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Left-right noncommutative Poisson algebras

Research Article

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Abstract: The notions of left-right noncommutative Poisson algebra ($\mathsf{NP}^{\mathsf{lr}}$-algebra) and left-right algebra with bracket $\mathsf{AWB}^{\mathsf{lr}}$ are introduced. These algebras are special cases of NLP-algebras and algebras with bracket AWB, respectively, studied earlier. An $\mathsf{NP}^{\mathsf{lr}}$-algebra is a noncommutative analogue of the classical Poisson algebra. Properties of these new algebras are studied. In the categories $\mathsf{AWB}^{\mathsf{lr}}$ and $\mathsf{NP}^{\mathsf{lr}}$-algebras the notions of actions, representations, centers, actors and crossed modules are described as special cases of the corresponding well-known notions in categories of groups with operations. The cohomologies of $\mathsf{NP}^{\mathsf{lr}}$-algebras and $\mathsf{AWB}^{\mathsf{lr}}$ (resp. of NP^{r}-algebras and $\mathsf{AWB}^{\mathsf{r}}$) are defined and the relations between them and the Hochschild, Quillen and Leibniz cohomologies are detected. The cases P is a free $\mathsf{AWB}^{\mathsf{r}}$, the Hochschild or/and Leibniz cohomological dimension of P is $\leq n$ are considered separately, exhibiting interesting possibilities of representations of the new cohomologies by the well-known ones and relations between the corresponding cohomological dimensions.

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Keywords: Poisson algebra • Algebras with bracket • Leibniz algebra • Representation • Left-right noncommutative Poisson algebra cohomology • Hochschild, Quillen, Leibniz cohomologies • Cohomological dimension • Extension • Action • Universal strict general actor • Center

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Dedicated to the memory of J.-L. Loday

1. Introduction

In [4] there are defined and studied noncommutative Leibniz–Poisson algebras, denoted as NLP-algebras. These are associative algebras P, generally noncommutative, over a ring $\mathbb{K}$ with unit, with bracket operation, according to which they are Leibniz algebras over $\mathbb{K}$ and for which the Poisson identity

$$[a\cdot b, c] = a\cdot[b, c] + [a, c]\cdot b \tag{1}$$

holds for all $a, b, c \in P$. In this paper this identity will be called the left Poisson identity, and the above defined algebra a left noncommutative Poisson algebra, shortly a left NP-algebra or NP^{l}-algebra. It is natural to consider right NP-algebras over a ring $\mathbb{K}$ (NP^{r} in what follows), which are defined in an analogous way replacing the left Poisson identity with the right one:

$$[a, b\cdot c] = b\cdot[a, c] + [a, b]\cdot c, \qquad a, b, c \in P. \tag{2}$$

A left-right NP-algebra ($\mathsf{NP}^{\mathsf{lr}}$) over a ring $\mathbb{K}$ is an algebra, which is an associative and Leibniz algebra and satisfies both (1) and (2); it is a noncommutative analogue of the classical Poisson algebra. In the same way, an algebra with bracket AWB defined in [9], see below Definition 2.1, is a left AWB, which will be denoted by $\mathsf{AWB}^{\mathsf{l}}$. Obviously, we can define an in analogous ways $\mathsf{AWB}^{\mathsf{r}}$ and $\mathsf{AWB}^{\mathsf{lr}}$. Thus we obtain the following commutative diagram of the corresponding categories and inclusion functors:

$$\begin{array}{ccccc}
\mathbf{AWB}^{\mathsf{r}} & \hookleftarrow & \mathbf{AWB}^{\mathsf{lr}} & \hookrightarrow & \mathbf{AWB}^{\mathsf{l}} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{NP}^{\mathsf{r}} & \hookleftarrow & \mathbf{NP}^{\mathsf{lr}} & \hookrightarrow & \mathbf{NP}^{\mathsf{l}}.
\end{array}$$

The purpose of this paper is to study properties of the above defined algebras, including the construction of appropriate complexes for the definition of cohomology, to investigate and to establish relations between them and with the properties of the underlying associative and Leibniz algebras and the corresponding Hochschild [16], Quillen [32] and Leibniz cohomologies [26]. We will see that left-right AWB do not inherit all the properties of left or right AWB. But nevertheless due to the specific way of construction of cohomology complexes, they have interesting intersections and relations with each other. An analogous picture we have for left-right NP-algebras.

In Section 2 we present definitions of new algebras and examples. For convenience of the reader we include the definition of category of interest and some examples as well. In Section 3 we construct free $\mathsf{AWB}^{\mathsf{r}}$. The construction of free $\mathsf{AWB}^{\mathsf{l}}$ was given in [9], our approach is different, which gives the construction of free $\mathsf{AWB}^{\mathsf{l}}$ as well. The properties of free objects are investigated, in particular, it is proved that if P is a free $\mathsf{AWB}^{\mathsf{r}}$, then the underlying associative algebra of P is free as well. We prove that analogous results for $\mathsf{AWB}^{\mathsf{lr}}$ and $\mathsf{NP}^{\mathsf{lr}}$-algebras are not true in general. In Section 4 we describe action conditions, we present definitions of derivation, extension, crossed module and representation in the categories of the new algebras. All these are special cases of the well-known definitions in categories of groups with operations. It turned out that the category of $\mathsf{NP}^{\mathsf{lr}}$-algebras is a category of interest, from which, applying the general result of [29], we conclude that this category is action accessible in the sense of [3]. We construct the universal strict general actor USGA(A) of an $\mathsf{NP}^{\mathsf{lr}}$-algebra A, defined in [6] in a category of interest; we describe center and define actor of $\mathsf{NP}^{\mathsf{lr}}$-algebras and, as a special case of the result in [6], we obtain the necessary and sufficient conditions for the existence of an actor of A in terms of USGA(A). We plan to consider the problem of the existence of an actor in $\mathbf{NP}^{\mathsf{lr}}$, or to find individual objects in this category with actor. According to [2] this problem in categories of interest is equivalent to the amalgamation property for protosplit monomorphisms. Here in $\mathbf{NP}^{\mathsf{lr}}$ we determine the full subcategory of commutative von Neumann regular rings with trivial bracket operations; by the result of [2] we have that in this category there always exists an actor for any algebra, and moreover, on the base of the result of the same paper and [10] we conclude that in $\mathbf{NP}^{\mathsf{lr}}$ there exists a subcategory which satisfies the amalgamation property. This result can be applied to the characterization of effective codescent morphisms in this subcategory. In Section 5 we construct complexes and define the corresponding cohomologies $H^n_{\mathsf{NP}^{\mathsf{lr}}}(P, M)$, $H^n_{\mathsf{AWB}^{\mathsf{lr}}}(P, M)$, where $P \in \mathbf{NP}^{\mathsf{lr}}$ ($P \in \mathbf{AWB}^{\mathsf{lr}}$, respectively), and M denotes the corresponding representations of P. In what follows under NP-algebras we will mean NP^{r}-, NP^{l}- and $\mathsf{NP}^{\mathsf{lr}}$-algebras, and under AWB we will mean $\mathrm{AWB}^{\mathrm{l}}$, $\mathrm{AWB}^{\mathrm{r}}$ and $\mathrm{AWB}^{\mathrm{lr}}$. We investigate the relation of the second cohomology with extensions. Like in the case of $\mathrm{AWB}^{\mathrm{l}}$ [9], we obtain the isomorphism $H^{n+1}_{\mathrm{AWB}^{\mathrm{r}}}(P,M) \approx H^{n}_{\mathrm{Q}}(P,M)$ with the Quillen cohomology. From the constructions of the cohomology complexes we detect short exact sequences, from which there follow long exact sequences involving cohomologies, relating NP, AWB, Hochschild and Leibniz cohomologies with each other. The special cases, where P is a free $\mathrm{AWB}^{\mathrm{r}}$, the Leibniz cohomological dimension or/and the Hochschild cohomological dimension of P is $n/\leq n$ give interesting results, in particular, in these cases we can represent the new cohomologies by means of the well-known ones and estimate cohomological dimensions of the corresponding AWB and NP-algebras. Note that an operadic approach to similar kind of investigations would be interesting, see e.g. [12, 14, 17, 28]. The cohomology of classical Poisson algebras was defined and studied by Huebschmann [18]. Different types of noncommutative Poisson algebras were studied in [20, 21, 33, 34].

