediately from 3.4 and the sequence above. □

Theorem 3.6. *The second strong and weak homology groups of a multiplicative Lie ring are isomorphic.*

Proof. In 3.5, let G denote the free group on the elements of $\mathfrak{h}$ and $\tau : G \longrightarrow \mathfrak{h}$ the canonical group homomorphism. Then there is a diagram of multiplicative Lie rings

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{\mathfrak{q}\cap\{\mathfrak{L}(G), \mathfrak{L}(G)\}}{\{\mathfrak{L}(G), \mathfrak{q}\}} & \longrightarrow & \dfrac{\{\mathfrak{L}(G), \mathfrak{L}(G)\}}{\{\mathfrak{L}(G), \mathfrak{q}\}} & \longrightarrow & \{\mathfrak{h}, \mathfrak{h}\} & \longrightarrow & 0 \\
 & & \mathfrak{L}(\tau)\Big\downarrow & & \Big\downarrow\mathfrak{L}(\tau) & & \Big\| & & \\
0 & \longrightarrow & \dfrac{\mathfrak{r}\cap\{\mathfrak{L}(\mathfrak{h}), \mathfrak{L}(\mathfrak{h})\}}{\{\mathfrak{L}(\mathfrak{h}), \mathfrak{r}\}} & \longrightarrow & \dfrac{\{\mathfrak{L}(\mathfrak{h}), \mathfrak{L}(\mathfrak{h})\}}{\{\mathfrak{L}(\mathfrak{h}), \mathfrak{r}\}} & \longrightarrow & \{\mathfrak{h}, \mathfrak{h}\} & \longrightarrow & 0
\end{array}$$

with exact rows. Hence 3.3.1, 3.5 and the "five lemma" complete the proof. □

Remark. As we mentioned above, if G is an abelian group then the multiplicative Lie ring $\mathfrak{L}(G)$ is the ordinary free Lie ring over G. Therefore the second Chevalley–Eilenberg homology group and the second Eilenberg–MacLane homology group of a Lie ring are isomorphic.

Lemma 3.7. *Let $\mathfrak{g}$ denote a multiplicative Lie ring. Then there is an epimorphism $HS_3^{mlr}(\mathfrak{g}) \twoheadrightarrow HW_3^{mlr}(\mathfrak{g})$.*

Proof. Let X_{**} denote the bisimplicial multiplicative Lie ring defined by applying a standard $(\mathfrak{F}\mathcal{U}, \tau, \delta)$-cotriple resolution to a $\mathcal{Q}$-projective simplicial resolution $\mathfrak{g}_* \to \mathfrak{g}$ of $\mathfrak{g}$. Now applying the functor $\mathfrak{Ab}$ dimension-wise to X_{**}, we obtain the bisimplicial group $\mathfrak{Ab}(X_{**})$. By [11] and 3.1.2(i), there is a spectral sequence

$$E^2_{pq} = \mathcal{L}^{\mathcal{Q}}_p HS^{mlr}_{q+1}(\mathfrak{g}) \Rightarrow HS^{mlr}_{p+q+1}(\mathfrak{g}).$$

It is clear that $E^2_{p0} = HW^{mlr}_{p+1}(\mathfrak{g})$, $p \geq 0$. By 3.6 and the fact that $HW^{mlr}_i(\mathfrak{h}) = 0$ for all $\mathfrak{h} \in \mathcal{Q}$ and $i \geq 2$, we get $E^2_{p1} = 0$, $p \geq 0$. Moreover since the spectral sequence is in the first quadrant, $E^\infty_{20} = E^2_{20} = HW^{mlr}_3(\mathfrak{g})$ and hence there is an epimorphism $HS^{mlr}_3(\mathfrak{g}) \twoheadrightarrow HW^{mlr}_3(\mathfrak{g})$. □

Remark. In general $HS^{mlr}_3(\mathfrak{g})$ is not isomorphic to $HW^{mlr}_3(\mathfrak{g})$. For example, if $\mathfrak{g} = \mathfrak{L}(G)$, where G is a group with $H_3(G) \neq 0$, then $HW^{mlr}_3(\mathfrak{g}) = 0$ and $HS^{mlr}_3(\mathfrak{g})$ maps onto $H_3(G)$.

Next we investigate the multiplicative Lie ring homology groups of multiplicative Lie rings $G_{[,]}$ (see 2.2(a)) where G is a group.

3.8. Non-abelian exterior products

Given a group G, the non-abelian exterior product $G \wedge G$ of Brown–Loday [1] is the group generated by all symbols $g \wedge h$, $(g, h \in G)$, subject to the following relations

$$gg' \wedge h = ({}^g g' \wedge {}^g h)(g \wedge h),$$
$$g \wedge hh' = (g \wedge h)({}^h g \wedge {}^h h'),$$
$$g \wedge g = 1$$

for all $g, g', h, h' \in G$. We give below another interpretation of this group.

Proposition 3.8.1. *The map*

$$G \wedge G \longrightarrow \Gamma_2(G), \quad g_1 \wedge g_2 \mapsto \{g_1, g_2\},$$

is an isomorphism of groups.

Proof. It is easy to check that the map is well defined and a homomorphism of groups. Now we establish that the map $\Gamma_2(G) \to G \wedge G$, $\{g_1, g_2\} \mapsto g_1 \wedge g_2$, is well defined and a homomorphism of groups. This will complete the proof, because the homomorphisms $G \wedge G \leftrightarrows \Gamma_2(G)$ are obviously mutually inverse.

