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Closed model category structures on the category of chain functors

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Abstract

In the category $\mathfrak{C}h$ of chain functors one can introduce fibrations (Section 3), cofibrations and weak equivalences (Section 4), satisfying all the properties of a closed model category as defined by D. Quillen except for the existence of finite limits and colimits. Nevertheless we show that there exists a canonically defined suspension—as well as a loop functor, which are invertible, turning the homotopy category $\mathfrak{C}h_h$ into a stable category (Section 8).

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0. Introduction

A chain functor is a pair $C'_* \subset C_*$ of functors from a topological or simplicial category into the category of chain complexes, together with some additional data (see Section 9 or [1] for further references). They are used to calculate homology groups of a given homology theory $h_*(\ )$ by means of chains, cycles and boundaries (i.e., by means of chain complexes) as in the case of ordinary singular, simplicial or cellular homology. On the other hand each spectrum $\boldsymbol{E}$ gives rise to a homology theory $\boldsymbol{E}_*(\ )$ (the homology theory

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with coefficients in $\boldsymbol{E}$), which in turn determines functorially a chain functor $\Phi(\boldsymbol{E})$ whose associated homology theory is $\boldsymbol{E}_*(\,)$. So the category $\mathfrak{C}h$ of chain functors constitutes some model of a stable category in which one can perform stable homotopy theory. In particular the question comes up whether $\mathfrak{C}h$ can be equipped with the structure of a *closed model category* or a *triangulated category*.

Closed model categories were introduced by Quillen [8]. We follow the exposition given in [5]:

A category $\mathfrak{C}$ is a *closed model category* whenever there are three distinguished classes of mappings (1) weak equivalences, (2) fibrations and (3) cofibrations, such that the following five conditions are fulfilled:

CM1: Finite limits and colimits exist in $\mathfrak{C}$.

CM2: If $f, g \in \mathfrak{C}$ are morphisms such that gf is defined, and two of the three maps f, g or gf are weak equivalences, then so is the third.

CM3: If a map f is a retract of a map g, and if g is either a fibration, a cofibration or a weak equivalence, then so is f.

CM4: Given a commutative square

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle F} \\ A & \xrightarrow[q]{} & C \end{array} \tag{1}$$

with fibration p and cofibration q. Then (1) has a *lifting* (i.e., a diagonal $\overline{F}: C \to E$, rendering everything commutative) whenever either q is a trivial cofibration (i.e., a cofibration and a weak equivalence) or p is a trivial fibration (i.e., a fibration and a weak equivalence).

CM5: Every morphism $f \in \mathfrak{C}$ can be factored as $f = pq$ in two ways: (1) q is a cofibration and p is a trivial fibration; (2) q is a trivial cofibration and p a fibration.

In Sections 3 and 4 we introduce fibrations, cofibrations and weak equivalences in $\mathfrak{C}h$ and present for the first two concepts several equivalent definitions. So we have, for example, Kan- and Hurewicz-fibrations, which turn out to be equivalent. Weak equivalences are simply chain homotopy equivalences between chain functors. The closed model structures are developed and described in such a way that the close analogy with the topological case becomes obvious.

In Section 5 we verify CM4 for $\mathfrak{C}h$, while in Sections 6 and 7 we deal with CM5. It turns out that for $\mathfrak{C}h$ the condition CM1 is not fulfilled. Although $\mathfrak{C}h$ has finite sums and products, there are apparently in general no kernels and cokernels. All these results are summarized in Section 8, where, in addition, the suspension and the loop functor are introduced. The necessary kernels and cokernels for this purpose are available. Since suspensions and loop functors are invertible and, up to an isomorphism, inverses to each other, $\mathfrak{C}h_h$ becomes a *stable* category (i.e., it allows not only suspensions but also arbitrary desuspensions).

Moreover we briefly refer to *basic model structures* respectively *Thomason model categories* in the sense of Weibel [11].

In Section 1 we describe a canonically defined cylinder and a dual cylinder functor, $(\boldsymbol{K} \times I)_*$ and $\boldsymbol{K}_*^I$ for given $\boldsymbol{K}_* \in \mathfrak{C}h$. The inclusion $\boldsymbol{K}_* \xrightarrow{i_0} (\boldsymbol{K} \times I)_*$ serves as the standard example of a trivial cofibration, while the projection $\boldsymbol{K}_*^I \xrightarrow{p_0} \boldsymbol{K}_*$ is the standard example of a trivial fibration.

In Section 2 we develop some auxiliary concepts needed for the concept of a Kan fibration (Definition 3.1(2)). We need these different concepts and their equivalence for settling the problems in Sections 5–7.

Concerning details about chain functors (definitions and motivations) the reader is referred to [1]. For convenience there is a short introduction to chain functors given in Section 9.

1. The cylinder construction and its dual

To each chain functor $\boldsymbol{K}_*$ we associate a new chain functor $(\boldsymbol{K} \times I)_*$, two morphisms $i_j : \boldsymbol{K}_* \to (\boldsymbol{K} \times I)_*$, $j = 0, 1$, as well as a morphism $r : (\boldsymbol{K} \times I)_* \to \boldsymbol{K}_*$ such that $ri_0 = 1_{\boldsymbol{K}_*}$, $i_0 r \simeq 1_{(\boldsymbol{K} \times I)_*}$. In other words, $\boldsymbol{K}_*$ appears as a deformation retract of $(\boldsymbol{K} \times I)_*$.

We set

$$(K \times I)_n(X, A) = K_n(X, A) \oplus K_{n-1}(X, A) \oplus K_n(X, A), \quad (X, A) \in \mathfrak{K} \tag{1}$$

and

$$(K \times I)'_n(X, A) = K'_n(X, A) \oplus K'_{n-1}(X, A) \oplus K'_n(X, A), \quad (X, A) \in \mathfrak{K}. \tag{2}$$

The boundary operator is defined by

$$d(a, b, c) = (da + b, -db, dc - b). \tag{3}$$

Let $f \in \mathfrak{K}((X, A), (Y, B))$ be a mapping, then $f_\#$ is defined componentwise

$$f_\#(a, b, c) = \big(f_\#(a), f_\#(b), f_\#(c)\big).$$

These are the usual mapping cylinders (see [11]). This yields a functor $(K \times I)_* : \mathfrak{K} \to \boldsymbol{ch}$, with subfunctor l: $(K \times I)'_* \subset (K \times I)_*$ and natural inclusion

$$i' : (K \times I)_*(A) \to (K \times I)'_*(X, A).$$

We will henceforth write x_c, $\tilde{c}$, c instead of respectively $(0, c, 0)$, $(0, 0, c)$, $(c, 0, 0)$. The “geometric” picture we have in mind is that we add to the original elements of $K_n(X, A)$ new elements x_c, $\dim x_c = n + 1$, $\tilde{c}$, $\dim \tilde{c} = n$, associated with given $c \in K_n(X, A)$, where x_c is the “cylinder over c”, c, $(\tilde{c})$ the “bottom” (the “top”) of this cylinder over c.

Using this notation we obtain:

(1) dc, $c \in K_{n+1}(X, A)$, as defined in K_*,
(2) $dx_c + x_{dc} = c - \tilde{c}$,
(3) $d\tilde{c} = \widetilde{dc}$.

Chain mappings φ, κ are defined by:

(1) $\varphi \mid K_*$ as already given,
(2) $\varphi(x_c) = x_{\varphi c}$,
(3) $\varphi\tilde{c} = \widetilde{\varphi c}$.

Respectively for κ. In K_* there exist chain homotopies $\varphi\kappa \simeq 1$, i.e., to each $c \in K_*$ a $D(c)$ satisfying

$$dD(c) + D(dc) = \varphi\kappa c - c.$$

In order to detect a chain homotopy $\varphi\kappa(w) \simeq w$, $w \in (K \times I)_*$, i.e., a $u(w) \in (K \times I)_{*+1}$ satisfying

$$du(w) + u(dw) = \varphi\kappa(w) - w \tag{4}$$

we set:

(1) $u(c) = D(c)$, $c \in K_*$,
(2) $u(x_c) = x_{D(c)}$,
(3) $u(\tilde{c}) = \widetilde{D(c)}$.

This provides us with a chain homotopy (4). The chain homotopy $j_\#\varphi \simeq l$ is established similarly.

The verification of the remaining properties of a chain functor for $(\boldsymbol{K} \times I)_*$ is routine (apply Lemma 9.2). So follows, for example, the excision property from the simple observation, that $\boldsymbol{K}_*$ and $(\boldsymbol{K} \times I)_*$ have isomorphic homology.

By setting $i_0(c) = c$, $i_1(c) = \tilde{c}$, $c \in K_*$, we obtain morphism in $\mathfrak{Ch}$, $i_j : \boldsymbol{K}_* \to (\boldsymbol{K} \times I)_*$, $j = 0, 1$, compatible with φ and κ.

We define $r(c) = c$, $r(\tilde{c}) = c$, $r(x_c) = 0$, $c \in K_*$ obtaining a morphism $r : (\boldsymbol{K} \times I)_* \to \boldsymbol{K}_*$ (compatible with φ and κ), satisfying

$$ri_0 = 1_{\boldsymbol{K}_*}.$$

We have

$$i_0 r(c) = c, \qquad i_0 r(\tilde{c}) = c, \qquad i_0 r(x_c) = 0.$$

Therefore

$$D(c) = 0, \quad c \in K_*, \qquad D(\tilde{c}) = x_c, \qquad D(x_c) = 0$$

furnishes a chain homotopy

$$D\colon\ i_0 r \simeq 1_{(\boldsymbol{K} \times I)_*}.$$

Let $\lambda : \boldsymbol{K}_* \to \boldsymbol{L}_*$ be a morphism between chain functors, then there exists a

$$\lambda \times 1 : (\boldsymbol{K} \times I)_* \to (\boldsymbol{L} \times I)_*$$

commuting with i_j and r.

We summarize:

Proposition 1.1. *To each chain functor $\boldsymbol{K}_*$ there exists a canonically defined cylinder $(\boldsymbol{K} \times I)_*$ which contains $\boldsymbol{K}_*$ as a deformation retract.*

Let $f_0, f_1 : \boldsymbol{K}_* \to \boldsymbol{L}_*$ be two morphisms in $\mathfrak{C}h$, then $f_0 \simeq f_1$ if and only if there exists a $H : (\boldsymbol{K} \times I)_* \to \boldsymbol{L}_*$ such that $Hi_j = f_j$, $j = 0, 1$.

Proof. Only the last assertion needs proof. Suppose D: $f_0 \simeq f_1$ is a chain homotopy, then we define H by

$$H(c) = f_0(c), \qquad H(\tilde{c}) = f_1(c), \qquad H(x_c) = D(c).$$

Reading this proof backwards furnishes the other direction of the assertion. □

Remark. Our intention is to be with our terminology and our notations (concerning definitions, assertions, proofs) as close as possible to the topological case. So we prefer to write, e.g., $(A \times I) \cup B$, i: $A \times 0 = A \in B$ whenever we mean algebraically $(A \times I) \oplus_i B$. This is, what is meant by "gluing $(A \times 0)$ and B together at $A \times 0$".