2. Preliminary definitions and examples

Let $\mathbb{K}$ be a commutative ring with unit. We recall that a *Leibniz algebra* [22, 23] A over $\mathbb{K}$ is a $\mathbb{K}$-module equipped with a $\mathbb{K}$-module homomorphism $[-,-]\colon A\otimes A \to A$, called a square bracket, satisfying the Leibniz identity

$$\big[a,[b,c]\big] = \big[[a,b],c\big] - \big[[a,c],b\big],$$

for all $a,b,c \in A$. Here and in what follows $\otimes$ means $\otimes_{\mathbb{K}}$.

Definition 2.1.

(i) A left (resp. right) algebra with bracket over $\mathbb{K}$, for short, $\mathrm{AWB}^{\mathrm{l}}$ (resp. $\mathrm{AWB}^{\mathrm{r}}$), is an associative algebra equipped with a $\mathbb{K}$-module homomorphism $[-,-]\colon A\otimes A \to A$, such that (1) (resp. (2)) holds.

(ii) A left-right algebra with bracket over $\mathbb{K}$ (for short, $\mathrm{AWB}^{\mathrm{lr}}$) is an associative algebra A equipped with a $\mathbb{K}$-module homomorphism $[-,-]\colon A\otimes A \to A$, such that (1) and (2) hold.

As we have noted in the introduction, $\mathrm{AWB}^{\mathrm{l}}$ is the same as the algebra with bracket AWB defined in [9], and the NP^{l}-algebra is the NLP-algebra defined in [4]. Morphisms between the above defined algebras are $\mathbb{K}$-module homomorphisms preserving the dot and bracket operations. The corresponding categories will be denoted by $\mathbf{NP}^{\mathrm{l}}$, $\mathbf{NP}^{\mathrm{r}}$, $\mathbf{NP}^{\mathrm{lr}}$, $\mathbf{AWB}^{\mathrm{l}}$, $\mathbf{AWB}^{\mathrm{r}}$ and $\mathbf{AWB}^{\mathrm{lr}}$. The sign "$\cdot$" of the dot operation will be often omitted, when it is clear from the context, which operation is meant between the elements, e.g. $a\cdot b$ will be written as ab.

Example 2.2.

(a) Every Poisson algebra is an $\mathrm{NP}^{\mathrm{lr}}$-algebra.

(b) Any Leibniz algebra A is an $\mathrm{NP}^{\mathrm{lr}}$-algebra with trivial dot operation, i.e. $ab=0$, $a,b\in A$.

(c) Any associative algebra A is an $\mathrm{NP}^{\mathrm{lr}}$-algebra with the usual bracket $[a,b]=ab-ba$, $a,b\in A$.

(d) Let A be an associative algebra and let $D\colon A\to A$ be a square zero derivation, i.e. $D^2=0$ and $D(ab)=(Da)b+a(Db)$. Define the bracket operation by $[a,b]=a(Db)-(Db)a$. It is easy to check that with this bracket operation A is an NP^{l}-algebra, but not NP^{r}-algebra.

(e) Let A be an associative algebra from the case (d), where the bracket operation is defined by $[a,b]=(Da)b-b(Da)$. Then A is an NP^{r}-algebra, but not NP^{l}-algebra.

(f) Let A be an associative algebra with the property that $abc=bac=acb$, for any $a,b,c\in A$, and let $D\colon A\to A$ be a square zero derivation. Then A is an $\mathrm{NP}^{\mathrm{lr}}$-algebra with respect to the rule $[a,b]=a(Db)-(Db)a$.

(g) Every NP-algebra is an AWB.

(h) The following algebra is $\mathrm{AWB}^{\mathrm{r}}$ (resp. $\mathrm{AWB}^{\mathrm{l}}$), but not an NP^{r}-algebra (resp. NP^{l}-algebra). Let A be an associative algebra with a linear application $D\colon A\to A$. Then A is $\mathrm{AWB}^{\mathrm{r}}$ (resp. $\mathrm{AWB}^{\mathrm{l}}$) where the bracket operation is defined by $[a,b]=(Da)b-b(Da)$ (resp. by $[a,b]=a(Db)-(Db)a$); for the left AWB this example was given in [9].

(i) Let A be an associative algebra with a linear application $D\colon A \to A$, satisfying the condition $(Da)b - b(Da) = a(Db) - (Db)a$, for any $a, b \in A$. Then the algebra defined in the case (h) is an $\mathrm{AWB}^{\mathrm{lr}}$.

(j) If the linear application $D\colon A \to A$ from the case (i) is a square zero derivation like in case (d), then the algebra with respect to the square bracket $[a, b] = (Da)b - b(Da)$ is an $\mathrm{NP}^{\mathrm{lr}}$-algebra.

(k) Any associative dialgebra [25] with respect to the operations $ab = a \vdash b$, $[a, b] = a \vdash b - b \dashv a$ (resp. $[a, b] = a \dashv b - b \vdash a$) is an $\mathrm{AWB}^{\mathrm{r}}$ (resp. $\mathrm{AWB}^{\mathrm{l}}$), but not an $\mathrm{AWB}^{\mathrm{l}}$ (resp. $\mathrm{AWB}^{\mathrm{r}}$).

(l) The algebras defined in the case (k) generally are not NP^{r} and NP^{l}-algebras, respectively. The greatest quotient of these algebras by the congruence relation generated by the relation $\big[a, [b, c]\big] \sim \big[[a, b], c\big] - \big[[a, c], b\big]$, for any $a, b, c \in A$, gives examples of NP^{r}- and NP^{l}-algebras, respectively. For NP^{l}-algebras this example was given in [4].

(m) The algebra defined in the case (k), under the additional condition $a \vdash b - b \dashv a = a \dashv b - b \vdash a$, for any $a, b \in A$, is an $\mathrm{NP}^{\mathrm{lr}}$-algebra.

(n) For an example of a graded version of NP^{l}-algebra coming from physics see [19].

(o) See Section 3 for the constructions of free $\mathrm{AWB}^{\mathrm{r}}$ and $\mathrm{AWB}^{\mathrm{l}}$.

Definition 2.3.

Let $P \in \mathbf{NP}^{\mathrm{lr}}$. A subalgebra of P is an associative and Leibniz subalgebra of P. A subalgebra R of P is called a two-sided ideal if $a \cdot r, r \cdot a, [a, r], [r, a] \in R$, for all $a \in P$, $r \in R$.

The inclusion functor $\mathrm{inc}\colon \mathbf{Poiss} \to \mathbf{NP}$ from the category of Poisson algebras to the category of NP-algebras, i.e. left, right or left-right noncommutative Poisson algebras, respectively, has a left adjoint $(-)_{\mathrm{Poiss}}\colon \mathbf{NP} \to \mathbf{Poiss}$. This functor assigns to an NP-algebra P the quotient algebra of P with the smallest two-sided ideal spanned by the elements $[x, x]$ and $xy - yx$, for all $x, y \in P$.

Lemma 2.4.

For a set S any word with the elements from S, brackets and dots as formal operations, which have a sense, can be rewritten in a unique way under the relations of associativity and (2) *(resp.* (1)*) for the dot operation and the bracket and the dot operations, respectively.*

Proof. It is sufficient to note that two different decompositions of the words of the type $[a, b \cdot c \cdot d]$ (resp. $[a \cdot b \cdot c, d]$) in any $\mathrm{AWB}^{\mathrm{r}}$ (resp. $\mathrm{AWB}^{\mathrm{l}}$) corresponding to the words $\big[a, b \cdot (c \cdot d)\big]$ and $\big[a, (b \cdot c) \cdot d\big]$ (resp. $\big[a \cdot (b \cdot c), d\big]$ and $\big[(a \cdot b) \cdot c, d\big]$) give the same expression

$$b \cdot c \cdot [a, d] + b \cdot [a, c] \cdot d + [a, b] \cdot c \cdot d$$

(resp. $a \cdot b \cdot [c, d] + a \cdot [b, d] \cdot c + [a, d] \cdot b \cdot c$). □

Consider the elements $\big[a, [b, c \cdot d]\big]$, $\big[a, [b \cdot c, d]\big]$, $\big[a \cdot b, [c, d]\big]$ and $[a \cdot b, c \cdot d]$ in the category of $\mathrm{NP}^{\mathrm{lr}}$-algebras. The two different decompositions of the first and the fourth elements give the identities

$$[a, c] \cdot [b, d] + [a, c] \cdot [d, b] + [b, c] \cdot [a, d] + [c, b] \cdot [a, d] = 0, \tag{3}$$

$$a \cdot c \cdot [b, d] + [a, c] \cdot d \cdot b = c \cdot a \cdot [b, d] + [a, c] \cdot b \cdot d. \tag{4}$$

The last identity is true in the category of $\mathrm{AWB}^{\mathrm{lr}}$ as well.