Let

$$0 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 0$$

denote a free presentation of G in the category of groups. By 3.4,

$$\prod_{n\geq 2} \Gamma_n(G) \cong \left(\prod_{n\geq 2} \Gamma_n(F)\right) \Big/ \left(\prod_{n\geq 2} \Gamma_n(R, F)\right). \tag{3.8.2}$$

There is a natural homomorphism $\theta : \prod_{n\geq 2} \Gamma_n(F) \to [F, F]$ induced by $\mathfrak{L}(F) \to F_{[,]}$. Since $\theta(\Gamma_n(R, F)) \subseteq [R, F]$ for $n \geq 2$, it follows from (3.8.2) that there is a natural homomorphism $\theta : \prod_{n\geq 2} \Gamma_n(G) \to [F, F]/[R, F]$. This defines a natural homomorphism

$$\theta : \Gamma_2(G) \longrightarrow [F, F]/[R, F]. \tag{3.8.3}$$

Moreover, it is well known by [8] that $F \wedge F \cong [F, F]$ under the map $f_1 \wedge f_2 \mapsto [f_1, f_2]$. Therefore

$$G \wedge G \cong (F \wedge F)/X \cong [F, F]/[R, F], \tag{3.8.4}$$

where X is the normal subgroup of $F \wedge F$ generated by all elements $r \wedge f$ such that $r \in R$ and $f \in F$. The composition of the homomorphisms in (3.8.3) and (3.8.4) is the required homomorphism $\Gamma_2(G) \to G \wedge G$. □

Remark. 3.8.1 implies that if G is a cyclic group then $HW_n^{mlr}(G_{[,]}) = 0$ for $n \geq 2$, because

$$(\Gamma_2(G) = G \wedge G = 1) \Rightarrow (\Gamma_n(G) = 1 \text{ for } n \geq 2) \Rightarrow \left(G = \mathfrak{L}^n(G) \text{ for } n \geq 1\right).$$

Proposition 3.9. *Let G denote a group. Then there is an isomorphism*

$$HS_2^{mlr}(G_{[,]}) \cong H_2(G)$$

and an epimorphism

$$HS_3^{mlr}(G_{[,]}) \twoheadrightarrow H_3(G).$$

Proof. Let $F_* \to G$ denote a free simplicial resolution of G. Let $\mathfrak{f}_{**}$ denote the bisimplicial multiplicative Lie ring defined by applying a standard $(\mathfrak{F}\mathcal{U}, \tau, \delta)$-cotriple resolution to a simplicial Lie ring $F_{*[,]}$. Applying the functor $\mathfrak{Ab}$ dimension-wise to $\mathfrak{f}_{**}$, we obtain the bisimplicial group $\mathfrak{Ab}(\mathfrak{f}_{**})$. By [11] there is a spectral sequence

$$E^1_{pq} = HS^{mlr}_{q+1}(F_{p[,]}) \Rightarrow HS^{mlr}_{p+q+1}(G_{[,]}).$$

It is clear that $E^2_{p0} = H_{p+1}(G)$, $p \geq 0$.

Thus in order to prove assertion, it suffices to show that $HW_2^{mlr}(F_{[,]}) = 0$ for any free group F. Let $\theta : \mathfrak{L}(F) \to F_{[,]}$ denote the canonical homomorphism induced by 1_F. We must show that

$$\ker\theta \cap \{\mathfrak{L}(F), \mathfrak{L}(F)\} = \{\mathfrak{L}(F), \ker\theta\}. \tag{3.9.1}$$

For any $x, y \in \mathfrak{L}(F)$,

$$\{x, y\} = \{x, \theta(y)\theta(y)^{-1}y\} = \{x, \theta(y)\}^{\theta(y)}\{x, \theta(y)^{-1}y\}.$$

Therefore

$$\{\mathfrak{L}(F), \mathfrak{L}(F)\} = \Gamma_2(F)\{\mathfrak{L}(F), \ker\theta\}.$$

Since the group F is free, the restriction of θ to $\Gamma_2(F)$ is injective, which proves the first assertion of the proposition. Hence by 3.6, $E^2_{p1} = 0$, $p \geq 0$ which clearly completes the proof. □

Remark. In general $HS^{mlr}_3(G_{[,]})$ is not isomorphic to $H_3(G)$. For example, if G is a free group, then it is not difficult to show that there is an epimorphism $HS^{mlr}_3(G_{[,]}) \twoheadrightarrow G^{ab} \wedge G^{ab} \wedge G^{ab}$.

4. Central extensions of multiplicative Lie rings

In this section we introduce the notion of central extension of a multiplicative Lie ring and prove for multiplicative Lie rings, the analogs of classical results for such extensions in the categories of groups, Lie algebras and Leibniz algebras [7].

Definition 4.1. A *central extension* of multiplicative Lie ring $\mathfrak{g}$ is an exact sequence

$$1 \longrightarrow \mathfrak{c} \longrightarrow \mathfrak{h} \xrightarrow{\phi} \mathfrak{g} \longrightarrow 1 \qquad (\mathfrak{h})$$

of multiplicative Lie rings such that $\{\mathfrak{c}, \mathfrak{h}\} = 1$, i.e. $\mathfrak{c} \subseteq \mathrm{ZL}(\mathfrak{h})$. A central extension $(\mathfrak{h})$ *splits* if it admits a section, that is, a multiplicative Lie ring homomorphism $s : \mathfrak{g} \to \mathfrak{h}$ such that $\phi s = 1_{\mathfrak{g}}$. A central extension $(\mathfrak{h})$ is called *universal* if, for every central extension $(\mathfrak{h}')$ of $\mathfrak{g}$ there exists one and only one homomorphism $f : \mathfrak{h} \to \mathfrak{h}'$ of multiplicative Lie rings satisfying $\phi = \phi' f$.