Suppose q: $\boldsymbol{A}_* \subset \boldsymbol{B}_*$ is an inclusion in $\mathfrak{C}h$, then q respects the entire structure of a chain functor (including φ and κ). This allows us to define $\boldsymbol{B}_* \cup (\boldsymbol{A} \times I)_* = \boldsymbol{B}_* \cup_q (\boldsymbol{A} \times I)_*$ by simply repeating the construction of $(\boldsymbol{B} \times I)_*$ but now only adding new x_a, $\tilde{a}$, for $a \in \boldsymbol{A}_*$ and not for all $b \in \boldsymbol{B}_*$. Application of Lemma 9.2 yields:

Corollary 1.2. *$\boldsymbol{B}_* \cup (\boldsymbol{A} \times I)_*$ is a chain functor with inclusions $j : \boldsymbol{B}_* \cup (\boldsymbol{A} \times I)_* \xrightarrow{\subset} (\boldsymbol{B} \times I)_*$, $i : \boldsymbol{B}_* \xrightarrow{\subset} \boldsymbol{B}_* \cup (\boldsymbol{A} \times I)_*$ and retraction $r : \boldsymbol{B}_* \cup (\boldsymbol{A} \times I)_* \to \boldsymbol{B}_*$.*

There is a dual construction, associating with each chain functor $\boldsymbol{K}_*$ in a functorial way (as in 1.1) a chain functor $\boldsymbol{K}_*^I$ together with three morphisms of chain functors

$$\boldsymbol{K}_*^I \xrightarrow{p_j} \boldsymbol{K}_* \xrightarrow{s} \boldsymbol{K}_*^I, \quad j = 0, 1, \tag{5}$$

such that

$$p_0 s = 1, \qquad s p_0 \simeq 1. \tag{6}$$

Two morphisms $f_0, f_1 : \boldsymbol{A}_* \to \boldsymbol{K}_*$ are chain homotopic whenever there exists a morphism $H : \boldsymbol{A}_* \to \boldsymbol{K}_*^I$ satisfying $p_j H = f_j$.

We can easily establish a model of $\boldsymbol{K}_*^I$ by giving a straightforward construction as we did for $(\boldsymbol{K} \times I)_*$:

$$K_n^I(X, A) = K_n(X, A) \oplus K_n(X, A) \oplus K_{n+1}(X, A) \tag{7}$$

with boundary

$$d(c, c_1, x) = \bigl(dc, dc_1, dx + (-1)^{n+1}(c - \tilde{c})\bigr).$$

Setting $p_0(c, c_1, x) = c$, $p_1(c, c_1, x) = c_1$, $s(c) = (c, c, 0)$, we obtain $p_0 s = 1$, D: $s p_0 \simeq 1$, with chain homotopy

$$D(c, c_1, x) = (-1)^n (0, x, 0).$$

This model of $\boldsymbol{K}_*^I$ is functorial and can be equipped with the structure of a chain functor as we did this for $(\boldsymbol{K}\times I)_*$.

However we prefer to present also a conceptual existence proof for $\boldsymbol{K}_*^I$: To this end we employ (1) tensor products between chain functors, even if one partner is an *irregular* chain functor (a concept which was introduced by the first author, see [1,2] concerning details); and (2) an irregular chain functor $\boldsymbol{Z}_*$ (see [2, §3]) having the property that

$$\boldsymbol{K}_*\otimes\boldsymbol{Z}_*\approx\boldsymbol{K}_*,\quad \boldsymbol{K}_*\in\mathfrak{C}h.$$

We form $(\boldsymbol{Z}\times I)_*$ and confirm very easily:

Lemma 1.3. *There exists a* (*with respect to* $\boldsymbol{K}_*$) *natural isomorphism*

$$(\boldsymbol{K}\times I)_*\approx\boldsymbol{K}_*\otimes(\boldsymbol{Z}\times I)_*.$$

We need a very special case of an internal Hom functor in $\mathfrak{C}h$, satisfying

$$\operatorname{Hom}(\boldsymbol{Z}_*,\boldsymbol{K}_*)\approx\boldsymbol{K}_*,\qquad \operatorname{Hom}\big((\boldsymbol{Z}\times I)_*,\boldsymbol{K}_*\big)=\boldsymbol{K}_*^I$$

where the assignment $\boldsymbol{K}_*\mapsto\boldsymbol{K}_*^I$ is adjoint to the assignment $\boldsymbol{K}_*\mapsto(\boldsymbol{K}\times I)_*$. Concerning the definition of $\operatorname{Hom}(\cdot,\cdot)$ for chain complexes see [4, p. 18]. Like for tensor products we set

$$\begin{aligned}K_n^I(X,A) = {}&\operatorname{Hom}\big((\mathbb{Z}\times I)_0(X),K_n(X,A)\big)\\ &\oplus\operatorname{Hom}\big((\mathbb{Z}\times I)_1(X),K_{n+1}(X,A)\big),\end{aligned}\tag{8}$$

observing that induced mappings for a $f\in\mathfrak{K}((X,A),(Y,B))$ are well-defined, since $f_\#:(\mathbb{Z}\times I)_j(X)\to(\mathbb{Z}\times I)_j(Y)$ is always the identity $(j=0,1)$. As functors into $\boldsymbol{ch}$, (8) and (7) are isomorphic. We equip (8) with the structure of an (irregular) chain functor (in the same way as this was done for the tensor product in [2, §1, §3]) such that the result is equivalent to the regular chain functor $\boldsymbol{K}_*^I$ defined in (7).

Moreover one obtains to the mappings $i_j:\boldsymbol{Z}_*\to(\boldsymbol{Z}\times I)_*$ $j=0,1$, $r:(\boldsymbol{Z}\times I)_*\to\boldsymbol{Z}_*$ the corresponding $p_j=\operatorname{Hom}(i_j,\boldsymbol{K}_*):\operatorname{Hom}(\boldsymbol{Z}\times I)_*,\boldsymbol{K}_*)\to\operatorname{Hom}(\boldsymbol{Z}_*,\boldsymbol{K}_*)$, respectively $s=\operatorname{Hom}(r,\boldsymbol{K}_*):\operatorname{Hom}(\boldsymbol{Z}_*,\boldsymbol{K}_*)\to\operatorname{Hom}((\boldsymbol{Z}\times I)_*,\boldsymbol{K}_*)$, exhibiting the previously mentioned properties (6). So we can summarize:

Proposition 1.4. *To each* $\boldsymbol{K}_*$ *there exists in a natural way a chain functor* $\boldsymbol{K}_*^I$, *together with mappings* $p_j:\boldsymbol{K}_*^I\to\boldsymbol{K}_*$, $j=0,1$, $s:\boldsymbol{K}_*\to\boldsymbol{K}_*^I$ *such that* $p_0s=1$, $sp_0\simeq 1$.

Two mappings $f_0,f_1:\boldsymbol{A}_*\to\boldsymbol{K}_*$ *are chain homotopic whenever there exists* $H:\boldsymbol{A}_*\to\boldsymbol{K}_*^I$, *satisfying* $f_j=p_jH$.

Remarks.

(1) The existence of cylinders and dual cylinders is usually deduced from property CM5 of a closed model category (see [5]). In our case we employ the existence of *canonically* defined cylinder and dual cylinder functors for establishing the properties of a closed model category.

(2) Cylinder and dual cylinder functors for chain complexes are of course well known in the literature (see [4,11]). However since chain functors have a much more involved structure than mere functors into a category of chain complexes and because explicit constructions are needed, we felt it necessary to describe everything in detail, not only referring to the literature.

2. Horns and their fillings

In simplicial homotopy theory (see [7]) one deals with *horns* (or funnels, in German: Trichter) and their *fillings*: A horn is a sequence of n-simplexes σ_i^n, $i = 0, \ldots, \hat{k}, \ldots, n$, which behave like the collection of n faces (all with the exception of the kth) of an $n+1$ simplex σ^{n+1}. A filling of this horn consists of this $(n+1)$-simplex together with the remaining n-simplex σ_k^n, i.e., one has

$$\partial_i \sigma^{n+1} = \sigma_i^n, \quad i = 0, \ldots, \hat{k}, \ldots, n.$$

Since we would like to have the same concepts available for chain functors, we must try to imitate all this algebraically:

Definition 2.1.

(1) Let $\boldsymbol{K}_*$ be a chain functor, $\boldsymbol{c}$ a collection of elements in some $K_*(X, A)$, satisfying

$$c \in \boldsymbol{c} \cap K_*(X, A),\ f \in \mathfrak{K}\big((X, A), (Y, B)\big) \implies f_\#(c) \in \boldsymbol{c} \cap K_*(Y, B). \qquad (*)$$

Then $\boldsymbol{c}$ is called a *prehorn* in $\boldsymbol{K}_*$.
If $\dim c = n\ \forall c \in \boldsymbol{c}$, then we set $\dim \boldsymbol{c} = n$.

(2) A *horn* is a natural mapping $\lambda : \boldsymbol{e} \to \boldsymbol{K}_*$, $\boldsymbol{K}_* \in \mathfrak{C}h$, $\boldsymbol{e} \subset \boldsymbol{L}_* \in \mathfrak{C}h$ a prehorn, such that there exists a chain functor $\boldsymbol{M}_*$, $\boldsymbol{e} \subset \boldsymbol{M}_* \subset \boldsymbol{L}_*$ and a $\bar{\lambda} \in \mathfrak{C}h(\boldsymbol{M}_*, \boldsymbol{L}_*)$ with $\bar{\lambda} \mid \boldsymbol{e} = \lambda$.

Remarks and examples.

(1) The 0-prehorn $\mathbf{0}$ consists only of the zero element $0 \in K_n(X, A)$ for each $(X, A) \in \mathfrak{K}$.
(2) $K_n(\cdot)(\subset K_*(\cdot))$, fixed n, is a prehorn; $1 : \boldsymbol{K}_* \to \boldsymbol{K}_*$ is a horn.
(3) Let a natural basis $\boldsymbol{b}$ of $C_n(\cdot)$ (for all n, see Lemma 9.1) and $f \in \mathfrak{C}h(\boldsymbol{C}_*, \boldsymbol{K}_*)$, be given, then $f \mid \boldsymbol{b} : \boldsymbol{b} \to \boldsymbol{K}_*$ is a horn.
(4) Let $\boldsymbol{c} \subset \boldsymbol{b}$ denote all bounding cycles then $f \mid \boldsymbol{c}$ is a horn.
(5) Let $\lambda : \boldsymbol{e} \to \boldsymbol{K}_*$ be a horn, $f \in \mathfrak{C}h(\boldsymbol{K}_*, \boldsymbol{L}_*)$, then $f\lambda$ is a horn, in particular f itself is a horn.
(6) Let $\boldsymbol{e} \subset \boldsymbol{M}_*$ be a prehorn, then $\boldsymbol{e} \times I \subset (\boldsymbol{M} \times I)_*$ is a prehorn (defined as $(\boldsymbol{M} \times I)_*$ but now only establishing e, x_e, $\tilde{e}$, for $e \in \boldsymbol{e}$); x_{de} exists in $(\boldsymbol{M} \times I)_*$. If $\lambda : \boldsymbol{e} \to \boldsymbol{K}_*$ is a horn, then $\lambda \times I : \boldsymbol{e} \times I \to (\boldsymbol{K} \times I)_*$ is a horn in $(\boldsymbol{K} \times I)_*$; if $\boldsymbol{e} = \boldsymbol{M}_*$ then $\boldsymbol{e} \times I = (\boldsymbol{M} \times I)_*$.
(7) If $\lambda : \boldsymbol{e} \to \boldsymbol{K}_*$ is a horn, then $d\lambda : d\boldsymbol{e} \to \boldsymbol{K}_*$ is a horn.

Definition 2.2.

(1) Let $\lambda, \tilde{\lambda} : \boldsymbol{e} \to \boldsymbol{K}_*$ be two horns in $\boldsymbol{K}_*$. A *chain homotopy* D: $\lambda \simeq \tilde{\lambda}$ is a mapping $\Lambda : \boldsymbol{e} \times I \to \boldsymbol{K}_*$ (i.e., to which there exists $\overline{\Lambda} : \boldsymbol{N}_* \to \boldsymbol{K}_*$, $\boldsymbol{e} \times I \subset (\boldsymbol{M} \times I)_* \subset \boldsymbol{N}_*$ satisfying $\overline{\Lambda} \mid \boldsymbol{e} \times I = \Lambda$) such that $\Lambda \mid \boldsymbol{e} = \lambda$, $\Lambda \mid \tilde{\boldsymbol{e}} = \tilde{\lambda}$. We set $D(e) = \overline{\Lambda}(x_e) = \Lambda(x_e)$, $D(de) = \overline{\Lambda}(x_{de})$, therefore we conclude $dD(e) + D(de) = \lambda(e) - \tilde{\lambda}(e)$.