The two different decompositions of the second and the third elements do not give identities. Analogously, considering two different decompositions of the first element in the category of NP^{r}-algebras, and the second element in the category of NP^{l}-algebras we obtain, respectively, the identities

$$\big[[a, c] \cdot d, b\big] = \big[[a, c], b\big] \cdot d - [a, c] \cdot [b, d] - [b, c] \cdot [a, d] + c \cdot \big[[a, d], b\big] - \big[c \cdot [a, d], b\big], \tag{5}$$

$$\big[a, b \cdot [c, d]\big] + \big[a, [b, d] \cdot c\big] = \big[[a, b \cdot c], d\big] - \big[[a, d], b \cdot c\big].$$

In the categories $\mathrm{AWB}^{\mathrm{lr}}$ and $\mathrm{NP}^{\mathrm{lr}}$-algebras we have the following identity as well:

$$[a\cdot b, c]-[a, c\cdot b]+[b\cdot c, a]-[b, a\cdot c]+[c\cdot a, b]-[c, b\cdot a]=0. \tag{6}$$

By decomposition of all summands except the first one in the right side of (5) according to the identity (2) we obtain the following:

$$\big[[a, c\cdot d], b\big]=-\big[b,[a, c]\cdot d\big]+\big[[b, a], c\big]\cdot d-\big[[b, c], a\big]\cdot d-\big[a,[b, c]\cdot d\big]+\big[[a, b], c\big]\cdot d+\big[[a, d], c\cdot b\big]-\big[[a, d], c\big]\cdot b.$$

These identities will be applied in the next section and in the construction of free objects in the new categories.

Recall that an action (a derived action in the sense of [30]) of P on M for associative algebras is given by two $\mathbb{K}$-module homomorphisms $-\cdot-\colon P\otimes M\to M$, $-\cdot-\colon M\otimes P\to M$ with the conditions

$$\begin{aligned} p\cdot(m_1\cdot m_2)&=(p\cdot m_1)\cdot m_2, & m_1\cdot(p\cdot m_2)&=(m_1\cdot p)\cdot m_2, & (m_1\cdot m_2)\cdot p&=m_1\cdot(m_2\cdot p),\\ p_1\cdot(p_2\cdot m)&=(p_1\cdot p_2)\cdot m, & p_1\cdot(m\cdot p_2)&=(p_1\cdot m)\cdot p_2, & m\cdot(p_1\cdot p_2)&=(m\cdot p_1)\cdot p_2. \end{aligned}$$

An action of P on M for Leibniz algebras is given by two $\mathbb{K}$-module homomorphisms $[-,-]\colon P\otimes M\to M$, $[-,-]\colon M\otimes P\to M$ with the conditions

$$\begin{aligned} \big[p,[m_1, m_2]\big]&=\big[[p, m_1], m_2\big]-\big[[p, m_2], m_1\big], & \big[p_1,[p_2, m]\big]&=\big[[p_1, p_2], m\big]-\big[[p_1, m], p_2\big],\\ \big[m_1,[p, m_2]\big]&=\big[[m_1, p], m_2\big]-\big[[m_1, m_2], p\big], & \big[p_1,[m, p_2]\big]&=\big[[p_1, m], p_2\big]-\big[[p_1, p_2], m\big],\\ \big[m_1,[m_2, p]\big]&=\big[[m_1, m_2], p\big]-\big[[m_1, p], m_2\big], & \big[m,[p_1, p_2]\big]&=\big[[m, p_1], p_2\big]-\big[[m, p_2], p_1\big]. \end{aligned}$$

Here we recall the definition of category of interest. Let $\mathbf{C}$ be a category of groups with a set of operations Ω and with a set of identities $\mathbf{E}$, such that $\mathbf{E}$ includes the group identities and the following conditions hold. If Ω_i is the set of i-ary operations in Ω, then

(a) $\Omega=\Omega_0\cup\Omega_1\cup\Omega_2$;

(b) the group operations (written additively: $(0,-,+)$) are elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega_2'=\Omega_2\setminus\{+\}$, $\Omega_1'=\Omega_1\setminus\{-\}$ and assume that if $*\in\Omega_2$, then Ω_2' contains $*^{\circ}$ defined by $x*^{\circ}y=y*x$. Assume further that $\Omega_0=\{0\}$;

(c) for each $*\in\Omega_2'$, $\mathbf{E}$ includes the identity $x*(y+z)=x*y+x*z$;

(d) for each $\omega\in\Omega_1'$ and $*\in\Omega_2'$, $\mathbf{E}$ includes the identities $\omega(x+y)=\omega(x)+\omega(y)$ and $\omega(x)*y=\omega(x*y)$.

Note that the group operation is denoted additively, but it is not commutative in general. A category $\mathbf{C}$ defined above is called a *category of groups with operations*. The idea of the definition comes from [15] and the axioms are from [30] and [31]. We formulate two more axioms on $\mathbf{C}$ [30, Axioms (7) & (8)].

If C is an object of $\mathbf{C}$ and $x_1, x_2, x_3\in C$ then:

Axiom 1. $x_1+(x_2*x_3)=(x_2*x_3)+x_1$ for each $*\in\Omega_2'$.

Axiom 2. For each ordered pair $(*,\overline{*})\in\Omega_2'\times\Omega_2'$ there is a word W such that

$$(x_1*x_2)\overline{*}x_3 = W\big(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1, (x_3x_2)x_1, x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2\big),$$

where each juxtaposition represents an operation in Ω_2'.

A category of groups with operations satisfying Axioms 1 and 2 is called a *category of interest* in [30].

Denote by $\mathbf{E}_G$ the subset of identities of $\mathbf{E}$ which includes the group laws and the identities (c) and (d). We denote by $\mathbf{C}_G$ the corresponding category of groups with operations. Thus we have $\mathbf{E}_G\hookrightarrow\mathbf{E}$, $\mathbf{C}=(\Omega,\mathbf{E})$, $\mathbf{C}_G=(\Omega,\mathbf{E}_G)$ and there is a full inclusion functor $\mathbf{C}\hookrightarrow\mathbf{C}_G$. The category $\mathbf{C}_G$ is called a *general category of groups with operations* of a category of interest $\mathbf{C}$ (see [6, 8]).

Example 2.5 (categories of interest).

The categories of groups, modules over a ring, associative algebras, associative commutative algebras, Lie algebras, Leibniz algebras are categories of interest. In the example of groups $\Omega_2' = \varnothing$. In the case of associative algebras with multiplication represented by $*$, we have $\Omega_2' = \{*, *^\circ\}$. For Lie algebras take $\Omega_2' = \{[\cdot,\cdot],[\cdot,\cdot]^\circ\}$ (where $[a,b]^\circ = [b,a] = -[a,b]$). For Leibniz algebras, take $\Omega_2' = \{[\cdot,\cdot],[\cdot,\cdot]^\circ\}$ (here $[a,b]^\circ = [b,a]$). The category of alternative algebras is a category of interest as well [30] (see also [7]). The categories of crossed modules and precrossed modules in the category of groups, respectively, are equivalent to categories of interests (see e.g. [5, 6]). According to [2] the category of commutative von Neumann regular rings is isomorphic to a category of interest. In [29] there are given new examples of categories of interest, these are associative dialgebras and associative trialgebras. Dialgebras and trialgebras were defined by Loday [24, 25, 27]. As it is noted in [30], Jordan algebras do not satisfy Axiom 2. It is easy to see that $\mathbf{NP}^{\mathrm{lr}}$ is a category of interest; while the categories $\mathbf{AWB}^{\mathrm{lr}}$, $\mathbf{AWB}^{\mathrm{r}}$, $\mathbf{AWB}^{\mathrm{l}}$, $\mathbf{NP}^{\mathrm{r}}$ and $\mathbf{NP}^{\mathrm{l}}$ are not categories of interest, they do not satisfy Axiom 2 of the definition.