Clearly, if a universal central extension exists then it is unique up to isomorphism.

Proposition 4.2. (i) *A central extension* $(\mathfrak{h})$ *of* $\mathfrak{g}$ *is universal if and only if* $\mathfrak{h}$ *is perfect and every central extension of* $\mathfrak{h}$ *splits.*

(ii) *A multiplicative Lie ring* $\mathfrak{g}$ *admits a universal central extension if and only if* $\mathfrak{g}$ *is perfect. Furthermore, the kernel of a universal central extension is canonically isomorphic to the second homology group* $HS^{mlr}_2(\mathfrak{g})$.

The proof of the proposition uses the following lemma.

Lemma 4.2.1. *Let* $(\mathfrak{h})$ *denote a central extension of a multiplicative Lie ring* $\mathfrak{g}$.

(a) *If* $(\mathfrak{h}')$ *is another central extension of* $\mathfrak{g}$ *and* $\mathfrak{h}$ *is perfect then there exists at most one homomorphism from* $\mathfrak{h}$ *to* $\mathfrak{h}'$ *over* $\mathfrak{g}$.
(b) *If* $\mathfrak{h}$ *is not perfect then for a suitable chosen* $(\mathfrak{h}')$ *there exists more than one homomorphism from* $\mathfrak{h}$ *to* $\mathfrak{h}'$ *over* $\mathfrak{g}$.
(c) *If* $\mathfrak{g}$ *is perfect then the Lie commutator* $\{\mathfrak{h}, \mathfrak{h}\}$ *is perfect and maps onto* $\mathfrak{g}$.

Proof. (a) Let $f_1, f_2 : \mathfrak{h} \to \mathfrak{h}'$ be homomorphisms of multiplicative Lie rings over $\mathfrak{g}$. Then for any $u, v \in \mathfrak{h}$

$$f_1(u) = f_2(u)c, \quad f_1(v) = f_2(v)d,$$

for some $c, d \in \ker(\phi') \subseteq \mathrm{ZL}(\mathfrak{h}')$. By (2.1.2) and (2.1.3) and the centrality of c and d in the multiplicative Lie ring $\mathfrak{h}'$, we have

$$f_1\{u, v\} = \{f_1(u), f_1(v)\} = \{f_2(u)c, f_2(v)d\} = \{f_2(u), f_2(v)\} = f_2\{u, v\}.$$

Since $\mathfrak{h}$ is perfect, it follows that $f_1 = f_2$.

(b) Since $\mathfrak{h}$ is not perfect, there exists a non-zero multiplicative Lie homomorphism $f : \mathfrak{h} \to \mathfrak{a}$ where $\mathfrak{a}$ is some abelian multiplicative Lie ring. Let $(\mathfrak{h}')$ denote the split multiplicative Lie ring extension such that $\mathfrak{h}' = \mathfrak{g} \times \mathfrak{a}$ and $\phi' : \mathfrak{g} \times \mathfrak{a} \to \mathfrak{g}$, $(x, a) \mapsto x$. Let

$$f_1, f_2 : \mathfrak{h} \longrightarrow \mathfrak{g} \times \mathfrak{a}$$

denote the homomorphisms defined by $f_1(u) = (\phi(u), 1)$ and $f_2(u) = (\phi(u), f(u))$. Clearly, they are over $\mathfrak{g}$ and are distinct.

(c) Since $\mathfrak{g}$ is perfect, every element u of $\mathfrak{h}$ can be expressed as a product $u'c$ with $u' \in \{\mathfrak{h}, \mathfrak{h}\}$ and $c \in \mathrm{ZL}(\mathfrak{h})$. By (2.1.2) and (2.1.3),

$$\{u_1, u_2\} = \{u'_1c_1, u'_2c_2\} = \{u'_1c_1, u'_2\}^{u'_2}\{u'_1c_1, c_2\} = {}^{u'_1}\{c_1, u'_2\}\{u'_1, u'_2\} = \{u'_1, u'_2\}. \quad \square$$

Proof of 4.2. (i) Suppose $\mathfrak{h}$ is perfect and every central extension of $\mathfrak{h}$ splits. Given a central extension $(\mathfrak{h}')$ of $\mathfrak{g}$, form the pullback

$$(pull)\qquad \begin{array}{ccc} \mathfrak{h} \underset{\mathfrak{g}}{\times} \mathfrak{h}' & \xrightarrow{\nu} & \mathfrak{h}' \\ {\scriptstyle\mu}\downarrow & & \downarrow {\scriptstyle\phi'} \\ \mathfrak{h} & \xrightarrow{\phi} & \mathfrak{g} \end{array},$$

in the category $\mathcal{LM}$. By definition, $\mathfrak{h} \underset{\mathfrak{g}}{\times} \mathfrak{h}'$ is the subring of the multiplicative Lie ring $\mathfrak{h} \times \mathfrak{h}'$ consisting of all (u, u') such that $\phi(u) = \phi'(u')$, and $\mu(u, u') = u$ and $\nu(u, u') = u'$. It is clear that $(\mathfrak{h} \underset{\mathfrak{g}}{\times} \mathfrak{h}', \mu)$ is a central extension of $\mathfrak{h}$, and hence possesses by assumption a section $s : \mathfrak{h} \to \mathfrak{h} \underset{\mathfrak{g}}{\times} \mathfrak{h}'$. The multiplicative Lie ring homomorphism $\nu s : \mathfrak{h} \to \mathfrak{h}'$ is clearly over $\mathfrak{g}$ and by 4.2.1(a) is unique.