(2) A *filling* $\Lambda = (\Lambda, \tilde{\lambda})$ of a horn λ in $\boldsymbol{K}_*$ is a chain homotopy D: $\lambda \simeq \tilde{\lambda}$ with $D(de) = 0$, $e \in \boldsymbol{e}$.

Remarks and examples.

(1) Every horn $\lambda : \boldsymbol{e} \to \boldsymbol{K}_*$ has a trivial filling $\Lambda = (\Lambda, \lambda)$.
Proof: Take

$$\Lambda : \boldsymbol{e} \times I \xrightarrow{\lambda \times I} (\boldsymbol{K} \times I)_* \xrightarrow{r} \boldsymbol{K}_*,$$

where r is the retraction (see Section 1).

(2) Let λ be a horn in $\boldsymbol{K}_*$, $\Lambda = (\Lambda, \alpha)$ a filling of the horn $d\lambda$, i.e., $\Lambda : d\boldsymbol{e} \times I \to \boldsymbol{K}_*$ satisfying $d\Lambda(x_{de}) = d\lambda(e) - \alpha(e)$, then $\gamma : \boldsymbol{e} \to \boldsymbol{K}_*$, $\gamma(e) = \lambda(e) - \Lambda(x_{de})$ is a horn. Let $\Gamma = (\Gamma, \tilde{\lambda})$ be a filling of γ, then we calculate

$$d\Gamma(x_e) = \lambda(e) - \Lambda(x_{de}) - \tilde{\lambda}(e).$$

In the same way we obtain:

Lemma 2.3. *Let*

$$D\colon\ \lambda \simeq \tilde{\lambda}. \tag{1}$$

be a chain homotopy, then by setting $\Gamma(x_e) = D(e)$, $\Lambda(x_{de}) = D(de)$ *we obtain a* 2*-stage filling of the horns* $d\lambda$ *and* γ.

Moreover we observe:

Lemma 2.4.

(1) *Let* $f \in \mathfrak{Ch}(\boldsymbol{K}_*, \boldsymbol{L}_*)$ *be a morphism,* λ *a horn in* $\boldsymbol{K}_*$, D: $\lambda \simeq \tilde{\lambda}$ *a chain homotopy, then* $f(D)$: $f\lambda \simeq f\tilde{\lambda}$ *is a chain homotopy between the images.*

(2) *Suppose* $f_0, f_1 \in \mathfrak{Ch}(\boldsymbol{K}_*, \boldsymbol{L}_*)$, *then* $f_0 \simeq f_1$, *whenever there exists a chain homotopy* D: $f_0(\boldsymbol{K}_*) \simeq f_1(\boldsymbol{K}_*)$ *between the horns* $f_i(\boldsymbol{K}_*) \subset \boldsymbol{L}_*$, $i = 0, 1$.

Let $\boldsymbol{c} \subset \boldsymbol{K}_*$ be a prehorn, then $\boldsymbol{c} \cap K_n(\cdot) = \boldsymbol{c}_n$ is a prehorn; we have $\boldsymbol{c} = \bigcup_{n \in \mathbb{Z}} \boldsymbol{c}_n$.

Let $\boldsymbol{e} \subset \boldsymbol{L}_*$ be a prehorn, then we define $\bar{\boldsymbol{e}} \subset \boldsymbol{L}_*$ as the smallest natural sub-chain complex of $\boldsymbol{L}_*$, containing $\boldsymbol{e}$, which is closed under the application of φ, κ, i', l and the chain homotopies $\varphi\kappa(\cdot) \simeq (\cdot)$, $j_\#\varphi(\cdot) \simeq l(\cdot)$. This $\bar{\boldsymbol{e}}$ is not necessarily a chain functor, but, according to Lemma 9.2, for any $\boldsymbol{e} \subset \boldsymbol{M}_* \subset \boldsymbol{L}_*$, $\boldsymbol{P}_* = \boldsymbol{M}_* \cup_q (\bar{\boldsymbol{e}} \times I)$, q: $\bar{\boldsymbol{e}} \subset \bar{\boldsymbol{e}} \times I$ is a chain functor, because the inclusion $\boldsymbol{M}_* \subset \boldsymbol{P}_*$ is a homotopy equivalence.

We have:

Lemma 2.5.

(1) *To each commutative diagram,*

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{p} & \boldsymbol{B}_* \\ \uparrow{\scriptstyle \lambda} & & \uparrow{\scriptstyle \Lambda} \\ \boldsymbol{e} & \xrightarrow[i_0]{} & \boldsymbol{e} \times I \end{array} \tag{2}$$

where $\boldsymbol{e}$ is a prehorn, with obvious inclusion $i_0 : \boldsymbol{e} \to \boldsymbol{e} \times I$, there exists a commutative diagram

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{p} & \boldsymbol{B}_* \\ \uparrow{\scriptstyle \bar{\lambda}} & & \uparrow{\scriptstyle \overline{\Lambda}} \\ \boldsymbol{M}_* & \xrightarrow[i]{} & \boldsymbol{M}_* \cup (\bar{\boldsymbol{e}} \times I) \end{array} \tag{3}$$

$\boldsymbol{e} \subset \boldsymbol{M}_*$, such that $\lambda = \bar{\lambda} \mid \boldsymbol{e}$, $\overline{\Lambda} \mid \boldsymbol{e} \times I = \Lambda$.

(2) *Let $\boldsymbol{K}_* \in \mathfrak{C}h$ and $\boldsymbol{b} \subset \boldsymbol{K}_*$ be a natural basis* (*in all dimensions, see Lemma* 9.1) *then $\bar{\boldsymbol{b}} = \boldsymbol{K}_*$ and $\boldsymbol{K}_* \cup (\bar{\boldsymbol{b}} \times I) = (\boldsymbol{K} \times I)_*$.*

Proof. We have $\boldsymbol{e} \subset \bar{\boldsymbol{e}} \subset \boldsymbol{M}_*$, $\bar{\lambda} : \boldsymbol{M}_* \to \boldsymbol{E}_*$ such that $\bar{\lambda} \mid \boldsymbol{e} = \lambda$, so that we can extend λ over $\bar{\boldsymbol{e}}$, obtaining a $\tilde{\lambda} : \bar{\boldsymbol{e}} \to \boldsymbol{E}_*$. On the other hand there exists $\widetilde{\Lambda} : \boldsymbol{N}_* \to \boldsymbol{B}_*$, $\boldsymbol{e} \times I \subset \bar{\boldsymbol{e}} \times I \subset \boldsymbol{N}_*$, extending Λ. So we detect $\overline{\Lambda} : \boldsymbol{M}_* \cup (\bar{\boldsymbol{e}} \times I) \to \boldsymbol{B}_*$ by setting

$$\overline{\Lambda} \mid \bar{\boldsymbol{e}} \times I = \widetilde{\Lambda} \mid \bar{\boldsymbol{e}} \times I, \qquad \overline{\Lambda} \mid \boldsymbol{M}_* = p\bar{\lambda}.$$

The second part of 2.5 is immediate. □

We deduce from Definition 2.2 of a homotopy and of a filling:

Lemma 2.6. *Let $p : \boldsymbol{E}_* \to \boldsymbol{B}_*$ be a mapping, λ a horn in $\boldsymbol{E}_*$:*

(1) *a chain homotopy D: $p\lambda \simeq \eta$ amounts to the existence of a commutative diagram* (2) (*as a restriction of a diagram* (3)).

(2) *$\Lambda = (\Lambda, \tilde{\lambda})$ is a filling of $p\lambda$, whenever we have*

$$\overline{\Lambda}(x_{de}) = 0, \qquad d\Lambda(x_e) = p\big(\lambda(e)\big) - \tilde{\lambda}(e).$$

3. Fibrations

We will define two different concepts of a fibration, the *Hurewicz*- and the *Kan*-fibrations. Hurewicz fibrations are modelled after the topological example, while Kan fibrations (see [7]) are defined as in the simplicial case by requiring that certain fillings can be lifted. Both concepts are needed in the course of the development of a closed model structure in $\mathfrak{C}h$. Fortunately both concepts turn out to equivalent. Let $p : \boldsymbol{E}_* \to \boldsymbol{B}_*$ be a morphism of chain functors.

Definition 3.1.

(1) p is a Hurewicz fibration whenever each commutative diagram in $\mathfrak{C}h$

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{p} & \boldsymbol{B}_* \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle F} \\ \boldsymbol{K}_* & \xrightarrow{i_0} & (\boldsymbol{K} \times I)_* \end{array} \tag{1}$$

admits a diagonal $\overline{F} : (\boldsymbol{K} \times I)_* \to \boldsymbol{E}_*$ such that $\overline{F} i_0 = f$ and $p\overline{F} = F$.

(2) p is a Kan fibration whenever to each horn λ in $\boldsymbol{E}_*$ with given filling $\widetilde{\Lambda}$ of $p\lambda$ (in $\boldsymbol{B}_*$) there exists a filling Λ of λ in $\boldsymbol{E}_*$ such that $p\Lambda = \widetilde{\Lambda}$.

Lemma 3.2. *p is a Kan fibration whenever to each horn λ in $\boldsymbol{E}_*$ and homotopy $\widetilde{\Lambda}$: $p\lambda \simeq \tilde{\lambda}$ there exists a homotopy Λ: $\lambda \simeq \gamma$ such that $p\Lambda = \widetilde{\Lambda}$, $p\gamma = \tilde{\lambda}$.*

Proof. $\Rightarrow$: Apply 2.3.

$\Leftarrow$: Follows because every filling of a horn is a special case of a chain homotopy (the first step in this 2-stage process is trivial). □

Lemma 3.3. *p is a Kan fibration whenever for any horn λ in $\boldsymbol{E}_*$, and commutative diagram* (*see Section* 2(2))

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{p} & \boldsymbol{B}_* \\ {\scriptstyle \lambda}\uparrow & & \uparrow{\scriptstyle \Lambda} \\ \boldsymbol{e} & \xrightarrow{i_0} & \boldsymbol{e} \times I \end{array} \tag{2}$$

there exists a diagonal $\widetilde{\Lambda} : \boldsymbol{e} \times I \to \boldsymbol{E}_*$, rendering (2) *commutative.*

Proof. This is according to Lemma 2.6 just a reformulation of Lemma 3.2. □

For the next assertion we need some arguments about cofibrations which are verified in the next section without using this present result:

Theorem 3.4. *p is a Kan fibration if and only if p is a Hurewicz fibration.*

Proof. $\Rightarrow$: Assume that p is a Kan fibration and let (1) be a commutative diagram. For any horn λ in $\boldsymbol{K}_*$ Lemma 3.3 guarantees the existence of a lifting $\widetilde{\Lambda} : \boldsymbol{e} \times \boldsymbol{I} \to \boldsymbol{E}_*$ in (2). We apply this to the horn $f : \boldsymbol{K}_* \to \boldsymbol{E}_*$. Since p is Kan fibration we find a diagonal $\overline{F} : (\boldsymbol{K} \times I) \to \boldsymbol{E}_*$.

$\Leftarrow$: Let a commutative diagram (2) be given, and assume that p is a Hurewicz fibration, then there exists a chain functor $\boldsymbol{K}_*$ (containing $\boldsymbol{e}$) and a commutative square

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{p} & \boldsymbol{B}_* \\ {\scriptstyle\bar\lambda}\uparrow & & \uparrow{\scriptstyle\overline{\Lambda}} \\ \boldsymbol{K}_* & \xrightarrow[i]{} & \boldsymbol{K}_* \cup (\bar{\boldsymbol{e}} \times I) \end{array} \qquad \boldsymbol{e} \subset \boldsymbol{K}_*$$

admitting a lifting $\overline{F} : \boldsymbol{K}_* \cup (\bar{\boldsymbol{e}} \times I) \to \boldsymbol{E}_* : i$ is obviously a trivial cofibration, hence the "$\Rightarrow$" proof of Theorem 5.1 (which does not use the present arguments) guarantees the existence of $\overline{F}$. The restriction of $\overline{F}$ to $\boldsymbol{e} \times I$ furnishes a diagonal of (2). □

In the future we will mostly talk about a *fibration* whenever we mean a Hurewicz or a Kan fibration.