3. Free objects in AWB

For any set X we shall build a free $\mathrm{AWB}^{\mathrm{r}}$ over a ring $\mathbb{K}$. Denote by $W(X)$ the set, which contains X and all formal combinations (words) of two operations $(\cdot,[-,-])$ with the elements from X, which have a sense, and which do not contain elements of the form $[a, b\cdot c]$, where a,b,c are from X or are combinations of elements of X and dot and bracket operations. Let $W_n(X)$ be the subset of those words of $W(X)$, which contain n elements of X, i.e. the number of both operations together is $n-1$, $n\geq 1$; we say that this word is of length n. Obviously, $W(X)=\bigcup_{n\geq 1} W_n(X)$. We define the following maps:

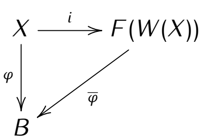
$$\sigma_{n,m},\tau_{n,m}\colon\ W_n(X)\times W_m(X)\to W_{n+m}(X).$$

The map $\sigma_{n,m}$ is defined for any pair $(a,b)\in W_n(X)\times W_m(X)$ by $\sigma_{n,m}(a,b)=a\cdot b$, where the right side denotes the word from $W_{n+m}(X)$, which is defined uniquely. The map $\tau_{n,m}$ is defined only on those pairs (a,b), for which the word $[a,b]\in W_{n+m}(X)$, and by definition $\tau_{n,m}(a,b)=[a,b]$. In the case $[a,b]\notin W_{n+m}(X)$, $\tau_{n,m}$ is not defined. Let $F(W(X))$ be the free $\mathbb{K}$-module generated by the set $W(X)$. Define the dot operation on $F(W(X))$ as a linear extension of $\sigma_{n,m}$ on the whole $F(W(X))$. For those words of $F(W(X))$ on which $\tau_{n,m}$ is defined, we define the bracket operation as a linear extension on $F(W(X))$ of $\tau_{n,m}$. If the element $[a,b]\notin W_{n+m}(X)$, for $a\in W_n(X)$, $b\in W_m(X)$, then we decompose $[a,b]$ according to the identity (2) and $\mathbb{K}$-linearity of the bracket operation, until we obtain the sum of the words, which contain bracket operations only on those pairs of words, on which the bracket is already defined. Therefore we will obtain the sum $c_1+\cdots+c_k$, with $c_i\in W_{n+m}(X)$, $i=1,\ldots,k$, and by definition $[a,b]=c_1+\cdots+c_k$. By Lemma 2.4 it follows that the results of the bracket operations are defined uniquely. By construction $F(W(X))$ has a structure of $\mathrm{AWB}^{\mathrm{r}}$.

Let $i\colon X\to F(W(X))$ be the natural injection of sets.

Proposition 3.1.

For any $B\in\mathbf{AWB}^{\mathrm{r}}$ and a map $\varphi\colon X\to B$, there exists a unique homomorphism $\overline{\varphi}\colon F(W(X))\to B$ such that the following diagram commutes:



Therefore $F(W(X))$ is a free $\mathrm{AWB}^{\mathrm{r}}$ on the set X.

Proof. For any word $Q(x_1,\ldots,x_k)\in W(X)$ define a map $\varphi'\colon W(X)\to B$ by $\varphi'(Q(x_1,\ldots,x_k))=Q(\varphi(x_1),\ldots,\varphi(x_k))$. The map $\overline{\varphi}$ is defined as a $\mathbb{K}$-linear extension of φ' to $F(W(X))$. By construction of $F(W(X))$ and by application of Lemma 2.4 any element $a\in F(W(X))$ is expressed in a unique way as a $\mathbb{K}$-linear combination of the words from $W(X)$. From this it follows that $\overline{\varphi}$ is defined correctly. By the definition it is a homomorphism of $\mathrm{AWB}^{\mathrm{r}}$ and it is a unique homomorphism with the property that the diagram commutes. □

The construction of a free $\mathrm{AWB}^{\mathrm{l}}$ is similar to the construction given above; in this case we take all formal combinations (words) of two operations $(\cdot, [-,-])$ with the elements from X, which have a sense, and do not contain the elements of the form $[a\cdot b, c]$ (cf. with the construction given in [9]). The constructions of free objects in other new defined categories are much more complicated; we plan to consider them in a separate paper.

It is easy to see that the given construction defines a functor $\mathbf{F}$ from the category $\mathbf{Set}$ of sets to $\mathbf{AWB}^{\mathrm{r}}$, where $\mathbf{F}(X) = F(W(X))$, which is a left adjoint to the underlying functor

$$\mathbf{Set} \underset{\mathsf{U}}{\overset{\mathsf{F}}{\rightleftarrows}} \mathbf{AWB}^{\mathrm{r}}.$$

Analogously, for left AWB.

Let $V_A^{\mathrm{lr}}\colon \mathbf{NP}^{\mathrm{lr}} \to \mathbf{Ass}$, $V_L^{\mathrm{lr}}\colon \mathbf{NP}^{\mathrm{lr}} \to \mathbf{Leib}$ and $T_A^{\mathrm{lr}}\colon \mathbf{AWB}^{\mathrm{lr}} \to \mathbf{Ass}$ be the forgetful functors, where $\mathbf{Ass}$ and $\mathbf{Leib}$ denote the categories of associative and Leibniz algebras, respectively. The analogous meaning will have the symbols V_A^{r}, V_A^{l}, V_L^{r}, V_L^{l}, T_A^{r}, T_A^{l}.

Proposition 3.2.
If P is a free $\mathrm{AWB}^{\mathrm{r}}$, then $T_A^{\mathrm{r}}(P)$ is a free associative algebra.

Proof. Let P be a free $\mathrm{AWB}^{\mathrm{r}}$ on the set X. Denote by X' the set of all kind of those words of the type $[\dots,\dots]$, which do not contain the words of the form $[a, b\cdot c]$. Let $X_1 = X \cup X'$. Applying Lemma 2.4 it is easy to see that every element of P is decomposed in a unique way as a linear combination of the words constructed from the elements of X_1 and the dot operation. From this fact, in a similar way as it is in the proof of Proposition 3.1, it follows that $T_A^{\mathrm{r}}(P)$ is a free associative algebra on the set X_1. □

An analogous statement for $\mathrm{AWB}^{\mathrm{l}}$ is proved in [9].

Proposition 3.3.
If P is a free $\mathrm{NP}^{\mathrm{lr}}$-algebra (resp. $\mathrm{AWB}^{\mathrm{lr}}$), then $V_A^{\mathrm{lr}}(P)$ (resp. $T_A^{\mathrm{lr}}(P)$) is not a free associative algebra and $V_L^{\mathrm{lr}}(P)$ is not a free Leibniz algebra.

Proof. Let P be the free $\mathrm{NP}^{\mathrm{lr}}$-algebra on the set X. A basis for $V_A^{\mathrm{lr}}(P)$ must contain all elements from X, and all elements of the form $[a, b]$, where $a, b \in X$. From identity (3) or (4) it follows that $V_A^{\mathrm{lr}}(P)$ is not a free associative algebra. Analogously, from identity (4) we see that $T_A^{\mathrm{lr}}(P)$ is not a free associative algebra. In the case of the Leibniz algebra $V_L^{\mathrm{lr}}(P)$, its basis must contain all elements from X and all kind of elements of the form $a_1\cdot\ldots\cdot a_n$, where $a_1,\ldots,a_n \in X$, $n \geq 1$. The identity (6) proves that $V_L^{\mathrm{lr}}(P)$ is not a free Leibniz algebra. □

4. Actions, representations and crossed modules in NP and AWB

Under action we will mean a set of actions derived from the corresponding split extension, i.e. a derived action in the sense of [30]. An action for NP^{l}-algebras is defined in [4] in the following way.

Definition 4.1 ([4]).
Let $M, P \in \mathbf{NP}^{\mathrm{l}}$. We say that P acts on M if we have an action of P on M as associative and Leibniz algebras given respectively by the $\mathbb{K}$-module homomorphisms

$$-\cdot-\colon P\otimes M \to M, \qquad -\cdot-\colon M\otimes P \to M, \tag{7}$$
$$[-,-]\colon P\otimes M \to M, \qquad [-,-]\colon M\otimes P \to M, \tag{8}$$

and the following conditions hold:

$$[p_1\cdot p_2, m] = p_1\cdot[p_2, m] + [p_1, m]\cdot p_2, \qquad [p_1\cdot m, p_2] = p_1\cdot[m, p_2] + [p_1, p_2]\cdot m,$$
$$[m\cdot p_1, p_2] = m\cdot[p_1, p_2] + [m, p_2]\cdot p_1, \qquad [m_1\cdot m_2, p] = m_1\cdot[m_2, p] + [m_1, p]\cdot m_2,$$
$$[m_1\cdot p, m_2] = m_1\cdot[p, m_2] + [m_1, m_2]\cdot p, \qquad [p\cdot m_1, m_2] = p\cdot[m_1, m_2] + [p, m_2]\cdot m_1,$$

for all $m, m_1, m_2 \in M$, $p, p_1, p_2 \in P$.