Conversely, let $(\mathfrak{h})$ be a universal central extension of $\mathfrak{g}$. It follows from 4.2.1(b) that $\mathfrak{h}$ is perfect. It remains to show that every central extension $(\mathfrak{f}, \psi)$ of $\mathfrak{h}$ splits. We show first that the composition

$$\mathfrak{f} \xrightarrow{\psi} \mathfrak{h} \xrightarrow{\phi} \mathfrak{g}$$

is a central extension of $\mathfrak{g}$. Let $x \in \mathfrak{f}$ such that $\phi\psi(x) = 1$. Then $\{x, u\} \in \mathrm{ZL}(\mathfrak{f})$ for all $u \in \mathfrak{f}$. Moreover by 4.2.1(c), $\mathfrak{f} = \{\mathfrak{f}, \mathfrak{f}\}\ker(\psi)$. Hence to complete the proof, it suffices to show that $\{\{y, z\}, x\} = 1$ for all $y, z \in \mathfrak{f}$. By (2.1.4) and (2.1.5)

$$\begin{aligned}\{\{y, z\}, x\} &= \{\{{}^{z^{-1}}x, y\}, {}^{y}z\}^{-1}\{\{z, {}^{z^{-1}}x\}, {}^{z^{-1}xz}y\}^{-1} = \{{}^{z^{-1}}\{x, {}^{z}y\}, {}^{y}z\}^{-1}\{{}^{z^{-1}}\{z, x\}, {}^{z^{-1}xz}y\}^{-1} \\ &= {}^{z^{-1}}(\{\{x, {}^{z}y\}, {}^{zy}z\}^{-1}\{\{z, x\}, {}^{xz}y\}^{-1}) = 1.\end{aligned}$$

(ii) Suppose $\mathfrak{g}$ admits a universal central extension $(\mathfrak{h})$. Then by (i), $\mathfrak{h}$ and consequently its homomorphic image $\mathfrak{g}$ are perfect.

Conversely, suppose $\mathfrak{g}$ is perfect. Choose a surjective homomorphism $\upsilon : F \to \mathfrak{g}$ where F is a free multiplicative Lie ring over some set. Let $R = \ker \upsilon$. Then there is a natural surjective homomorphism of multiplicative Lie rings

$$\tilde{\upsilon} : \frac{F}{\{F, R\}} \longrightarrow \frac{F}{R} \cong \mathfrak{g}.$$

Clearly $\ker \tilde{\upsilon} = \frac{R}{\{F,R\}}$ is included in the center of the multiplicative Lie ring $\frac{F}{\{F,R\}}$. Thus by 4.2.1(c), the canonical homomorphism

$$\psi : \frac{\{F, F\}}{\{F, R\}} \longrightarrow \mathfrak{g}\ (\cong F/R)$$

is a perfect central extension of $\mathfrak{g}$.

We claim that $\psi : \frac{\{F,F\}}{\{F,R\}} \to \mathfrak{g}$ is universal. Let $\phi : \mathfrak{h} \to \mathfrak{g}$ denote any central extension of $\mathfrak{g}$. Since F is free, there exists a homomorphism $h : F \to \mathfrak{h}$ such that $\upsilon = \phi h$. Since $h(R) \subseteq \mathrm{ZL}(\mathfrak{h})$, it follows that $h\{F, R\} = 1$. The induced homomorphism $h : \frac{\{F,F\}}{\{F,R\}} \to \mathfrak{h}$ has the property that $\phi h = \psi$ and is unique by 4.2.1(a).

It is clear $\ker \psi = \frac{R \cap \{F,F\}}{\{F,R\}}$, and by 3.3.1, $\frac{R \cap \{F,F\}}{\{F,R\}} \cong HS_2^{mlr}(\mathfrak{g})$. □

4.3. Perfect groups

If G is a perfect group then the multiplicative Lie ring $G_{[,]}$ (see 2.2(a)) is perfect. There is a natural question: what is the universal central extension of $G_{[,]}$ in the category of multiplicative Lie rings?

Proposition 4.3.1. *If $U \xrightarrow{\Phi} G$ is a universal central extension of a perfect group G then $U_{[,]} \xrightarrow{\Phi} G_{[,]}$ is a universal central extension of multiplicative Lie rings.*

First we need the following lemma.

Lemma 4.3.2. *Let G be a perfect group and $R = \ker(\mathfrak{L}(G) \to G_{[,]})$. Then $\{R, \mathfrak{L}(G)\} = 1$.*

Proof. Since G is perfect, it follows from (2.1.9) that $\Gamma_n(G) \subset \Gamma_2(G)$, $n \geq 2$. Therefore, R is a normal subgroup of $\mathfrak{L}(G)$ generated by all $[g_1, g_2]\{g_1, g_2\}^{-1}$ such that $g_1, g_2 \in G$. Hence, it suffices to prove that

$$\{[g_1, g_2]\{g_1, g_2\}^{-1}, x\} = 1, \quad \text{and} \quad \{[g_1, g_2]\{g_1, g_2\}^{-1}, \{x, y\}\} = 1,$$

for all $g_1, g_2 \in G$ and $x, y \in \mathfrak{L}(G)$. Since G is perfect, it suffices by (2.1.2) and (2.1.3) to consider the case when $x = [x_1, x_2]$ and $y = [y_1, y_2]$ where $x_1, x_2, y_1, y_2 \in G$. We compute