Examples.

(1) Let $\boldsymbol{A}_*, \boldsymbol{B}_* \in \mathfrak{C}h$ be given, then $\boldsymbol{A}_* \oplus \boldsymbol{B}_* = \boldsymbol{E}_*$ is a chain functor and the projection $p : \boldsymbol{E}_* \to \boldsymbol{B}_*$ is a fibration.
(2) Suppose $\boldsymbol{K}_*$ is a chain functor, then

Lemma 3.5.

$$p_0 : \boldsymbol{K}_*^I \to \boldsymbol{K}_* \tag{3}$$

is a trivial fibration, *i.e., a fibration which is at the same time a weak equivalence* (= *a chain homotopy equivalence, see Definition* 4.7).

Proof. p_0 is by construction a weak equivalence. It is easy to see that the proof that $p_0 : \boldsymbol{K}_*^I \to \boldsymbol{K}_*$ is a fibration reduces to the following question: Let $c, \tilde{c} \in K_n(\cdot)$, $y_c \in K_{n+1}(\cdot)$ be given such that $dy_c = c - \tilde{c}$ and take a prescribed $\gamma = (c, \tilde{c}_1, x_1) \in K_n^I(\cdot)$ such that $p_0(\gamma) = c$. We seek a $\bar{\gamma} = (y_c, \bar{\tilde{c}}_1, \bar{x}_1) \in K_{n+1}^I(\cdot)$, $\tilde{\gamma} = (a, b, e) \in K_n^I(\cdot)$, satisfying

$$p_0\bar{\gamma} = y_c, \qquad d\bar{\gamma} = \gamma - \tilde{\gamma}.$$

This is accomplished by setting

$$a = \tilde{c}, \qquad \bar{\tilde{c}}_1 = 0, \qquad b = \tilde{c}_1, \qquad \bar{x}_1 = 0, \qquad e = (-1)^n y_c + x_1. \qquad \square$$

4. Cofibrations and weak equivalences

Let $q \in \mathfrak{C}h(\boldsymbol{A}_*, \boldsymbol{B}_*)$, $\boldsymbol{L}_* \in \mathfrak{C}h$ be given.

Definition 4.1. q is a cofibration, if it is an inclusion of chain functors and if the following condition is fulfilled*:*

Suppose $\boldsymbol{L}_* \in \mathfrak{C}h$ is any chain functor and let $f : \boldsymbol{B}_* \to \boldsymbol{L}_*$, $f_{1A} : \boldsymbol{A}_* \to \boldsymbol{L}_*$ be mappings in $\mathfrak{C}h$, with given chain homotopy D_A: $fq \simeq f_{1A} \in \mathfrak{C}h(\boldsymbol{A}_*, \boldsymbol{L}_*)$. Then there exists a chain homotopy $D\colon\ f \simeq f_1$ for some $f_1 \in \mathfrak{C}h(\boldsymbol{B}_*, \boldsymbol{L}_*)$ (extending f_{1A} over $\boldsymbol{B}_*$), such that $D \mid \boldsymbol{A}_* = D(q \times 1) = D_A$.

Remark. This is the translation of the topological *homotopy extension property*, which describes cofibrations for topological spaces. Since we are trying to be as close as possible to the topological case, we pronounce this as the definition of a cofibration.

There are some equivalent conditions describing a cofibration:

Lemma 4.2. *q is a cofibration, if and only if every commutative diagram*

$$\begin{array}{ccc} \boldsymbol{L}_*^I & \xrightarrow{p_0} & \boldsymbol{L}_* \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle G} \\ \boldsymbol{A}_* & \xrightarrow[q]{} & \boldsymbol{B}_* \end{array} \tag{1}$$

has a diagonal $\overline{G} : \boldsymbol{B}_* \to \boldsymbol{L}_*^I$, rendering both triangles commutative.

Proof. Take the adjoint $H : (\boldsymbol{A} \times I)_* \to \boldsymbol{L}_*$ of g, then the commutativity of (1) describes the basic situation of 4.1: There exists a homotopy $H\colon\ Gq \simeq G_{1A}$; since q is a cofibration, there exists an extension $\overline{H} : (\boldsymbol{B} \times I)_* \to \boldsymbol{L}_*$ of H, $\overline{H}\colon\ G \simeq G_1$. The adjoint $\overline{G} : \boldsymbol{B}_* \to \boldsymbol{L}_*^{\boldsymbol{I}}$ of $\overline{H}$ is the required diagonal of (1).

If on the other hand each diagram (1) has a diagonal, then this proof can be read backwards, ensuring that q is a cofibration. □

Remark. We observed already in Section 1 that if q is an inclusion of chain functors (implying that q is compatible not only with l and i', but also with $\varphi,\ \kappa$ and the chain homotopies $\varphi\kappa \simeq 1,\ j_\#\varphi \simeq l$) $\boldsymbol{S}_* = \boldsymbol{B}_* \cup_q (\boldsymbol{A} \times I)_*$ is not only a functor into the category of free chain complexes, but carries the structure of a chain functor. One can define $\boldsymbol{S}_*$ either as we did in Section 1 or by gluing $\boldsymbol{A}_* \subset \boldsymbol{B}_*$ to the basis of $(\boldsymbol{A} \times I)_*$ (see remark following Proposition 1.1). This can be easily verified.

Lemma 4.3. *Suppose q is an inclusion of chain functors; q is a cofibration if and only if $\boldsymbol{B}_* \cup_q (\boldsymbol{A} \times I)_*$ is a retract of $(\boldsymbol{B} \times I)_*$, i.e., if there exists a $r : (\boldsymbol{B} \times I)_* \to \boldsymbol{B}_* \cup_q (\boldsymbol{A} \times I)_*$ satisfying $rj = 1$, $j : \boldsymbol{B}_* \cup_q (\boldsymbol{A} \times I)_* \to (\boldsymbol{B} \times I)_*$.*

Proof. $\Rightarrow$: If q is a cofibration, then we set in Definition 4.1. $\boldsymbol{L}_* = \boldsymbol{B}_* \cup_q (\boldsymbol{A} \times I)_*$ and notice that $D : (\boldsymbol{A} \times \boldsymbol{I})_* \xrightarrow{\subset} \boldsymbol{L}_*,\ f : \boldsymbol{B}_* \xrightarrow{\subset} \boldsymbol{L}_*$ can be put together, giving the identity $1 : \boldsymbol{B}_* \cup_q (\boldsymbol{A} \times I)_* \to \boldsymbol{B}_* \cup_q (\boldsymbol{A} \times I)_*$. According to Definition 4.1 this identity can be extended to a $r : (\boldsymbol{B} \times I)_* \to \boldsymbol{B}_* \cup_q (\boldsymbol{A} \times I)_*$, $rj = 1$.

$\Leftarrow$: Suppose there are given $D_A,\ f$ as in Definition 4.1, then they determine a mapping $h : \boldsymbol{B}_* \cup_q (\boldsymbol{A} \times I)_* \to \boldsymbol{L}_*$, $\boldsymbol{L}_* \in \mathfrak{C}h$ (= an arbitrary chain functor), and vice-versa. Then the existence of a retraction $r : (\boldsymbol{B} \times I)_* \to \boldsymbol{B}_* \cup_q (\boldsymbol{A} \times I)_*$ yields a $\bar{h} = hr : (\boldsymbol{B} \times I)_* \to \boldsymbol{L}_*$, guaranteeing that q is a cofibration. □

Lemma 4.4. *An inclusion $q\colon \boldsymbol{A}_* \subset \boldsymbol{B}_*$ is a cofibration if and only if for all n each functor $A_n(\cdot)$ is a direct summand of the functor $B_n(\cdot)$.*

Proof. $\Rightarrow$: Suppose q is a cofibration, then we have according to Lemma 4.3 a retraction $r:(\boldsymbol{B}\times I)_* \to \boldsymbol{B}_* \cup (\boldsymbol{A}\times I)_*$. We have:

$$r(\tilde{b}) = x_{a_1} + \tilde{a} + b_1, \quad b, b_1 \in B_*(\cdot),\ a \in A_*(\cdot),$$

and set $\alpha_n(\tilde{b}) = \tilde{a}$ and $\beta_n(b) = a$. Since $r(\tilde{a}) = \tilde{a}$, we deduce $\beta_n(a) = a$, hence $\beta_n : B_n(\cdot) \to A_n(\cdot)$ is a retraction, which is not necessarily compatible with boundaries. The existence of a direct sum decomposition

$$B_n(\cdot) \approx A_n(\cdot) \oplus C_n(\cdot) \tag{2}$$

follows.

$\Leftarrow$: Suppose q is an inclusion allowing a direct sum decomposition (2) for all n, then we define a retraction $r:(\boldsymbol{B}\times I)_* \to \boldsymbol{B}_* \cup (\boldsymbol{A}\times I)_*$ in the following way:

$$r(b) = b, \quad b \in B_*(\cdot).$$

Suppose $b = a + c$ according to (2), then we set

$$r(x_b) = x_a,$$
$$r(\tilde{b}) = \tilde{a} + c - r(x_{dc}).$$

Let $dc = a_1 + c_1$ be the representation of dc, then

$$d\tilde{b} = \tilde{a}_1 + d\tilde{a} + \tilde{c}_1$$

and

$$r(x_{dc}) = r(x_{a_1}) + r(x_{c_1}) = x_{a_1}.$$

This r is compatible with boundaries: $r(db) = dr(b) = b$, $b \in B_n(\cdot)$;

$$dr(\tilde{b}) = d\tilde{a} + dc - dx_{a_1} = d\tilde{a} + dc + x_{da_1} - a_1 + \tilde{a}_1,$$
$$r(d\tilde{b}) = r(\tilde{a}_1 + d\tilde{a} + \tilde{c}_1) = \tilde{a}_1 + d\tilde{a} + c_1 + x_{da_1},$$

since

$$-x_{dc_1} = x_{da_1}.$$

Hence

$$dr(\tilde{b}) = r(d\tilde{b}),$$
$$dr(x_b) = dx_a = -x_{da} + a - \tilde{a},$$
$$r(dx_b) = r(-x_{db} + b - \tilde{b}) = -x_{a_1} - x_{da} + a + c - \tilde{a} - c + x_{a_1} = -x_{da} + a - \tilde{a}.$$

As a result we have

$$dr(x_b) = r(dx_b).$$

Since r is compatible with all structures of a chain functor, natural and additive, it is a morphism of chain functors. Moreover $r \mid \boldsymbol{B}_* \cup (\boldsymbol{A}\times I)_* = 1$. This completes the proof of the lemma. □

Corollary 4.5. *An inclusion q is a cofibration if and only if there exists a retraction as in* 4.3 *such that $r(x_b) = x_a$ for all $b \in \boldsymbol{B}_*$ and suitable $a \in \boldsymbol{A}_*$.*

Proof. The retraction constructed in the $\Leftarrow$ part of the proof of Lemma 4.4 is of that kind. The other direction follows from 4.3. □

Corollary 4.6. *Suppose we have mappings $f : \boldsymbol{B}_* \to \boldsymbol{L}_*$, D_A: $fq \simeq f_{A1} \in \mathfrak{C}h(\boldsymbol{A}_*, \boldsymbol{L}_*)$ as in Definition* 4.1*, assume furthermore, that there exists a natural subcomplex $K_* \subset \boldsymbol{L}_*$* (*not necessarily a sub-chain functor*) *such that $D_A(\cdot) \subset K_*(\cdot)$. If q is a cofibration, we detect a homotopy D extending D_A such that $D(\cdot) \subset K_*(\cdot)$.*

Proof. Take a retraction r as in 4.5, then the mapping $\bar{h} = hr : (\boldsymbol{B} \times I)_* \to \boldsymbol{L}_*$ in the $\Leftarrow$ part of the proof of Lemma 4.3 has the required property. □

Example. Let $\boldsymbol{K}_*$ be any chain functor, then

$$i_0 : \boldsymbol{K}_* \to (\boldsymbol{K} \times I)_* \tag{3}$$

is a cofibration. This is of course dual to the corresponding result in Section 3, Example 2.