Definition 4.2.

Let $M, P \in \mathbf{NP}^{\mathrm{r}}$. We say that P acts on M if we have an action of P on M as associative and Leibniz algebras given by the $\mathbb{K}$-module homomorphisms (7) and (8), respectively, and the following conditions hold:

$$[m, p_1\cdot p_2] = p_1\cdot[m, p_2] + [m, p_1]\cdot p_2, \qquad [p_1, p_2\cdot m] = p_2\cdot[p_1, m] + [p_1, p_2]\cdot m,$$
$$[p_1, m\cdot p_2] = m\cdot[p_1, p_2] + [p_1, m]\cdot p_2, \qquad [p, m_1\cdot m_2] = m_1\cdot[p, m_2] + [p, m_1]\cdot m_2,$$
$$[m_1, m_2\cdot p] = m_2\cdot[m_1, p] + [m_1, m_2]\cdot p, \qquad [m_1, p\cdot m_2] = p\cdot[m_1, m_2] + [m_1, p]\cdot m_2,$$

for all $m, m_1, m_2 \in M$ and $p, p_1, p_2 \in P$.

Definition 4.3.

Let $M, P \in \mathbf{NP}^{\mathrm{lr}}$. We say that P acts on M if we have an action of P on M as left and right NP-algebras.

Actions in the categories $\mathbf{AWB}^{\mathrm{l}}$, $\mathbf{AWB}^{\mathrm{r}}$ and $\mathbf{AWB}^{\mathrm{lr}}$ are defined in similar ways as in the previous definitions, but obviously, the Leibniz algebra action conditions are not required. If an NP-algebra P acts on M, and M is singular, or equivalently abelian, i.e. $M\cdot M = [M, M] = 0$, then M will be called a *representation* of P. Representation in the category $\mathbf{AWB}$ (for $\mathbf{AWB}^{\mathrm{l}}$ see [9]) is defined in a similar way. These definitions coincide with the special cases of the general definition of module given in categories of groups with operations in [30]. If M is a representation of P in $\mathbf{NP}$, then M is a P-P-bimodule, P considered as the underlying associative algebra; analogously, M is an AWB representation of P and a Leibniz representation of P defined in [26]. In the case of Poisson algebras we obtain the representation defined in [13].

A homomorphism between two representations over P is a linear map $f\colon M \to M'$ satisfying

$$f(p\cdot m) = p\cdot f(m), \qquad f(m\cdot p) = f(m)\cdot p, \qquad f[p, m] = [p, f(m)], \qquad f[m, p] = [f(m), p],$$

for all $p \in P$ and $m \in M$.

Definition 4.4.

Let $P \in \mathbf{NP}$ and M be a representation of P. A derivation from P to M is a linear map $d\colon P \to M$ satisfying the conditions

$$d(p_1\cdot p_2) = d(p_1)\cdot p_2 + p_1\cdot d(p_2), \qquad d[p_1, p_2] = [d(p_1), p_2] + [p_1, d(p_2)].$$

(We can give the analogous definition for AWB.)

Denote by $\mathrm{Der}_{\mathrm{NP}}(P, M)$ the $\mathbb{K}$-module of such derivations; analogously we will use the notation $\mathrm{Der}_{\mathrm{AWB}}(P, M)$. Any NP-algebra P is a representation of P acting on itself by the operations in P (see [4, Example 2.3.2]). For $p \in P$, the application $\mathrm{ad}_p\colon P \to P$ defined by $\mathrm{ad}_p(p') = -[p', p]$ is an example of derivation. The following definition is a special case of the definitions given in [30, 31].

Definition 4.5.

Let $P, M \in \mathbf{NP}$. An abelian extension of P by M is a short exact sequence

$$E\colon\ 0 \to M \xrightarrow{i} Q \xrightarrow{j} P \to 0,$$

where $Q \in \mathbf{NP}$ and M is abelian.

Any abelian extension defines on M a unique representation of P in such a way that

$$i(j(q)\cdot m) = q\cdot i(m), \qquad i(m\cdot j(q)) = i(m)\cdot q, \qquad i[j(q), m] = [q, i(m)], \qquad i([m, j(q)]) = [i(m), q],$$

for any $m \in M, q \in Q$. Two abelian extensions E and E' of P by M are called *equivalent* if there exists a homomorphism of NP-algebras $f\colon Q \to Q'$ inducing the identity morphisms on M and P. Note that in this case f is an isomorphism. Let M be any representation of P. Denote by $\mathrm{Ext}_{\mathrm{NP}}(P, M)$ the set of all equivalence classes of those abelian extensions of P by M, which induce the given representation M of P.

Definition 4.6.

Let $M, P \in \mathbf{NP}$ with an action of P on M. A crossed module is a morphism $\mu\colon M \to P$ in $\mathbf{NP}$ satisfying the following axioms:

$$\begin{aligned} \mu(p\cdot m) &= p\cdot\mu(m), & \mu(m\cdot p) &= \mu(m)\cdot p,\\ \mu[p, m] &= [p, \mu(m)], & \mu[m, p] &= [\mu(m), p],\\ \mu(m)\cdot m' = m\cdot m' &= m\cdot\mu(m'), & [\mu(m), m'] = [m, m'] &= [m, \mu(m')]. \end{aligned}$$

A homomorphism of crossed modules is a pair $(\phi, \psi)\colon (M, P, \mu) \to (M', P', \mu')$ where ϕ, ψ are morphisms in $\mathbf{NP}$ such that $\psi\mu = \mu'\phi$ and $\phi(p\cdot m) = \psi(p)\cdot\phi(m)$; $\phi(m\cdot p) = \phi(m)\cdot\psi(p)$; $\phi[p, m] = [\psi(p), \phi(m)]$; $\phi[m, p] = [\phi(m), \psi(p)]$, for all $p \in P$, $m \in M$.

Examples of representations and crossed modules and the construction of semi-direct products in the category of NP-algebras and AWB are analogous to those given for NP^{l}-algebras and $\mathrm{AWB}^{\mathrm{l}}$, therefore for these subjects we refer the reader to [4] and [9], respectively.

It is proved in [29] that every category of interest is action accessible in the sense of [3]. Since $\mathbf{NP}^{\mathrm{lr}}$ is a category of interest (see Section 2) we obtain

Theorem 4.7.

The category $\mathbf{NP}^{\mathrm{lr}}$ is action accessible.

In [6] for any category of interest $\mathbf{C}$ and for any object $A \in \mathbf{C}$ there is defined and constructed the universal strict general actor USGA(A) of A, which is generally an object of $\mathbf{C}_G$. Here we give this construction for the category $\mathbf{NP}^{\mathrm{lr}}$. In this case we have three binary operations: the addition, denoted by "+", the dot and the (square) bracket operations. Ω_2' from the definition of category of interest is a set with three elements $\Omega_2' = \{\cdot, [-,-], [-,-]^{\circ}\}$. Since the addition is commutative, the action corresponding to this operation is trivial. Thus we will deal only with actions, which are defined by dot and bracket operations; the actions of b on a will be denoted as $a\cdot b, b\cdot a, [b, a]$ and $[a, b]$. Below under $*$ operation we will mean either dot or bracket operations. Let $A \in \mathbf{NP}^{\mathrm{lr}}$; consider all split extensions of A,

$$E_j\colon\ 0 \to A \xrightarrow{i_j} C_j \xrightarrow{p_j} B_j \to 0, \qquad j \in \mathbb{J}.$$

Let $\{b_j * : b_j \in B_j, * \in \Omega_2'\}$ be the corresponding set of derived actions for $j \in \mathbb{J}$. For any element $b_j \in B_j$ denote $\mathbf{b}_j = \{b_j * : * \in \Omega_2'\}$. Let $\mathbb{B} = \{\mathbf{b}_j : b_j \in B_j, j \in \mathbb{J}\}$. Thus each element $\mathbf{b}_j \in \mathbb{B}$, $j \in \mathbb{J}$, is the special type of a function