$$\begin{aligned}\{[g_1, g_2]\{g_1, g_2\}^{-1}, x\} &= \{[g_1, g_2]\{g_1, g_2\}^{-1}, [x_1, x_2]\} = (\text{by } (2.1.9)) \\ &= [[g_1, g_2]\{g_1, g_2\}^{-1}, \{x_1, x_2\}] = \left({}^{[g_1,g_2]\{g_1,g_2\}^{-1}}\{x_1, x_2\}\right)\{x_1, x_2\}^{-1} \\ &= (\text{by } (2.1.8)) = 1,\end{aligned}$$

and

$$\begin{aligned}\{[g_1, g_2]\{g_1, g_2\}^{-1}, \{x, y\}\} &= \{[g_1, g_2]\{g_1, g_2\}^{-1}, \{x, [y_1, y_2]\}\} = (\text{by } (2.1.9)) \\ &= [[g_1, g_2]\{g_1, g_2\}^{-1}, \{x, \{y_1, y_2\}\}] \\ &= \left({}^{[g_1,g_2]\{g_1,g_2\}^{-1}}\{x, \{y_1, y_2\}\}\right)\{x, \{y_1, y_2\}\}^{-1} = (\text{by } (2.1.8)) = 1. \quad \square\end{aligned}$$

Proof of 4.3.1. It is well known by [1] that $U \cong G \wedge G$, where $\wedge$ is the non-abelian exterior product. On the other hand, according to the proof of 4.2(ii), there is the universal central extension

$$\frac{\{\mathfrak{L}(F(G)), \mathfrak{L}(F(G))\}}{\{Q, \mathfrak{L}(F(G))\}} \longrightarrow G_{[,]}$$

of $G_{[,]}$ where $F(G)$ is a free group over G and Q is the kernel of the natural homomorphism $\mathfrak{L}(F(G)) \to G_{[,]}$. Since G is perfect, by 3.5 and 4.3.2 we have

$$\frac{\{\mathfrak{L}(F(G)), \mathfrak{L}(F(G))\}}{\{Q, \mathfrak{L}(F(G))\}} \longrightarrow \{\mathfrak{L}(G), \mathfrak{L}(G)\}.$$

Identify G with its canonical image in $\mathfrak{L}(G)$. Since the extension $\mathfrak{L}(G) \to G_{[,]}$ is central (by 4.3.2) and maps G onto $G_{[,]}$, it follows that $\{\mathfrak{L}(G), \mathfrak{L}(G)\} = \{G, G\}$. By definition, $\Gamma_2(G) = \{G, G\}$ and by 3.8.1, $\Gamma_2(G) \cong G \wedge G \cong$ (by [1]) U.

To complete the proof, we must show that $\{x, y\} = [x, y]$ for all $x, y \in \{G, G\}$. It suffices by (2.1.2) and (2.1.3) to consider the case when $x = \{g_1, g_2\}$ and $y = \{g_3, g_4\}$ where $g_1, g_2, g_3, g_4 \in G$. In fact, by (2.1.3), (2.1.9) and 4.3.2

$$\begin{aligned}\{\{g_1, g_2\}, \{g_3, g_4\}\} &= \{[g_1, g_2][g_1, g_2]^{-1}\{g_1, g_2\}, \{g_3, g_4\}\} \\ &= {}^{[g_1,g_2]}\{[g_1, g_2]^{-1}\{g_1, g_2\}, \{g_3, g_4\}\}\{[g_1, g_2], \{g_3, g_4\}\} \\ &= \{[g_1, g_2], \{g_3, g_4\}\} = [\{g_1, g_2\}, \{g_3, g_4\}]. \quad \square\end{aligned}$$

Remark. A similar result is true for Lie rings, i.e. if $U \xrightarrow{\Phi} G$ is a universal central extension in the category of Lie rings then it is also a universal central extension in the category of multiplicative Lie rings.

5. Application to K-theory and cyclic homology

In this section we introduce the Steinberg multiplicative Lie ring, $St^{mlr}(R)$, of a unital ring R and describe it as a universal central extension of the multiplicative Lie ring $E(R) \times sl(R)$ where $sl(R) = \varinjlim sl_n(R)$ and $sl_n(R)$ denotes the Lie subring of $gl_n(R)$ generated by all strictly upper and lower triangular matrices, i.e. by all matrices whose diagonal coefficients are zero. Then we show that the kernel of this universal central extension is isomorphic to $K_2(R) \times HC_1(R)$ where $HC_1(R)$ denotes the first cyclic homology group of R.

5.1. Steinberg multiplicative Lie rings

Let R be a ring with unit.

Definition 5.1.1. For $n \geq 3$ the n-th Steinberg multiplicative Lie ring $St_n^{mlr}(R)$ is the multiplicative Lie ring defined by generators $x_{ij}(s)$, $s \in R$, $1 \leq i \neq j \leq n$, subject to the relations

$$x_{ij}(s)x_{ij}(t) = x_{ij}(s+t), \tag{5.1.2}$$

$$\{x_{ij}(s), x_{kl}(t)\} = \begin{cases} 1 & \text{if } i \neq l,\ j \neq k \\ x_{il}(st) & \text{if } i \neq l,\ j = k. \end{cases} \tag{5.1.3}$$

Define the Steinberg multiplicative Lie ring $St^{mlr}(R)$ by

$$St^{mlr}(R) = \varinjlim_n St_n^{mlr}(R).$$

By (5.1.3), every generator $x_{ij}(s)$ of $St_n^{mlr}(R)$ satisfies

$$x_{ij}(s) = \{x_{ik}(s), x_{kj}(1)\} \quad \text{for } k \neq i, j.$$

Thus $St_n^{mlr}(R)$ and hence $St^{mlr}(R)$ are perfect multiplicative Lie rings.