The following definition has already been used:

Definition 4.7.

(1) A morphism $w \in \mathfrak{C}h(\boldsymbol{A}_*, \boldsymbol{B}_*)$ is a weak equivalence, whenever there exists a $\tilde{w} \in \mathfrak{C}h(\boldsymbol{B}_*, \boldsymbol{A}_*)$ and chain homotopies $w\tilde{w} \simeq 1_{\boldsymbol{B}_*}$, $\tilde{w}w \simeq 1_{\boldsymbol{A}_*}$.
(2) A trivial cofibration (fibration) is a $w \in \mathfrak{C}h(\boldsymbol{A}_*, \boldsymbol{B}_*)$ which is a weak equivalence and a cofibration (respectively a fibration).

Example. (3) is a trivial cofibration.

Lemma 4.8. *Let $q : \boldsymbol{A}_* \to \boldsymbol{B}_*$ be a trivial cofibration, $\tilde{q} : \boldsymbol{B}_* \to \boldsymbol{A}_*$ a homotopy inverse of q, then there exists a $\hat{q} \simeq \tilde{q}$ such that $\hat{q}q = 1_{\boldsymbol{A}_*}$. Moreover the homotopy D: $q\tilde{q} \simeq 1_{\boldsymbol{B}_*}$ can be assumed to be stationary on $\boldsymbol{A}_*$, i.e., one has $D(q \times I) = 0$.*

Proof. The first assertion is proved as in the topological case. Assume that already $\tilde{q}q = 1_{\boldsymbol{A}_*}$ and let D: $q\tilde{q} \simeq 1_{\boldsymbol{B}_*}$ be a given homotopy. According to 4.4, t: $\widehat{\boldsymbol{A}}_* = \boldsymbol{B}_* \cup (\boldsymbol{A} \times I)_* \cup \widetilde{\boldsymbol{B}}_* \subset (\boldsymbol{B} \times I)_*$ is a cofibration. We detect a mapping $F : (\widehat{\boldsymbol{A}} \times I)_* \to (\boldsymbol{B} \times I)_*$ which is either D or an inclusion and on $x_{(x_a)}$ zero. This homotopy can be extended to a $\overline{F} : ((\boldsymbol{B} \times I) \times I)_* \to (\boldsymbol{B} \times I)_*$ and $\overline{F}i_1 : (\boldsymbol{B} \times I)_* \to (\boldsymbol{B} \times I)_*$ reveals itself as a homotopy $\overline{D}$: $q\tilde{q} \simeq 1$ which is stationary on $\boldsymbol{A}_*$. □

Dually we have:

Lemma 4.9. *Let $p : \boldsymbol{E}_* \to \boldsymbol{B}_*$ be a trivial fibration, $\tilde{p} : \boldsymbol{B}_* \to \boldsymbol{E}_*$ the homotopy inverse of p, then there exists a $\hat{p} \simeq \tilde{p}$ such that $p\hat{p} = 1_{\boldsymbol{E}_*}$ and a homotopy G: $\hat{p}p \simeq 1$ such that $G \subset \ker p$.*

Proof. The existence of $\hat{p}$ is dual to that of $\hat{q}$ in 4.8. Suppose that $H\colon\ \hat{p}p \simeq 1$ is any homotopy, then $\hat{p}pH\colon\ \hat{p}p \simeq \hat{p}p$ and $G = \hat{p}pH - H\colon\ \hat{p}p \simeq 1$ satisfies $pG = 0$, hence it is a homotopy in $\ker p$. □

5. Relations between fibrations and cofibrations

Let

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{p} & \boldsymbol{B}_* \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle F} \\ \boldsymbol{A}_* & \xrightarrow[q]{} & \boldsymbol{C}_* \end{array} \tag{1}$$

be a commutative square.

The following theorem establishes the cofibration half of axiom CM4:

Theorem 5.1. *p is a fibration if and only if every commutative square* (1) *with q being a trivial cofibration admits a diagonal $\overline{F}: \boldsymbol{C}_* \to \boldsymbol{E}_*$, rendering the diagram commutative.*

Proof. $\Rightarrow$: Assume p is a Hurewicz fibration and q a trivial cofibration. According to 4.8 we can assume that there exists a homotopy inverse $\tilde{q}$ to q, such that $\tilde{q}q = 1_{\boldsymbol{A}_*}$ and that the homotopy $D\colon\ q\tilde{q} \simeq 1_{\boldsymbol{C}_*}$, $D: (\boldsymbol{C} \times I)_* \to \boldsymbol{C}_*$ has the property that $D \mid \boldsymbol{A}_*$ is stationary, i.e., that $D(q \times I) = 0$.

Setting $\hat{f} = f\tilde{q}: \boldsymbol{C}_* \to \boldsymbol{E}_*$, $\widehat{F} = FD$, we obtain a commutative diagram

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{p} & \boldsymbol{B}_* \\ {\scriptstyle \hat{f}}\uparrow & & \uparrow{\scriptstyle \widehat{F}} \\ \boldsymbol{C}_* & \xrightarrow[i_0]{} & (\boldsymbol{C} \times I)_* \end{array}$$

which admits a diagonal $G: (\boldsymbol{C} \times I)_* \to \boldsymbol{E}_*$. We define $\widetilde{F} = Gi_1$ and deduce $p\widetilde{F} = \widehat{F}i_1 = FDi_1 = F$.

On the other hand $G(q \times I)$ is a homotopy $G(q \times I)\colon\ G(q \times I)i_0 \simeq G(q \times I)i_1$. Since

$$G(q \times I)i_0 = Gi_0q = \hat{f}q = f, \qquad G(q \times I)i_1 = Gi_1q = \widetilde{F}q,$$

$G(q \times I)\colon\ f \simeq \widetilde{F}q$ is a homotopy, satisfying $pG(q \times I) = FD(q \times I) = 0$. Since q is a cofibration, we can apply Corollary 4.6 to the result that we detect a homotopy $H\colon\ \widetilde{F} \simeq \overline{F}$, such that $\overline{F}q = f$, $H(\cdot) \subset \ker p$, hence we have also $p\overline{F} = F$. So (1) has a diagonal.

$\Leftarrow$: If every diagram (1) admits a diagonal, then in particular each commutative

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{p} & \boldsymbol{B}_* \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle F} \\ \boldsymbol{K}_* & \xrightarrow[i_0]{} & (\boldsymbol{K} \times I)_* \end{array} \tag{2}$$

has this property, ensuring that according to Definition 3.1(1), p is a Hurewicz fibration. □

The following theorem establishes the fibration half of axiom CM4:

Theorem 5.2. *q is a cofibration if and only if each commutative square* (1), *p a trivial fibration, admits a diagonal $\overline{F}:\boldsymbol{C}_* \to \boldsymbol{E}_*$ rendering the diagram commutative.*

Proof. $\Rightarrow$: Suppose q is a cofibration, p a trivial (Hurewicz-) fibration and $p\tilde{p} = 1$, G: $\tilde{p}p \simeq 1$ with G in $\ker p$ (see Lemma 4.9). We define $\tilde{p}F = \widetilde{F}:\boldsymbol{C}_* \to \boldsymbol{E}_*$ so that $p\widetilde{F} = F$. Since $\tilde{p}p \simeq 1$ in $\ker p$, we conclude $p\widetilde{F}q = Fq = pf$, $\tilde{p}p\widetilde{F}q = \tilde{p}pf$. Therefore we detect a chain homotopy $\widetilde{F}q \simeq f$ in $\ker p$. Since q is a cofibration, this yields a chain homotopy H: $\widetilde{F} \simeq \overline{F}$ such that according to Corollary 4.6 pH: $p\widetilde{F} = p\overline{F} = F$ and $\overline{F}q = f$.

$\Leftarrow$: Follows because of Lemma 4.2. □

6. Decompositions of mappings (I)

Let $f \in \mathfrak{C}h(\boldsymbol{K}_*, \boldsymbol{L}_*)$ be a morphism, then we have:

Theorem 6.1. *There exists a trivial cofibration $q:\boldsymbol{K}_* \to \boldsymbol{M}_{f}*$ and a fibration $p:\boldsymbol{M}_{f*} \to \boldsymbol{L}_*$ such that*

$$f = pq.$$

Proof. Our objective is to convert f into a fibration. What keeps f from being a fibration? There are eventually horns λ in $\boldsymbol{K}_*$, having fillings Λ of $f\lambda$ which cannot be lifted to $\boldsymbol{K}_*$.

According to Lemma 2.6 the existence of λ with filling Λ of $f\lambda$ yields a commutative diagram, where $\boldsymbol{e} \subset \boldsymbol{C}_*$, for some $\boldsymbol{C}_* \in \mathfrak{C}h$:

$$\begin{array}{ccc} \boldsymbol{K}_* & \xrightarrow{f} & \boldsymbol{L}_* \\ \uparrow{\scriptstyle\lambda} & & \uparrow{\scriptstyle\Lambda} \\ \boldsymbol{e} & \xrightarrow[i_0]{} & (\boldsymbol{e} \times I). \end{array} \tag{1}$$

We enlarge $K_*(X, A)$ by (1) new free generators $x(\lambda, \Lambda, e)$, $e \in \boldsymbol{e}$, $\dim x(\lambda, \Lambda, e) = \dim \lambda(e) + 1$, Λ a filling of $f\lambda$ in $\boldsymbol{L}_*$, (2) new free generators $y(\lambda, \Lambda, e)$, $\dim y(\lambda, \Lambda, e) = \dim \lambda(e)$, satisfying

$$dx(\lambda, \Lambda, e) = \lambda(e) - y(\lambda, \Lambda, e). \tag{2}$$

We assume that for any $g \in \mathfrak{K}((X, A), (Y, B))$ one has

$$g_\#\big(x(\lambda, \Lambda, e)\big) = x\big(\lambda, \Lambda, g_\#(e)\big), \tag{3}$$

respectively for $y(\cdots)$.

This defines naturally a chain complex $\widetilde{M}^1_{f*}(X, A)$. Now we enlarge $\widetilde{M}^1_{f*}(X, A)$ again such that the larger $\boldsymbol{M}^1_{f*}$ carries the structure of a chain functor. We define

$$x(\lambda, \Lambda, e), y(\lambda, \Lambda, e) \in \boldsymbol{M}^{1\prime}_{f*}$$

whenever e is contained in $\boldsymbol{C}'_*$, implying that also $\Lambda(x_e)$ is contained in $\boldsymbol{L}'_*$. We set

$$i'\bigl(x(\lambda, \Lambda, e)\bigr) = x\bigl(\lambda, \Lambda, i'(e)\bigr) \tag{4}$$

respectively for $y(\cdots)$, whenever this is defined (see Section 9 concerning i'). Now we deal with φ, κ, i' and the chain homotopies

$$h(\cdot)\colon\ \varphi\kappa(\cdot) \simeq (\cdot), \qquad \bar{h}(\cdot)\colon\ j_\#\varphi(\cdot) \simeq l(\cdot)$$

and form words $w = w_1, \ldots, w_k$, where either w_i is one of the symbols φ, κ, h, $\bar{h}$, i' or a map induced by a $g \in \mathfrak{K}(\cdot,\cdot)$. Here we have to assume that $w_i(\cdot)$ and $w_{i-1}w_i$ is only defined whenever this makes sense, e.g., $w_k(\cdot) = \varphi(\cdot)$ only if $(\cdot) \in M'^1_{f*}$, respectively $w_{i-1} = \varphi$, $w_i = \kappa$.