$\mathbf{b}_j\colon \Omega_2' \longrightarrow \mathrm{Maps}(A\to A)$, $\mathbf{b}_j(*) = b_j * -\colon A \to A$. According to Axiom 2 of the definition of a category of interest, we define $*$ operation, $\mathbf{b}_i * \mathbf{b}_k$, $* \in \Omega_2'$, for the elements of $\mathbb{B}$ by the equalities

$$(\mathbf{b}_i * \mathbf{b}_k)\,\overline{*}\,(a) = W(b_i, b_k; a; *, \overline{*}).$$

We define

$$\begin{aligned}(\mathbf{b}_i + \mathbf{b}_k) * (a) &= b_i * a + b_k * a, & (-\mathbf{b}_k) * (a) &= -(b_k * a),\\ (-b) * (a) &= -(b * (a)), & -(b_1 + \cdots + b_n) &= -b_n - \cdots - b_1,\end{aligned}$$

where $* \in \Omega_2'$, $b, b_1, \ldots, b_n$ are certain combinations of the dot and the bracket operations on the elements of $\mathbb{B}$, i.e. the elements of the type $\mathbf{b}_{i_1} *_1 \cdots *_{n-1} \mathbf{b}_{i_n}$, where $n > 1$. We do not know if the new functions defined by us are again in $\mathbb{B}$. Denote by $\mathfrak{B}(A)$ the set of functions $(\Omega_2' \longrightarrow \mathrm{Maps}(A\to A))$ obtained by performing all kinds of the above defined operations on elements of $\mathbb{B}$ and the new obtained elements as results of operations. Let $b \sim b'$ in $\mathfrak{B}(A)$ if $b * a = b' * a$, for any $a \in A$, $* \in \Omega_2'$. It is an equivalence relation; denote by $\mathrm{USGA}(A)$ be the corresponding quotient algebra. Let $\mathbf{NP}^{\mathrm{lr}}_G$ be a general category of groups with operations of the category of interest $\mathbf{NP}^{\mathrm{lr}}$.

By direct checking of identities one can prove the following proposition.

Proposition 4.8.

$\mathrm{USGA}(A)$ *is an object in* $\mathbf{NP}^{\mathrm{lr}}_G$.

As above, we will write for simplicity $b * (a)$ instead of $(b(*))(a)$, for $b \in \mathrm{USGA}(A)$ and $a \in A$. Define a set of actions of $\mathrm{USGA}(A)$ on A in the following natural way. For $b \in \mathrm{USGA}(A)$ we define $b * a = b * (a)$, $* \in \Omega_2'$. Thus if $b = \mathbf{b}_{i_1} *_1 \cdots *_{n-1} \mathbf{b}_{i_n}$, where we mean certain round brackets, we have

$$b\,\overline{*}\,a = (\mathbf{b}_{i_1} *_1 \cdots *_{n-1} \mathbf{b}_{i_n})\,\overline{*}\,(a).$$

The right side of the equality is defined inductively according to Axiom 2. For $b_k \in B_k$, $k \in \mathbb{J}$, we have

$$\mathbf{b}_k * a = \mathbf{b}_k * (a) = b_k * a.$$

Also

$$(b_1 + b_2 + \cdots + b_n) * a = b_1 * (a) + \cdots + b_n * (a).$$

Proposition 4.9.

The set of actions of $\mathrm{USGA}(A)$ *on* A *is an action in the category* $\mathbf{NP}^{\mathrm{lr}}_G$.

Proof. It is a special case of the proof of the general statement for categories of interest given in [6]. The checking shows that the set of actions of $\mathrm{USGA}(A)$ on A satisfies conditions of [11, Proposition 1.1], which proves that it is an action in $\mathbf{NP}^{\mathrm{lr}}_G$. □

Note that this is an action in $\mathbf{NP}^{\mathrm{lr}}_G$, which in general does not satisfy the action conditions in $\mathbf{NP}^{\mathrm{lr}}$. Define a map $d\colon A \to \mathrm{USGA}(A)$ by $d(a) = \mathfrak{a}$, where $\mathfrak{a} = \{a\,\cdot, a\,*, * \in \Omega_2'\}$. Thus we have by definition

$$d(a) * a' = a * a', \qquad a, a' \in A, \quad * \in \Omega_2'.$$

Proofs of the following two statements are special cases of those given in [6].

Lemma 4.10.
The map d is a homomorphism in $\mathbf{NP}^{\mathrm{lr}}_G$.

Proposition 4.11.
The map $d\colon A \to \mathrm{USGA}(A)$ is a crossed module in $\mathbf{NP}^{\mathrm{lr}}_G$.

According to the general definition of center [30] (cf. with the definition in [6]) we describe the center of an object in $\mathbf{NP}^{\mathrm{lr}}$ as follows.

Definition 4.12.
The center of $P \in \mathbf{NP}^{\mathrm{lr}}$ is $Z(P) = \big\{z \in P : z \cdot p = p \cdot z = [z, p] = [p, z] = 0,\ p \in P\big\}$.

It is easy to see that $Z(P) = \mathrm{Ker}\, d$. Next we give the definition of an actor in $\mathbf{NP}^{\mathrm{lr}}$ (for the case of a category of interest see [6]).

Definition 4.13.
For any object A in $\mathbf{NP}^{\mathrm{lr}}$ an actor of A is an object $\mathrm{Act(A)} \in \mathbf{NP}^{\mathrm{lr}}$, which has an action on A in the same category (i.e. satisfying the conditions of Definition 4.3), such that for any object C in $\mathbf{NP}^{\mathrm{lr}}$ with an action on A, there is a unique morphism $\varphi\colon C \to \mathrm{Act}(A)$ with

$$c \cdot a = \varphi(c) \cdot a, \qquad a \cdot c = a \cdot \varphi(c), \qquad [c, a] = [\varphi(c), a], \qquad [a, c] = [a, \varphi(c)],$$

for any $a \in A$ and $c \in C$.

According to the same paper, an actor of A is a split extension classifier for A in the sense of [1]. From the results of [6] we obtain.

Theorem 4.14.
For any element $A \in \mathbf{NP}^{\mathrm{lr}}$ there exists an actor of A if and only if the semidirect product $\mathrm{USGA}(A) \ltimes A \in \mathbf{NP}^{\mathrm{lr}}$. If it is the case, then $\mathrm{Actor}(A) = \mathrm{USGA}(A)$.

At the end of this section we give an example of a subcategory in $\mathbf{NP}^{\mathrm{lr}}$, which satisfies the amalgamation property. This result can be applied to the description of effective codescent morphisms in the corresponding subcategory. For the definition of amalgamation property one can see [2].

Recall that a ring R (generally without a unit) is von Neumann regular if for any $r \in R$ there exists an element $r' \in R$ such that $rr'r = r$.

Proposition 4.15.
In the category of $\mathrm{NP}^{\mathrm{lr}}$-algebras there exists a subcategory, which satisfies the amalgamation property.

Proof. Consider the full subcategory in $\mathbf{NP}^{\mathrm{lr}}$, whose objects are commutative von Neumann regular rings with trivial bracket operations. Now it remains to apply the result from [2], where it is proved that the category of (not necessarily unital) commutative von Neumann regular rings satisfies the amalgamation property. □

5. Cohomology

We recall the constructions of complexes for Hochschild and Leibniz cohomologies, for cohomologies of left algebras with bracket and left NP-algebras, i.e. $\mathrm{AWB}^{\mathfrak{l}}$ and $\mathrm{NP}^{\mathfrak{l}}$-algebras according to [4, 9], respectively. Below for $P \in \mathrm{NP}$ instead of underlying associative and Leibniz algebras $V_A(P), V_L(P)$ and underlying AWB we will write for simplicity just P and will note what kind of algebras we mean, similarly for $P \in \mathrm{AWB}$ and $T_A(P), T_L(P)$.