We deduce next the following identity in $St_n^{mlr}(R)$, $n \geq 5$:

$$[x_{ij}(s), x_{kl}(t)] = 1 \quad \text{for all } i \neq l,\ j \neq k. \tag{5.1.4}$$

Choose an index h distinct from i, j, k, l. Then using (2.1.3), (2.1.9) and (5.1.3), we compute

$$\begin{aligned}[x_{ij}(s), x_{kl}(t)] &= [\{x_{ih}(s), x_{hj}(1)\}, x_{kl}(t)] = \{[x_{ih}(s), x_{hj}(1)], x_{kl}(t)\} \\ &= {}^{x_{ih}(s)x_{hj}(1)x_{ih}(-s)}\{x_{hj}(-1), x_{kl}(t)\}\,{}^{x_{ih}(s)x_{hj}(1)}\{x_{ih}(-s), x_{kl}(t)\} \\ &\quad {}^{x_{ih}(s)}\{x_{hj}(1), x_{kl}(t)\}\{x_{ih}(s), x_{kl}(t)\} = 1.\end{aligned}$$

Proposition 5.2. *For $n \geq 5$, every central extension of the multiplicative Lie ring $St_n^{mlr}(R)$ splits.*

Proof. Consider any central extension

$$1 \longrightarrow \mathfrak{c} \longrightarrow \mathfrak{h} \xrightarrow{\ \phi\ } St_n^{mlr}(R) \longrightarrow 1.$$

Let $x, x' \in St_n^{mlr}(R)$. Then the set $\{\phi^{-1}(x), \phi^{-1}(x')\} = \{\{y, y'\} | y \in \phi^{-1}(x), y' \in \phi^{-1}(x')\}$ has precisely one element. In fact, by (2.1.2) and (2.1.3) we have

$$\{yc, y'c'\} = \{yc, y'\}\,{}^{y'}\{yc, c'\} = {}^{y}\{c, y'\}\{y, y'\} = \{y, y'\}$$

for all $c, c' \in \mathfrak{c}$. By 4.2.1(c), we can suppose, without loss of generality, that $\mathfrak{h}$ is perfect as a multiplicative Lie ring. Then the set $[\phi^{-1}(x), \phi^{-1}(x')] = \{[y, y'] | y \in \phi^{-1}(x), y' \in \phi^{-1}(x')\}$ has also precisely one element. In fact, by 2.4(a) we have $\mathrm{ZL}(\mathfrak{g}) \subseteq Z(\mathfrak{g})$ and thus

$$[yc, y'c'] = [yc, y']\,{}^{y'}[yc, c'] = {}^{y}[c, y'][y, y'] = [y, y'].$$

Construct a section $\psi : St_n^{mlr}(R) \to \mathfrak{h}$ as follows: for each generator $x_{ij}(r)$ of $St_n^{mlr}(R)$, let

$$\psi(x_{ij}(r)) = \{\phi^{-1}(x_{ik}(1)), \phi^{-1}(x_{kj}(r))\}$$

where $k \neq i, j$.

We should show that ψ does not depend on the choice of index k and commutes with the relations (5.1.2) and (5.1.3).

The following lemma will be needed.

Lemma 5.2.1. *For any $j \neq k$ and $i \neq l$ we have the following equalities:*

(a) $\{\phi^{-1}(x_{ij}(r)), \phi^{-1}(x_{kl}(s))\} = 1$,
(b) $[\phi^{-1}(x_{ij}(r)), \phi^{-1}(x_{kl}(s))] = 1$.

Proof. Since $n \geq 5$, we can choose an index $h \neq i, j, k, l$. Let $y \in \phi^{-1}(x_{ih}(1))$, $y' \in \phi^{-1}(x_{hj}(r))$ and $y'' \in \phi^{-1}(x_{kl}(s))$. Clearly $\{y, y'\} \in \phi^{-1}(x_{ij}(r))$ and $\{y, y''\}, \{y', y''\} \in \mathfrak{c}$. Moreover, by (2.1.2), (2.1.5) and (2.1.10) we deduce easily from the above that $\{y'^{-1}, y''\}, \{y^{-1}, y''\}, \{y', {}^{y'^{-1}}y''\}, \{{}^{y'^{-1}}y'', y\} \in \mathfrak{c}$. Thus

$$\{\phi^{-1}(x_{ij}(r)), \phi^{-1}(x_{kl}(s))\} = \{\{y, y'\}, y''\} = \{\{y, y'\}, {}^{y'}({}^{y'^{-1}}y'')\} \overset{(2.1.4)}{=} \{\{{}^{y'^{-1}}y'', y\}, {}^{y}y'\}^{-1} \{\{y', {}^{y'^{-1}}y''\}, {}^{y'^{-1}y''y'}y\}^{-1} = 1$$

and

$$[\phi^{-1}(x_{ij}(r)), \phi^{-1}(x_{kl}(s))] = [\{y, y'\}, y''] \overset{(2.1.9)}{=} \{[y, y'], y''\} \overset{(2.1.3)}{=} {}^{yy'y^{-1}}\{y'^{-1}, y''\}\, {}^{yy'}\{y^{-1}, y''\}\, {}^{y}\{y', y''\}\{y, y''\} \overset{2.4(a)}{=} 1. \quad \square$$

We return to the proof of 5.2. Choose four distinct indices i, j, k, l and consider any elements $y \in \phi^{-1}(x_{ik}(1))$, $y' \in \phi^{-1}(x_{kl}(r))$, $y'' \in \phi^{-1}(x_{lj}(s))$.