Detecting $M^1_{f*}(X, A)$, we define in addition to the chains $x(\cdots)$, $y(\cdots)$ new chains $wx(\cdots)$, $wy(\cdots)$ as new free generators of $M^1_{f*}(X, A)$, where we have to take into account (3) for $w = g_\#$ and (4) for $w = i'$. Concerning the boundary we have $h(x(\lambda, \Lambda, e)) \in M^1_{f(n+2)}(X)$ satisfying

$$dh\bigl(x(\lambda, \Lambda, e)\bigr) + h\bigl(x(d\lambda, d\Lambda, de)\bigr) = \varphi\kappa\bigl(x(\lambda, \Lambda, e)\bigr) - x(\lambda, \Lambda, e),$$

while $\bar{h}(x(\lambda, \Lambda, e)) \in M^1_{f(n+2)}(X, A)$ satisfies

$$d\bar{h}\bigl(x(\lambda, \Lambda, e)\bigr) + \bar{h}\bigl(x(d\lambda, d\Lambda, de)\bigr) = j_\#\varphi\bigl(x(\lambda, \Lambda, e)\bigr) - lx(\lambda, \Lambda, e),$$

whenever $x(\cdots) \in M^1_{f(n+1)}(X, A)$, respectively for $y(\cdots)$.

The verification that this new $\boldsymbol{M}^1_{f*}$ (= $\boldsymbol{K}_*$ together with the complex generated by all these $w(\cdot)$'s) becomes a chain functor, is now an easy routine (see, e.g., Lemma 9.2). So the excision property, for example, holds for $\boldsymbol{M}^1_{f*}$, because we can assert that the inclusion $q^1\colon\ \boldsymbol{K}_* \subset \boldsymbol{M}^1_{f*}$ is a homotopy equivalence (therefore inducing an isomorphism of homology groups) and $\boldsymbol{K}_*$ is by assumption a chain functor:

There exists a deformation retraction $r^1 : \boldsymbol{M}^1_{f*} \to \boldsymbol{K}_*$ by mapping all new $w(x(\cdots))$ into zero and $w(y(\cdots))$ into $w(\lambda(e))$. Since K_n is a direct summand of M^1_{fn}, q^1 is, according to Lemma 4.6, a cofibration, hence a trivial cofibration.

We define $p^1 : \boldsymbol{M}^1_{f*} \to \boldsymbol{L}_*$ by setting

$$p^1(c) = f(c), \quad c \in K_*(X, A), \qquad p^1\bigl(x(\lambda, \Lambda, e)\bigr) = \Lambda(x_e),$$
$$p^1\bigl(y(\lambda, \Lambda, e)\bigr) = \Lambda(\tilde{e}) = \Lambda i_1(e), \qquad p^1\bigl(w(\cdot)\bigr) = wp^1(\cdot),$$

whenever this is defined. This p^1 commutes with boundaries; since p^1 is compatible with l and i', it is a transformation of chain functors. Observe that p^1 is in general, as an extension of f, *not* compatible with φ, κ and the relevant chain homotopies.

Suppose λ is a horn in $\boldsymbol{K}_* \subset \boldsymbol{M}^1_{f*}$, such that $f\lambda$ has a filling in $\boldsymbol{L}_*$ (i.e., such that there exists a commutative diagram (1)), then we establish a diagonal $\overline{\Lambda} : \boldsymbol{e} \times I \to \boldsymbol{M}^1_{f*}$ by:

$$\overline{\Lambda}(x_e) = x(\lambda, \Lambda, e), \qquad \overline{\Lambda}(\tilde{e}) = y(\lambda, \Lambda, e)$$

to the effect that $\overline{\Lambda}$ is a filling of λ in $\boldsymbol{M}^1_{f*}$, satisfying

$$p^1(\overline{\Lambda}) = \Lambda.$$

If λ extends over some $\boldsymbol{C}_* \in \mathfrak{C}h$, then Λ extends over $\boldsymbol{C}_* \cup \tilde{\boldsymbol{e}} \times \boldsymbol{I}$ (see 2.5) and $\overline{\Lambda}$ can be seen to extend over the same chain functor.

However it happens that we have new horns λ in $\boldsymbol{M}^1_{f*}$, with fillings of $p^1\lambda$ in $\boldsymbol{L}_*$, which cannot be lifted. So we must iterate the preceding process, constructing an increasing sequence

$$\cdots \supset \boldsymbol{M}^k_{f*} \supset \cdots \supset \boldsymbol{M}^1_{f*} \supset \boldsymbol{K}_*$$

and form the union

$$\boldsymbol{M}_{f*} = \bigcup_{k=1}^{\infty} \boldsymbol{M}^k_{f*},$$

which carries again the structure of a chain functor and comes together with morphisms

$$q\colon\ \boldsymbol{K}_* \subset \boldsymbol{M}_{f*}, \qquad p: \boldsymbol{M}_{f*} \to \boldsymbol{L}_*.$$

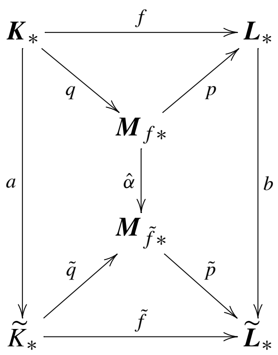
We have $pq = f$ and confirm that q is still a trivial cofibration. Since each horn λ in $\boldsymbol{M}_{f*}$ can be split into horns which are contained in some separate $\boldsymbol{M}^k_{f*}$, we conclude that every filling of $p\lambda$ can be lifted, assuring us that p is a fibration. $\square$

The previous construction immediately implies:

Corollary 6.2. *The decomposition $f = pq$ in Theorem* 6.1 *is canonical*: *If $\alpha = (a, b) : f \to \tilde{f}$*

$$\begin{array}{ccc} \boldsymbol{K}_* & \xrightarrow{f} & \boldsymbol{L}_* \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ \widetilde{\boldsymbol{K}}_* & \xrightarrow[\tilde{f}]{} & \tilde{\boldsymbol{L}}_* \end{array}$$

is a commutative diagram (*i.e., a morphism between morphisms*), *then there exists an induced mapping $\hat{\alpha} : \boldsymbol{M}_{f*} \to \boldsymbol{M}_{\tilde{f}*}$ rendering the corresponding diagram*



commutative.

7. Decompositions of mappings (II)

Let $f \in \mathfrak{C}h(\boldsymbol{K}_*, \boldsymbol{L}_*)$ be a morphism, then we have:

Theorem 7.1. *There exists a cofibration $q : \boldsymbol{K}_* \to \boldsymbol{N}_{f*}$ and a trivial fibration $p : \boldsymbol{N}_{f*} \to \boldsymbol{L}_*$ such that $f = pq$.*

Proof. In a first step we consider the chain functor $\boldsymbol{K}_* \oplus \boldsymbol{L}_*$ which is enlarged by new chains in dimension $n+1$ (i.e., by new free generators), the "connecting chains", $w(k)$, $k \in K_n(X, A)$ satisfying

$$\mathrm{d}w(k) + w(dk) = k - f(k). \tag{1}$$

with relations

$$g_\#\big(w(k)\big) = w\big(g_\#(k)\big), \quad g \in \mathfrak{K}\big((X, A), (Y, B)\big).$$

These new chains $w(k)$ are assumed to be contained in $N_{f*}(X, A)$, hence in particular in $N_{f*}(X, \emptyset)$ if $A = \emptyset$, but never in $N'_{f*}(X, A)$ nor in $N_{f*}(X)$ whenever $A \neq \emptyset$.

The idea is that we never have to define $\varphi w(k)$ nor $\kappa w(k)$, unless $A = \emptyset$. More precisely we are erecting the cone over the subcomplex $B_* \subset \boldsymbol{K}_* \oplus \boldsymbol{L}_*$, generated by all chains of the form $k - f(k)$, $k \in \boldsymbol{K}_*$.

We define:

$$\boldsymbol{N}_{f*}(\cdot) = \boldsymbol{K}_*(\cdot) \oplus \boldsymbol{L}_*(\cdot) \cup \operatorname{cone} B_*(\cdot). \tag{2}$$

For $k \in \boldsymbol{K}_*$, $l \in \boldsymbol{L}_*$, we take φ, κ as defined in these chain functors. We define B'_* to be the subcomplex generated by all $k - f(k)$ for $k \in K'_*$ and

$$\boldsymbol{N}'_{f*}(X, A) = \boldsymbol{K}'_*(X, A) \oplus \boldsymbol{L}'_*(X, A) \cup B'_*(X, A) \cup N'_{f*}(X, \emptyset)$$

with

$$\boldsymbol{N}'_{f*}(X, \emptyset) = \boldsymbol{K}'_*(X) \oplus \boldsymbol{L}'_*(X) \cup \operatorname{cone} B'_*(X).$$

For $A = \emptyset$ we define $\kappa w(k) = w(k)$, $k \in B'_*(X)$ and $\varphi\kappa w(k) = w(k)$. In particular no $w(k) \in \operatorname{cone} B_*(\cdot)$ are contained in $N'_{f*}(\cdot)$, unless $A = \emptyset$. Now it is easy to verify all properties of a chain functor for N_{f*}.

There exists a $p : \boldsymbol{N}_{f*} \to \boldsymbol{L}_*$ which is defined by

$$p(k) = f(k), \quad k \in \boldsymbol{K}_*, \qquad p(l) = l, \quad l \in \boldsymbol{L}_*, \qquad p \mid \operatorname{cone} B_* = 0.$$

Let λ be a horn in $\boldsymbol{N}_{f*}$ and $\Lambda = (\Lambda, \tilde{\gamma})$ a filling of $p\lambda$, then we have to determine a filling $\widehat{\Lambda}$ of λ such that $p\widehat{\Lambda} = \Lambda$, hence a diagonal in Section 6(1).

We can do this for three different cases separately:

(1) Suppose λ is a horn in $\boldsymbol{L}_* \subset \boldsymbol{N}_{f*}$, then we set $\widehat{\Lambda} = \Lambda$ in $\boldsymbol{L}_* \subset \boldsymbol{N}_{f*}$.
(2) Suppose λ is a horn in $\boldsymbol{K}_* \subset \boldsymbol{N}_{f*}$, we set

$$\widehat{\Lambda}(x_e) = w\big(\lambda(e)\big) + \Lambda(x_e),$$

$$\widehat{\Lambda}(x_{de}) = 0.$$

Hence

$$d\widehat{\Lambda}(x_e) = \lambda(e) - \bigl(w\bigl(d\lambda(e) + \tilde{\gamma}(e)\bigr),\ \tilde{\gamma}(e) = \Lambda i_1(e)\bigr)$$

where

$$d\Lambda(x_e) = f\bigl(\lambda(e)\bigr) - \tilde{\gamma}(e).$$

So

$$\bigl(\widehat{\Lambda}, \hat{\gamma}\bigr),\ \hat{\gamma}(e) = w\bigl(d\lambda(e)\bigr) + \tilde{\gamma}(e)$$

is a filling of λ satisfying $p\widehat{\Lambda} = \Lambda$.

(3) Suppose λ is a horn in cone B_*, then

$$p\lambda = \mathbf{0},$$

the 0-horn in $\boldsymbol{L}_*$.

Let $a(c)$, $c \in$ cone B_* be the natural cone over c in cone B_*, i.e., one has

$$da(c) + a(dc) = c,$$

and let $(\Lambda, \tilde{\gamma})$ be a filling of $p\lambda = \mathbf{0}$ in $\boldsymbol{L}_*$, i.e., one has $d\Lambda(x_e) = 0 - \tilde{\gamma}(e)$. We set

$$\widehat{\Lambda}(x_e) = a\bigl(\lambda(e)\bigr) + \Lambda(x_e),$$

$$\widehat{\Lambda}(x_{de}) = 0,$$

$$d\widehat{\Lambda}(x_e) = \lambda(e) - a\bigl(d\lambda(e)\bigr) - \tilde{\gamma}(e)$$

so that $(\widehat{\Lambda}, \hat{\gamma})$, $\hat{\gamma}(e) = a(d\lambda(e)) + \tilde{\gamma}(e)$ is a filling of λ with $p\widehat{\Lambda} = \Lambda$.