Let P be a left NP-algebra over a field $\mathbb{K}$ and M a representation of P. In particular, P is an associative algebra and M is a P-P-bimodule and, on the other hand, P is a Leibniz algebra and M is a representation of P in the category of Leibniz algebras. Let $\big(C^*_H(P,M), \partial^n_H\big)$ be the Hochschild complex and $\big(C^*_L(P,M), \partial^n_L\big)$ be the Leibniz complex. We recall that for $n \geq 0$,

$$C^n_H(P,M) = C^n_L(P,M) = \mathrm{Hom}(P^{\otimes n}, M)$$

and coboundary maps ∂^n_H and ∂^n_L are given by

$$\partial^n_H(f)(p_1,\dots,p_{n+1}) = (-1)^{n+1}\Big\{p_1 f(p_2,\dots,p_{n+1}) + \sum_{i=1}^{n}(-1)^i f(p_1,\dots,p_ip_{i+1},\dots,p_{n+1}) + (-1)^{n+1} f(p_1,\dots,p_n)p_{n+1}\Big\},$$

$$\partial^n_L(f)(p_1,\dots,p_{n+1}) = \big[p_1, f(p_2,\dots,p_{n+1})\big] + \sum_{i=2}^{n+1}(-1)^i\big[f(p_1,\dots,\widehat{p}_i,\dots,p_{n+1}), p_i\big]$$
$$+ \sum_{1\leq i<j\leq n+1}(-1)^{j+1} f\big(p_1,\dots,p_{i-1},[p_i,p_j],p_{i+1},\dots,\widehat{p}_j,\dots,p_{n+1}\big).$$

Thus $C^n_H(P,M)$ and $C^n_L(P,M)$ are complexes of $\mathbb{K}$-vector spaces. We will need below the P-P-bimodule M^e, defined by $M^e = \mathrm{Hom}(P,M)$ as a $\mathbb{K}$-vector space, and a bimodule structure on M^e given by $(p_1\cdot f)(p_2) = p_1\cdot f(p_2)$, $(f\cdot p_1)(p_2) = f(p_2)\cdot p_1$. We have an isomorphism of $\mathbb{K}$-vector spaces $\theta_n\colon C^{n+1}_H(P,M) \to C^n_H(P,M^e)$, $n \geq 1$, defined in an obvious way $\theta_n(f)(p_1,\dots,p_n)(p) = f(p_1,\dots,p_n,p)$. Denote the coboundary maps of the complex $C^*_H(P,M^e)$ by $\partial^{e,*}_H$. Let

$$\overline{C}^*_H(P,M) = \big(C^n_H(P,M), \partial^n_H : n \geq 1\big), \qquad \overline{C}^*_H(P,M^e) = \big(C^n_H(P,M^e), \partial^{e,n}_H,\ n \geq 1\big), \qquad \overline{C}^*_L(P,M) = \big(C^n_L(P,M), \partial^n_L : n \geq 1\big)$$

Consider the following homomorphisms of cochain complexes, defined in [4, 9], respectively:

$$\alpha^*\colon \overline{C}^*_H(P,M) \to \overline{C}^*_H(P,M^e), \qquad \beta^*\colon \overline{C}^*_L(P,M) \to \overline{C}^*_H(P,M^e)$$

and given by

$$\alpha^1(f)(p_1)(p_2) = [p_1, f(p_2)] + [f(p_1), p_2] - f([p_1,p_2]),$$

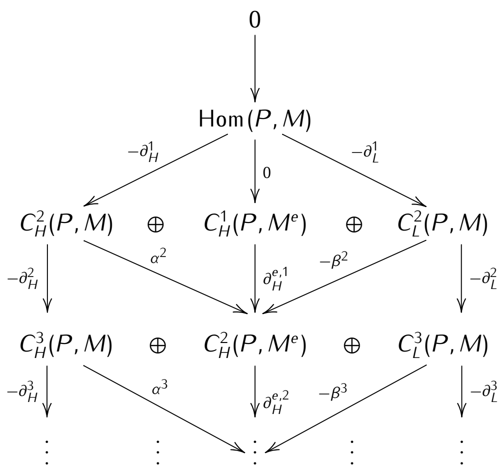
and for $n > 1$,

$$\alpha^n(f)(p_1,\dots,p_n)(p_{n+1}) = \big[f(p_1,\dots,p_n), p_{n+1}\big] - f([p_1,p_{n+1}],p_2,\dots,p_n) - f(p_1,[p_2,p_{n+1}],\dots,p_n)$$
$$- \cdots - f(p_1,\dots,p_{n-1},[p_n,p_{n+1}]),$$
$$\beta^{2k+1} = \theta_{2k+1}\partial^{2k+1}_L, \qquad k \geq 0, \qquad\qquad \beta^{2k} = \partial^{e,2k-1}_H\theta_{2k-1}, \qquad k \geq 1.$$

Note that $\alpha^1 = \beta^1$. α^* and β^* are homomorphisms of complexes (see resp. [4, 9]). Let $\mathrm{cone}\,\alpha^*$ and $\mathrm{cone}(-\beta^*)$ be the mapping cones and $C^*(P,M) = \mathrm{cone}\,\alpha^* \bigsqcup_{(i_1,i_2)} \mathrm{cone}(-\beta^*)$ the pushout, where i_1 and i_2 are the following injections of complexes:

$$\mathrm{cone}\,\alpha^* \xleftarrow{i_1} C^{*-1}_H(P,M^e) \xrightarrow{i_2} \mathrm{cone}(-\beta^*).$$

Define $C^0_{\mathrm{NP^l}}(P,M) = 0$, $C^1_{\mathrm{NP^l}}(P,M) = \mathrm{Hom}(P,M)$, $C^n_{\mathrm{NP^l}}(P,M) = C^n(P,M)$, $n \geq 2$; $\partial^0_{\mathrm{NP^l}} = 0$, $\partial^1_{\mathrm{NP^l}} = (\partial^1_H, 0, \partial^1_L)$, $\partial^n_{\mathrm{NP^l}} = \partial^n$, $n \geq 2$. We have $\partial^{n+1}_{\mathrm{NP^l}}\partial^n_{\mathrm{NP^l}} = 0$, $n \geq 0$, so $\left\{C^n_{\mathrm{NP^l}}(P,M), \partial^n_{\mathrm{NP^l}} : n \geq 0\right\}$ is a complex which has the form



The cohomology vector spaces $H^n_{\mathrm{NP^l}}(P,M)$, $n \geq 0$, of an $\mathrm{NP^l}$-algebra P with coefficients in a representation M of P are defined by

$$H^n_{\mathrm{NP^l}}(P,M) = H^n\left(C^*_{\mathrm{NP^l}}(P,M), \partial^n_{\mathrm{NP^l}}\right), \qquad n \geq 0.$$

According to [9] the cohomology of AWB is defined by $H^{n-1}_{\mathrm{AWB}}(P,M) = H^n(\mathrm{cone}\,\alpha^*)$, for $n \geq 1$, where $P \in \mathrm{AWB}$. Note that in cone α^* the zero term $\mathrm{cone}\,(\alpha^*)^0$ is zero, and the first one is $C^1_H(P,M) \oplus C^0_H(P,M^e)$. In this paper the cohomology of $\mathrm{AWB^l}$ are defined as $H^0_{\mathrm{AWB^l}}(P,M) = 0$ and $H^n_{\mathrm{AWB^l}}(P,M) = H^n(\mathrm{cone}\,\alpha^*)$, for $n \geq 1$.

Now we shall define the cohomology vector spaces of $\mathrm{NP^r}$- and $\mathrm{NP^{lr}}$-algebras. Let $\theta'_n \colon C^{n+1}_H(P,M) \to C^n_H(P,M^e)$, for $n \geq 1$, be the homomorphism defined by $\theta'_1(f)(p_1)(p_2) = f(p_2,p_1)$ and $\theta'_n(f)(p_1,p_2,\dots,p_n)(p_{n+1}) = f(p_{n+1},p_1,\dots,p_n)$, $n > 1$. It is easy to see that θ'_n is an isomorphism for each $n \geq 1$. Define the homomorphisms

$$\alpha'^* \colon \overline{C}^*_H(P,M) \to \overline{C}^*_H(P,M^e), \qquad \beta'^* \colon \overline{C}^*_L(P,M) \to \overline{C}^*_H(P,M^e),$$

by

$$\alpha'^1(f)(p_1)(p_2) = [f(p_2),p_1] + [p_2,f(p_1)] - f([p_2,p_1])$$

and for $n > 1$ by

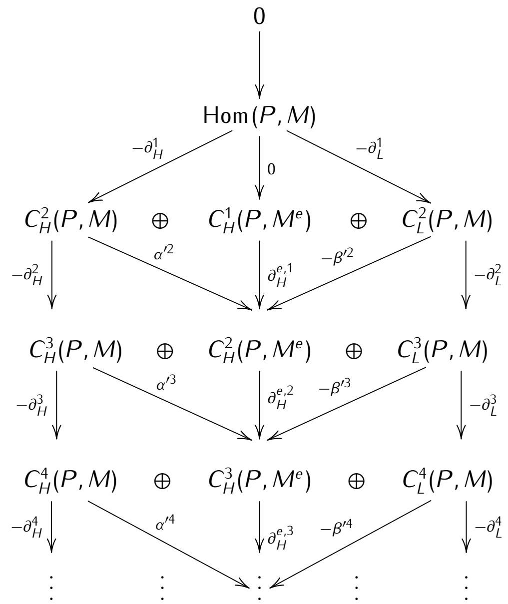
$$\begin{aligned} \alpha'^n(f)(p_1,\dots,p_n)(p_{n+1}) &= [p_{n+1}, f(p_1,\dots,p_n)] - f([p_{n+1},p_1],p_2,\dots,p_n) - f(p_1,[p_{n+1},p_2],\dots,p_n) \\ &\quad - \dots - f(p_1,\dots,p_{n-1},[p_{n+1},p_n]), \end{aligned}$$
$$\beta'^{2k+1} = \theta'_{2k}\partial^{2k+1}_L, \qquad k \geq 0, \qquad\qquad \beta'^{2k} = \partial^{e,2k-1}_H\theta'_{2k-1}, \qquad k \geq 1.$$