It follows from 5.2.1 that $\{y, y''\} = 1$ and ${}^{y''}y = y$. Hence from (2.1.4) we have the equality

$$\{\{y, y'\}, {}^{y'}y''\} = \{y, \{y', y''\}\}.$$

Since $\{y, y'\} \in \phi^{-1}(x_{il}(r))$ and $\{\{y, y'\}, y''\} \in \phi^{-1}(x_{ij}(rs))$, it follows from (2.1.5) and 5.2.1(b) that

$$\{\{y, y'\}, {}^{y'}y''\} = {}^{y'}\{{}^{y'^{-1}}\{y, y'\}, y''\} = \{\{y, y'\}, y''\}.$$

Thus

$$\{\{y, y'\}, y''\} = \{y, \{y', y''\}\}$$

or in other words

$$\{\phi^{-1}(x_{il}(r)), \phi^{-1}(x_{lj}(s))\} = \{\phi^{-1}(x_{ik}(1)), \phi^{-1}(x_{kj}(rs))\}. \tag{5.2.2}$$

Taking $r = 1$, we obtain that the element

$$\{\phi^{-1}(x_{ik}(1)), \phi^{-1}(x_{kj}(r))\}$$

does not depend on the choice of k.

Moreover (5.2.2) implies that $\{\psi(x_{il}(r)), \psi(x_{lj}(s))\} = \psi(x_{ij}(rs))$. This equality and 5.2.1(a) show that ψ commutes with relation (5.1.3).

To show that ψ commutes with the relation (5.1.2), we apply the identity (2.1.11) to elements $x \in \phi^{-1}(x_{ik}(1))$, $y \in \phi^{-1}(x_{kj}(r))$ and $z \in \phi^{-1}(x_{kj}(s))$, i.e.

$$\{x, y\}\{x, z\} = \{x, yz\}[y, \{z, x\}].$$

By definition,

$$\{x, y\} = \psi(x_{ij}(r)), \quad \{x, z\} = \psi(x_{ij}(s)), \quad \{x, yz\} = \psi(x_{ij}(r + s)).$$

By 5.2.1(b)

$$[y, \{z, x\}] = [\phi^{-1}(x_{kj}(r)), \phi^{-1}(x_{ij}(-s))] = 1.$$

Thus

$$\psi(x_{ij}(r))\psi(x_{ij}(s)) = \psi(x_{ij}(r + s)). \quad \square$$

5.3. Steinberg groups and Steinberg Lie rings

Here we recall the classical definitions of stable and non-stable Steinberg groups and Lie algebras of a ring with unit.

Definition 5.3.1. Let R denote a ring with unit. For $n \geq 3$, the *n-th Steinberg group* $St_n(R)$ is the group defined by generators $x_{ij}(s)$, $s \in R$, $1 \leq i \neq j \leq n$, subject to the relations

$$x_{ij}(s)x_{ij}(t) = x_{ij}(s+t),$$
$$[x_{ij}(s), x_{kl}(t)] = \begin{cases} 1 & \text{if } i \neq l,\ j \neq k \\ x_{il}(st) & \text{if } i \neq l,\ j = k. \end{cases}$$

The *Steinberg group* $St(R)$ is defined by

$$St(R) = \varinjlim_n St_n(R).$$

Definition 5.3.2. Let R denote a ring with unit. For $n \geq 3$, the n-th *Steinberg Lie ring* $st_n(R)$ is the Lie ring defined by generators $x_{ij}(s)$, $s \in R$, $1 \leq i \neq j \leq n$, subject to the relations

$$x_{ij}(s) + x_{ij}(t) = x_{ij}(s+t),$$
$$[x_{ij}(s), x_{kl}(t)] = \begin{cases} 0 & \text{if } i \neq l,\ j \neq k \\ x_{il}(st) & \text{if } i \neq l,\ j = k. \end{cases}$$

The *Steinberg Lie ring* $st(R)$ is defined by

$$st(R) = \varinjlim_n st_n(R).$$

It is easy to see there are natural homomorphisms of multiplicative Lie rings

$$\begin{array}{c} \Theta : St^{mlr}(R) \longrightarrow St(R)_{[,]} \\ x_{ij}(t) \mapsto x_{ij}(t) \end{array} \quad \text{and} \quad \begin{array}{c} \Theta' : St^{mlr}(R) \longrightarrow st(R) \\ x_{ij}(t) \mapsto x_{ij}(t). \end{array}$$

Lemma 5.3.3. *For $n \geq 5$ the multiplicative Lie homomorphism $\Theta : St_n^{mlr}(R) \to St_n(R)_{[,]}$ admits a group theoretic section.*

Proof. Since any central extension of $St_n(R)$ in the category of groups splits (see [9,4]), if suffices to check that $\Theta : St_n^{mlr}(R) \to St_n(R)_{[,]}$ is a central extension in the category of groups. Since $St_n^{mlr}(R)$ is a perfect multiplicative Lie ring, it suffices to show that

$$[a, \{b_1, b_2\}] = 1 \quad \text{for all } a \in \ker \Theta, b_1, b_2 \in St_n^{mlr}(R). \tag{5.3.4}$$