In all cases $\widehat{\Lambda}$ extends over some chain functor $\boldsymbol{C}_* \cup \bar{\boldsymbol{e}} \times \boldsymbol{I}$ as in the proof of 6.1.

(4) Every horn in $\boldsymbol{N}_{f*}$ splits into horns of the form $\lambda = \lambda_1, \lambda_2, \lambda_3$, where $\lambda_i,\ i = 1, 2, 3$, are horns of the form (1), (2) or (3).

This follows from the construction.

As a result p turns out to be a fibration. The inclusion α: $\boldsymbol{L}_* \subset \boldsymbol{N}_{f*}$ is easily recognized to be a homotopy inverse of p, so that p becomes a trivial fibration.

The inclusion q: $\boldsymbol{K}_* \subset \boldsymbol{N}_{f*}$, $q(k) = k$ is a cofibration, because for each n, $K_n(\cdot)$ is naturally a direct summand of $N_{fn}(\cdot)$ (see Lemma 4.6). Since $f = pq$, this yields the desired decomposition of f into a cofibration and a trivial fibration, thereby completing the proof of Theorem 7.1. □

We deduce immediately:

Corollary 7.2. *The decomposition of* f *in Theorem* 7.1 *is* (*in the same sense as the decomposition in Corollary* 6.1) *canonical.*

8. The remaining properties of a closed model category for $\mathfrak{C}h$ and the suspension functor

Following [5, p. 12], see also Section 0 of the present paper, we are dealing with the following axioms of a closed model category:

CM1. *Finite limits and colimits exist.*

This is not true for $\mathfrak{C}h$. There exist finite products and sums in $\mathfrak{C}h$, however if $f:\boldsymbol{K}_*\to\boldsymbol{L}_*$ is a morphism in $\mathfrak{C}h$, then $\ker f$ is not necessarily a chain functor: With $c\in\ker f$ we cannot be sure that $\varphi(c), \kappa(c)\in\ker f$ (provided this makes sense, i.e., $c\in K'_*(X,A)$, respectively $c\in K_*(X)$), unless we require that f commutes with φ and κ. However even under this condition we do not know that for a cycle $z\in(\ker f)_n(X,A)$ one detects a $l(z')+q_{\#}\bar{a}\sim z$ in $\ker f$ (see Section 9(3)). There are similar problems with cokernels: If f is an inclusion, then $\boldsymbol{L}_*\cup_f \operatorname{cone}\boldsymbol{K}_*$ is a chain functor, but not the categorical cokernel of f. However we will soon encounter interesting cases where kernels and cokernels exist.

CM2. *If f, g are maps and gf is defined, then, if two of these three maps are weak equivalences, then so is the third.*

This is obvious.

CM3. *Let f be a retract of g and g is* (1) *a fibration,* (2) *a weak equivalence, or* (3) *a fibration, then f has the same property.*

Proof. Ad(1): We use Definition 3.1(1) and have to ensure that for any $\boldsymbol{K}_*\in\mathfrak{C}h$ and commutative diagram

$$\begin{array}{ccc} \boldsymbol{E}_* & \xrightarrow{f} & \boldsymbol{B}_* \\ \uparrow{\scriptstyle m} & & \uparrow{\scriptstyle M} \\ \boldsymbol{K}_* & \xrightarrow[i_0]{} & (\boldsymbol{K}\times I)_* \end{array} \tag{1}$$

there exists a diagonal $\overline{M}:(\boldsymbol{K}\times I)_*\to\boldsymbol{E}_*$. We have

$$\begin{array}{ccccccc} \boldsymbol{K}_* & \xrightarrow{m} & \boldsymbol{E}_* & \xrightarrow{r} & \widetilde{\boldsymbol{E}}_* & \xrightarrow{s} & \boldsymbol{E}_* \\ \downarrow{\scriptstyle i_0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ (\boldsymbol{K}\times I)_* & \xrightarrow[M]{} & \boldsymbol{B}_* & \xrightarrow[\tilde{r}]{} & \widetilde{\boldsymbol{B}}_* & \xrightarrow[\tilde{s}]{} & \boldsymbol{B}_* \end{array} \tag{2}$$

with commutative squares and $sr=1$, $\tilde{s}\tilde{r}=1$. Since g is a fibration we find a diagonal $\widehat{M}:(\boldsymbol{K}\times I)_*\to\widetilde{\boldsymbol{E}}_*$ satisfying $g\widehat{M}=\tilde{r}M$, $\widehat{M}i_0=rm$. We set $\overline{M}=s\widehat{M}$ and calculate:

$$f\overline{M}=fs\widehat{M}=\tilde{s}g\widehat{M}=\tilde{s}\tilde{r}M=M,$$

$$\overline{M}i_0=s\widehat{M}i_0=srm=m.$$

Ad(2): If $\bar{g}:\widetilde{\boldsymbol{B}}_* \to \widetilde{\boldsymbol{E}}_*$ is a homotopy inverse of g, then $s\bar{g}\tilde{r} = \bar{f}$ is a homotopy inverse of f.

Ad(3): Use Lemma 4.2 as a characterization of a cofibration, then the proof that f is a cofibration is entirely dual to that of (1). □

CM4 is the objective of Section 5, Theorems 5.1, 5.2.

CM5 is settled by Theorems 6.1 and 7.1.

Remark. The decompositions of a mapping in Theorems 6.1, and in 7.1 are according to Corollary 6.2, respectively 7.2 canonical. This is more than it was required in CM5.

Although not every morphism in $\mathfrak{C}h$ has a kernel or a cokernel, there are significant cases, where kernels and cokernels exist:

The morphism $i_0 \oplus i_1 : \boldsymbol{K}_* \oplus \boldsymbol{K}_* \to (\boldsymbol{K} \times I)_*$ has a cokernel, the suspension of $\boldsymbol{K}_*$ (see [8]):

$$\boldsymbol{K}_* \oplus \boldsymbol{K}_* \xrightarrow{i_0 \oplus i_1} (\boldsymbol{K} \times I)_* \xrightarrow{\varrho} (\Sigma \boldsymbol{K})_*.$$

On the other hand the morphism $p_0 \otimes p_1 : \boldsymbol{K}_*^I \to \boldsymbol{K}_* \oplus \boldsymbol{K}_*$ has a kernel $(\Omega \boldsymbol{K})_*$ (see [8])

$$\Omega \boldsymbol{K}_* \to \boldsymbol{K}_*^I \xrightarrow{p_0 \oplus p_1} \boldsymbol{K}_* \oplus \boldsymbol{K}_*.$$

Define a functor $\overline{\Sigma} : \mathfrak{C}h \to \mathfrak{C}h$ by $(\overline{\Sigma} K)_*(X, A) = K_{*-1}(X, A)$, then we deduce:

Lemma 8.1. *There exist natural isomorphisms*

$$\overline{\Sigma} \boldsymbol{K}_* \approx \Sigma \boldsymbol{K}_*, \tag{3}$$

$$\Sigma \boldsymbol{K}_* \approx \boldsymbol{K}_* \otimes \Sigma \boldsymbol{Z}_*, \tag{4}$$

$$(\Omega K)_*(X, A) = K_{*+1}(X, A). \tag{5}$$

Proof. Suppose $\varrho(x_c) = y_c$, $c \in K_*$, then $dy_c = -y_{dc}$. Therefore the assignment $c \mapsto (-1)^{\dim c} y_c$ yields an isomorphism (3).

The existence of an isomorphism $\overline{\Sigma} \boldsymbol{K}_* \approx \boldsymbol{K}_* \otimes \overline{\Sigma} \boldsymbol{Z}_*$ is obvious (see [2] for the definition of the tensor product). The existence of an isomorphism (4) follows now from (3).

(5) follows immediately from the description of $\boldsymbol{K}_*^I$ in Section 1(7). □

We summarize:

Theorem 8.2.

(1) Σ (Ω) *are the suspension* (*loop*) *functors, associated with the given closed model structure* (*see* [8]).
(2) *They are invertible and, up to an isomorphism, inverses to each other, turning* $\mathfrak{C}h_h$ *into a stable category* (*i.e., one, allowing arbitrary desuspensions*).

Proof. (1) is obvious; (2) follows, because $\overline{\Sigma}$ is invertible, from Lemma 8.1. □

Remarks.

(1) Using the notation $\boldsymbol{K}_*^{(\cdot)} = \mathrm{Hom}((\cdot), \boldsymbol{K}_*)$ (see Section 1 concerning the dual cylinder) and (5), we can express the relationship between Σ and Ω by the following commutative diagram:

$$\begin{array}{ccccc}
\Omega \boldsymbol{K}_* & \longrightarrow & \boldsymbol{K}_*^I & \xrightarrow{p_0 \oplus p_1} & \boldsymbol{K}_* \oplus \boldsymbol{K}_* \\
\| & & \| & & \| \\
\boldsymbol{K}_*^{\Sigma \boldsymbol{Z}_*} & \xrightarrow[\boldsymbol{K}_*^{\varrho}]{} & \boldsymbol{K}_*^{(\boldsymbol{Z} \times I)_*} & \xrightarrow[\boldsymbol{K}_*^{i_0 \oplus i_1}]{} & \boldsymbol{K}_*^{\boldsymbol{Z}_* \oplus \boldsymbol{Z}_*}
\end{array}$$

(2) According to [9] (see also [6, 7.1.6]) the homotopy category of a closed model category satisfying 8.2(2) inherits in a natural way the structure of a *triangulated category*. The consequences of this fact in the case of $\mathfrak{Ch}_h$ will be studied elsewhere.

Let $\mathfrak{K}$ be any category with distinguished classes of fibrations, cofibrations and weak equivalences. Apart from D. Quillen's axioms CM1–CM5 there is R. Thomason's approach, to a closed model structure which is described in Weibel [10], leading to a *basic model category* respectively a *Thomason model category*.

Here axiom CM1 is replaced by a weaker statement, which deals with the existence and special properties of pushouts (pullbacks) along cofibrations (fibrations).

We do not know if and eventually under what restrictions this axiom holds for $\mathfrak{Ch}$.

Moreover CM5 is replaced by a factorization of any map $f = pe$ $(= em)$, with weak equivalence e, fibration p and cofibration m.

If this factorization turns out to be functorial, this basic model structure is called a *Thomason model structure*. According to our results in Sections 6, 7, these functorial factorizations exist (at least for special morphisms).

The concept of a *simplicial closed model structure* goes back to Quillen [8]. As can be expected from our constructions of $(\boldsymbol{K} \times I)_*$ and $\boldsymbol{K}_*^I$ as well as the functorial factorization in CM5, the model structure of $\mathfrak{Ch}$ will be (as long as it is defined) a simplicial one. Details will be given elsewhere.

9. Chain functors and associated homology theories

In this appendix we present for the convenience of the reader some material about the definition and the motivation of chain functors without proofs. Concerning details as well as further references, we refer to [1].

It would be advantageous to define a homology theory $h_*(\,)$ as the derived homology of a functor

$$C_* : \mathfrak{K} \to \boldsymbol{ch},$$

$\mathfrak{K}$ = the category on which h_* is defined. For us this will be always either a subcategory of the category of all pairs of topological spaces, or of pairs of spectra or of pairs of CW spaces, of CW spectra, or their simplicial counterparts. ***ch*** denotes the category of chain complexes (i.e., $C_* = \{C_n, d_n,\ n \in \mathbb{Z},\ d^2 = 0\} \in \boldsymbol{ch}$).

Let $(X, A) \in \mathfrak{K}$ be a pair, then one would like to have an exact sequence (writing $C_*(X)$ instead of $C_*(X, \emptyset)$)

$$0 \to C_*(A) \xrightarrow{i_\#} C_*(X) \xrightarrow{j_\#} C_*(X, A) \to 0 \tag{1}$$

such that the associated boundary $\bar{\partial} : H_n(C_*(X, A)) \to H_{n-1}(C_*(A))$ induces the boundary $\partial : h_n(X, A) \to h_{n-1}(A)$ of the homology theory $h_*(\ )$.

In accordance with [2] we call a homology with this property *flat*. Due to a result of Burdick, Conner and Floyd (see [1] or [3] for further reference) this implies for $\mathfrak{K}$ = category of CW pairs, that $h_*(\ )$ is a sum of ordinary homology theories, i.e., of those satisfying a dimension axiom, although not necessarily in dimension 0.

We call a functor C_* being equipped with a short exact sequence (1), determining the boundary operator, a *chain theory* for h_*. The non-existence of such a chain theory gives rise to the theory of chain functors.

A chain functor $\boldsymbol{C}_* = \{C_*, C'_*, l, i', \kappa, \varphi\}$ is a pair of functors $C_*, C'_* : \mathfrak{K} \to \boldsymbol{ch}$, natural inclusions $i' \colon C_*(A) \subset C'_*(X, A)$, $l \colon C'_*(X, A) \subset C_*(X, A)$, non-natural chain mappings

$$\varphi : C'_*(X, A) \to C_*(X), \qquad \kappa : C_*(X) \to C'_*(X, A),$$

satisfying conditions CH(1)–CH(7) below:

CH(1). *There exist* (*of course in general non-natural*) *chain homotopies* $\varphi\kappa \simeq 1$, $j_\#\varphi \simeq l$ $(j\colon X \subset (X, A))$, *as well as an identity*

$$\kappa i_\# = i', \qquad i\colon A \subset X.$$

CH(2). *All inclusions* $k\colon (X, A) \subset (Y, B)$ *are supposed to induce monomorphisms on* C_*. *All* $C_*(X, X)$ *are acyclic.*

It should be observed, that the chain complexes $C_*(X, A)$ appearing in (1) are not identical with the chain complexes $C_*(X, A)$ appearing in a chain functor. The latter have the property that for all pairs (X, A) one has inclusions $C_*(X) = C_*(X, \emptyset) \subset C_*(X, A) \subset C_*(X, X)$. These groups cannot be members of a short exact sequence (1).

Needless to say, that C'_*, as well as ϕ, κ are *not* determined by the functor $C_*(\cdots, \cdots)$ but are additional ingredients of the structure of a chain functor.

Instead of the exact sequence (1) for *chain theories* we are now, in the case of a *chain functor* dealing with the sequence

$$0 \to C_*(A) \xrightarrow{i'} C'_*(X, A) \xrightarrow{p} C'_*(X, A)/\operatorname{im} i' \to 0 \tag{2}$$

and there exists a homomorphism

$$\psi : H_*\big(C'_*(X, A)/\operatorname{im} i'\big) \to H_*\big(C_*(X, A)\big), \tag{3}$$
$$[z'] \mapsto \big[l(z') + q_\#(\bar{a})\big],$$

where $z' \in C'_*(X, A)$, $dz' \in \operatorname{im} i'$, $q\colon (A, A) \subset (X, A)$, $\bar{a} \in C_*(A, A)$, $d\bar{a} = -dz'$. By this assignment ψ is readily defined.

CH(3). *It is assumed that ψ is epic.*

Since $C_*(A, A)$ is acyclic and $dz' \in \operatorname{im} i'$, there exists an $\bar{a}$ with $q_\#(\bar{a}) = -dl(z')$ and $[l(z') + q_\#(\bar{a})]$ turns out to become independent of the choice of $\bar{a}$.

This assumption implies that each cycle $z \in C_*(X, A)$ is homologous to a cycle of the form $l(z') + q_\#(\bar{a})$, with z' being a *relative* cycle, the analogue of a classical relative cycle $z \in C_*(X)$ with $dz \in \operatorname{im} i_\#$, whenever (1) holds, i.e., whenever we are dealing with a chain theory.

Suppose $\bar{\partial} : H_n(C'_*(X, A)/\operatorname{im} i') \to H_{n-1}(C_*(A))$ is the boundary induced by the exact sequence (2).

CH(4). *We assume*

$$\ker \psi \subset \ker \bar{\partial}, \tag{4}$$

Moreover

$$\ker j_* \subset \ker p_*\kappa_*, \tag{5}$$

with, e.g., κ_* denoting the mapping induced by κ for the homology groups.

CH(5). *Homotopies $H : (X, A) \times I \to (Y, B)$ induce chain homotopies $D(H) : C_*(X, A) \to C_{*+1}(Y, B)$ naturally and compatible with i' and l.*

The derived (or associated) homology of a chain functor

$$h_*(X, A) = H_*\big(C_*(X, A)\big),$$

respectively for the induced mappings, is endowed with a boundary operator

$$\partial : H_n\big(C_*(X, A)\big) \to H_{n-1}\big(C_*(A)\big),$$

determined by $\bar{\partial}$:

Given $\zeta \in H_n(C_*(X, A))$ we choose a lift z', which exists by CH(3), a representative $l(z') + q_\#(\bar{a}) \in \zeta$ and set

$$\partial\zeta = \bar{\partial}[z'] = \big[i'^{-1} dz'\big].$$

This turns out to be independent of the choices involved.

This $h_*(\ )$ satisfies all properties of a homology theory eventually with the exception of an excision. Let us assume that in $\mathfrak{K}^2$ there are some mappings $p : (X, A) \to (X', A')$ serving as *excision maps* (of some kind, e.g., $p : (X, A) \to (X/A, \star)$). Then it is convenient to add:

CH(6). *Let p be an excision map then $p_* = H_*(C_*(p))$ is required to be an isomorphism.*

This $H_*(C_*(\,)) = h_*(\,)$ turns out to be a homology theory. Moreover under very general conditions on $\mathfrak{K}$, every homology theory $h_*(\,)$ is isomorphic to the derived homology of some chain functor (see [1] for further references).

Let $\lambda: C_* \to L_*$, $\lambda': C'_* \to L'_*$ be natural transformations, where $\boldsymbol{C}_*$, $\boldsymbol{L}_*$ are chain functors, compatible with i', l and the natural homotopies of CH(5), then we call $\lambda: \boldsymbol{C}_* \to \boldsymbol{L}_*$ a *transformation of chain functors*. Such a transformation induces obviously a transformation $\lambda_*: H_*(\boldsymbol{C}_*) \to H_*(\boldsymbol{L}_*)$ of the derived homology. This furnishes a category $\mathfrak{Ch}$ of chain functors. A *weak equivalence* in $\mathfrak{Ch}$ is a $\lambda: \boldsymbol{C}_* \to \boldsymbol{L}_*$ which has a homotopy inverse.

Furthermore we can introduce the homotopy category $\mathfrak{Ch}_h$ with chain homotopy classes of transformations of chain functors as morphisms (alternatively: $\mathfrak{Ch}_h = \mathfrak{Ch}/\{\mathfrak{W}\}$, $\mathfrak{W} =$ class of weak equivalences, i.e., all *weak* equivalences are becoming *strict* equivalences (hence isomorphisms) in $\mathfrak{Ch}_h$, see [5, Theorem 6.2], in a slightly different notation).

In order to establish all this it becomes sometimes necessary to assume that a chain functor $\boldsymbol{C}_*$ satisfies:

CH(7). *All chain complexes $C_*(X, A)$ are free (i.e., all $C_n(X, A)$ are free abelian groups).*

However this is not a severe restriction as the following lemma ensures:

Lemma 9.1. *To any chain functor $\boldsymbol{C}_*$ (satisfying* CH(1)–CH(6)) *there exists a canonically defined chain functor $\boldsymbol{L}_*$ and a transformation of chain functors $\lambda: \boldsymbol{L}_* \to \boldsymbol{C}_*$ compatible with φ and κ, inducing an isomorphism of homology, such that*:

(L1) *All $L_*(X, A)$ have a natural basis $\boldsymbol{b}$ in all dimensions*;

(L2) $b \in \boldsymbol{b} \Rightarrow db \in \boldsymbol{b}$; $b \in \boldsymbol{b} \Rightarrow i'(b) \in \boldsymbol{b}$, $l(b) \in \boldsymbol{b}$, *whenever this is defined and makes sense*;

(L3) *For every homology class $\zeta \in H_*(C_*(X, A))$ there exists a basic (with respect to the basis in* (L1)) $z \in (\lambda_*)^{-1}\zeta$.

Proof. Consider the free abelian group $F(C_n(X, A))$ generated by the elements of $C_*(X, A)$ and convert this into a chain complex $F_*(X, A)$ in an obvious way. To each $a \in C_n(X, A)$ corresponds a basic $\bar{a} \in F(C_n(X, A))$. Let i: $M_* \subset F_*$ be the subcomplex generated by all elements of the form $\sum m_i \bar{a}_i - \overline{\sum m_i a_i}$ and define

$$L_*(X, A) = F_*(X, A) \cup_i \operatorname{cone} M_*(X, A).$$

This furnishes evidently a functor into the category of chain complexes. We set $\lambda(\sum m_i \bar{a}_i) = \sum m_i a_i$, and $\lambda \mid M_* = 0$.

Moreover $\sum m_i \bar{a}_i \in L'_*$ whenever all $a_i \in C'_*$, respectively for the elements of cone M_*. This implies that (L2) holds. One can immediately equip $\boldsymbol{L}_*$ and λ with the structure of a chain functor, respectively of a transformation between chain functors.

Every cycle $z \in Z_n(C_*(X, A))$ is of the form $\lambda(\bar{z}) = z$, hence λ_* is epic. Any cycle $\tilde{z} \in Z_n(L_*(X, A))$ is homologous to a $\bar{z}$, $z \in Z_n(C_*(X, A))$:

Suppose $\tilde{z} = \sum m_i \bar{a}_i + c,\ c \in \operatorname{cone} M_*$, then we have $\tilde{z} = \bar{a} + c_1,\ c_1 \in \operatorname{cone} M_*$, hence $d\bar{a} = \overline{da} \in \operatorname{cone} M_*$, implying that $d\bar{a} = \overline{da} = 0$. So $\bar{a}$ and c_1 are cycles, and since c_1 is bounding in cone M_*, we conclude that $\tilde{z} \sim \bar{a}$.

If $z = dx$, then $\bar{z} = d\bar{x}$ and λ_* is therefore monic.

This completes the proof of the lemma. □

We will in the present paper without further mentioning assume, that all chain functors have such a natural basis satisfying (L1)–(L3) *eventually with the exception of the first property in* (L2) ($b \in \boldsymbol{b} \Rightarrow db \in \boldsymbol{b}$).

The following assertion is needed at some occasions in the present paper:

Lemma 9.2. *Suppose* $\{C_*, C'_*, i', l, \varphi, \kappa\}$ *satisfies all properties of a chain functor eventually without* CH(3), CH(4), CH(6). *Assume that there exists a chain functor* $\boldsymbol{L}_* \in \mathfrak{C}h$, q: $\boldsymbol{L}_* \subset C_*$ *such that* q *preserves all structure and induces an isomorphism of homology, then* C_* *is a chain functor.*

Proof. Follows immediately by checking the properties of a chain functor. □

Finally we repeat the definition of an *irregular chain functor* (see [1]) Definition 4.1 for more details or [2, Section 3] for an example): $\{C_*, C'_*, \varphi, \kappa, i', l\}$ satisfies all conditions of a chain functor, but we do no longer require (a) that all inclusions induce isomorphisms; (b) nor that i', l are necessarily monomorphisms; (c) nor any excision properties. Whenever we talk about a *regular* chain functor, we mean that it is not irregular. The role of the unnatural mappings φ and κ seems at the first glance to be a little mysterious.

A chain functor $\boldsymbol{K}_*$ is called *flat* whenever φ, κ and the chain homotopies $\varphi\kappa \simeq 1$, $j_\# \varphi \simeq l$ are natural. In the beginning we introduced the concept of a flat homology theory.

Theorem 9.3 [4, Theorem 3.3]. *The following conditions for a homology theory are equivalent*:

(1) h_* *is flat*;
(2) *there exists a flat chain functor associated with* h_*.

Corollary 9.4 [4, Corollary 3.4]. *For a homology theory defined on the category of* CW *spaces the conditions* (1), (2) *are equivalent to* (3) h_* *is the direct sum of ordinary homology theories.*

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