Let N denote the normal subgroup of $St_n^{mlr}(R)$ generated by all elements $\{x, y\}[x, y]^{-1}$ such that $x, y \in St_n^{mlr}(R)$. N is an ideal of $St_n^{mlr}(R)$, since by (2.1.3), (2.1.9) and (2.1.10)

$$\begin{aligned} \{\{x, y\}[x, y]^{-1}, z\} &= {}^{\{x,y\}}\{[x, y]^{-1}, z\}\{\{x, y\}, z\} = {}^{\{x,y\}}[\{x, y\}^{-1}, z]\{\{x, y\}, z\} \\ &= [z, \{x, y\}]\{z, \{x, y\}\}^{-1} \end{aligned}$$

for all $x, y, z \in St_n^{mlr}(R)$. Clearly $N \subseteq \ker \Theta$. Moreover, it is straightforward to check that the assignment $x_{ij}(r) \mapsto |x_{ij}(r)| \in St_n^{mlr}(R)/N$ defines an inverse for the natural map $St_n^{mlr}(R)/N \to St_n(R)_{[,]}$. Thus $N = \ker \Theta$. Hence, in order to check (5.3.4), we can take $a = \{x, y\}[x, y]^{-1}$. By (2.1.8) we have

$$[\{x, y\}[y, x], \{b_1, b_2\}] = {}^{\{x,y\}[y,x]}\{b_1, b_2\}\{b_1, b_2\}^{-1} = 1. \quad \square$$

Proposition 5.3.5. *For $n \geq 5$ there is an isomorphism of multiplicative Lie rings $\Theta \times \Theta' : St_n^{mlr}(R) \to St_n(R)_{[,]} \times st_n(R)$.*

Proof. Since Θ maps $[St_n^{mlr}(R), St_n^{mlr}(R)](\subseteq \ker \Theta')$ onto $St_n(R)$, it is clear that $\Theta \times \Theta'$ is surjective. Thus we only have to check injectivity. $[St_n^{mlr}(R), St_n^{mlr}(R)]$ is an ideal (see Section 2) of $St_n^{mlr}(R)$ and since the assignment $x_{ij}(r) \mapsto |x_{ij}(r)| \in St_n^{mlr}(R)/[St_n^{mlr}(R), St_n^{mlr}(R)]$ defines an inverse to the natural homomorphism $St_n^{mlr}(R)/[St_n^{mlr}(R), St_n^{mlr}(R)] \to st_n(R)$ of multiplicative Lie rings, it follows that $\ker \Theta' = [St_n^{mlr}(R), St_n^{mlr}(R)]$. Moreover, by 5.3.3 there is a group theoretic homomorphism $s : St_n(R) \to St_n^{mlr}(R)$ such that $\Theta s = 1$. Therefore, we get the following group theoretic isomorphism

$$St_n^{mlr}(R) \cong \ker \Theta \rtimes St_n(R)$$

where $St_n(R)$ acts on $St_n^{mlr}(R)$ via s. Hence there is an isomorphism

$$[St_n^{mlr}(R), St_n^{mlr}(R)] \cong [\ker \Theta, St_n^{mlr}(R)] \rtimes [St_n(R), St_n(R)].$$

Thus

$$\ker \Theta \cap \ker \Theta' = \ker \Theta \cap [St_n^{mlr}(R), St_n^{mlr}(R)] = [\ker \Theta, St_n^{mlr}(R)] \overset{(5.3.4)}{=} 1. \quad \square$$

Let K_2^{ad} and K_3^{ad} denote the additive analogs [5] of algebraic K-functors. In particular, $K_2^{ad}(R) = HW_2^{mlr}(sl(R)) \cong HC_1(R)$, $K_3^{ad}(R) = HW_3^{mlr}(st(R))$. Now we come to the main result of the section.

Theorem 5.4. *The center of multiplicative Lie ring $St^{mlr}(R)$ is isomorphic to $K_2(R) \times HC_1(R)$ and for $n \geq 5$ there is an exact sequence*

$$1 \longrightarrow K_{2,n}(R) \times HC_1(R) \longrightarrow St_n^{mlr}(R) \longrightarrow E_n(R)_{[,]} \times sl_n(R) \longrightarrow 1.$$

Proof. By 5.3.5 we deduce an isomorphism

$$\mathrm{ZL}(St_n^{mlr}(R)) \cong Z(St_n(R)) \times \mathrm{ZL}(st_n(R)).$$

It is well known (we refer the reader to [9,4–6]) that

$$Z(St(R)) = K_2(R) \quad \text{and} \quad \mathrm{ZL}(st_n(R)) \cong HC_1(R).$$

This implies the first part of the theorem. Moreover by definition, $K_{2,n}(R)$ is the kernel of the natural homomorphism $St_n(R) \to E_n(R)$ and for $n \geq 5$, the sequence

$$0 \longrightarrow HC_1(R) \longrightarrow st_n(R) \longrightarrow sl_n(R) \longrightarrow 0$$

is exact. Hence 5.3.5 completes the proof. $\square$

Remark. For an arbitrary unital ring R, it is not known (and is probably false) that $K_{2,n}(R) \subseteq Z(St_n(R))$. But if R is commutative, or more generally module finite over its center, and $n \geq 4$ then $K_{2,n}(R) = Z(St_n(R))$ by [12].

Corollary 5.5. *There is an isomorphism*

$$HS_2^{mlr}(E(R)_{[,]} \times sl(R)) \cong K_2(R) \times HC_1(R).$$

Proof. The corollary follows directly from 4.2(ii), 5.2 and 5.4. $\square$

Remark. We do not know if there is an isomorphism between $HS_3^{mlr}(St^{mlr}(R))$ and $K_3(R) \times K_3^{ad}(R)$. But by 3.9, 3.7, 5.3.5 and [5] we have at least an epimorphism $HS_3^{mlr}(St^{mlr}(R)) \twoheadrightarrow K_3(R) \times K_3^{ad}(R)$.

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