We have $\alpha'^1 = \beta'^1$. Easy checking shows that α'^* and β'^* are homomorphisms of complexes.

By taking the pushout $C'^*(P,M) = \mathrm{cone}\,\alpha'^* \sqcup_{(i'_1,i'_2)} \mathrm{cone}\,(-\beta'^*)$, where i'_1 and i'_2 are the following injections of complexes:

$$\mathrm{cone}\,\alpha'^* \overset{i'_1}{\longleftarrow} C^{*-1}_H(P,M^e) \overset{i'_2}{\longrightarrow} \mathrm{cone}\,(-\beta'^*),$$

we construct the complex analogous to $\left\{C^n_{\mathrm{NP^l}}(P,M),\partial^n_{\mathrm{NP^l}} : n\geq 0\right\}$, which will be denoted by $\left\{C^n_{\mathrm{NP^r}}(P,M),\partial^n_{\mathrm{NP^r}} : n\geq 0\right\}$. The complex has the form



The cohomology vector spaces of an $\mathrm{NP^r}$-algebra P with coefficients in a representation M of P are defined as the cohomologies of this complex and denoted as $H^n_{\mathrm{NP^r}}(P,M)$, $n\geq 0$.

Now we construct the complex for the cohomology of an $\mathrm{NP^{lr}}$-algebra P. Consider the following pairs of homomorphisms of complexes:

$$(\alpha^*,\alpha'^*)\colon\ \overline{C}^*_H(P,M)\to\overline{C}^*_H(P,M^e)\oplus\overline{C}^*_H(P,M^e),\qquad (\beta^*,\beta'^*)\colon\ \overline{C}^*_L(P,M)\to\overline{C}^*_H(P,M^e)\oplus\overline{C}^*_H(P,M^e).$$

From these homomorphisms we obtain two cones: $\mathrm{cone}(\alpha^*,\alpha'^*)$ and $\mathrm{cone}(\beta^*,\beta'^*)$. We have the following homomorphisms of complexes:

$$\mathrm{cone}(\alpha^*,\alpha'^*)\ \overset{j_1}{\longleftarrow}\ C^{*-1}_H(P,M^e)\oplus C^{*-1}_H(P,M^e)\ \overset{j_2}{\longrightarrow}\ \mathrm{cone}(-\beta^*,-\beta'^*).$$

The pushout of the pair (j_1,j_2) gives the desired complex. In particular, we take $C^0_{\mathrm{NP^{lr}}}(P,M)=0$, $C^1_{\mathrm{NP^{lr}}}(P,M)=\mathrm{Hom}(P,M)$, $C^n_{\mathrm{NP^{lr}}}(P,M)=C^n_H(P,M)\oplus C^n_H(P,M^e)\oplus C^n_H(P,M^e)\oplus C^n_L(P,M)$, for $n\geq 2$, moreover $\partial^0_{\mathrm{NP^{lr}}}=0$, $\partial^1_{\mathrm{NP^{lr}}}=\left(-\partial^1_H,0,0,-\partial^1_L\right)$, $\partial^n_{\mathrm{NP^{lr}}}$ is induced by $\alpha^n,\alpha'^n,\partial^{e,n-1}_H,\partial^{e,n-1}_H,\beta^n,\beta'^n$, for $n\geq 2$.

We have $\partial^{n+1}_{\mathrm{NP^{lr}}}\partial^n_{\mathrm{NP^{lr}}}=0$, $n\geq 0$, therefore $\left\{C^n_{\mathrm{NP^{lr}}}(P,M),\partial^n_{\mathrm{NP^{lr}}} : n\geq 0\right\}$ is a complex; it has the following form:

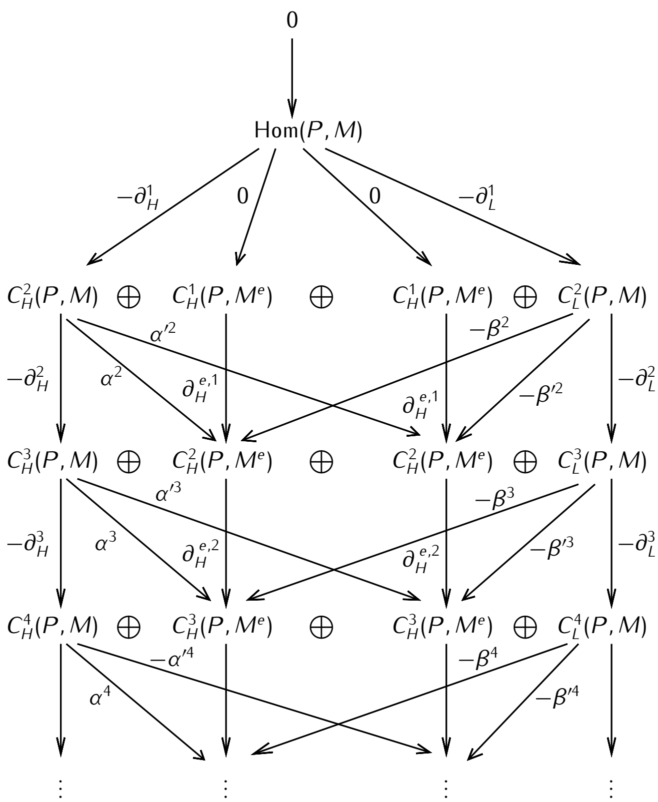
The cohomology vector spaces of an $\mathrm{NP^{lr}}$-algebra P with coefficients in a representation M of P are defined as the cohomologies of this complex and denoted as $H^n_{\mathrm{NP^{lr}}}(P,M)$, $n\geq 0$.

As in the case of $\mathrm{NP^l}$-algebras in [4], we define restricted second cohomology of $\mathrm{NP^{lr}}$-algebras. We have the natural injection $C^2_H(P,M)\oplus C^2_L(P,M)\longrightarrow C^2_{\mathrm{NP^{lr}}}(P,M)$ on to the first and the fourth summands; the image of this injection will be denoted again by the sum $C^2_H(P,M)\oplus C^2_L(P,M)$. Consider the restriction

$$d^2_{\mathrm{NP^{lr}}}=\partial^2_{\mathrm{NP^{lr}}}\restriction_{C^2_H(\,)\oplus C^2_L(\,)}.$$

We define the 2-dimensional restricted cohomology of the $\mathrm{NP^{lr}}$-algebra P with coefficients in M by

$$\mathbb{H}^2_{\mathrm{NP^{lr}}}(P,M)=\mathrm{Ker}\,d^2_{\mathrm{NP^{lr}}}/\,\mathrm{Im}\,\partial^1_{\mathrm{NP^{lr}}}.$$



The obvious injection $\kappa \colon \operatorname{Ker} d^2_{\mathrm{NP^{lr}}} \to \operatorname{Ker} \partial^2_{\mathrm{NP^{lr}}}$ induces the injection of the corresponding cohomologies

$$\chi \colon \mathbb{H}^2_{\mathrm{NP^{lr}}}(P,M) \to H^2_{\mathrm{NP^{lr}}}(P,M).$$

$\mathbb{H}^2_{\mathrm{NP^r}}(P,M)$ is defined in analogous way as for $\mathrm{NP^l}$-algebras. The cohomologies of $\mathrm{AWB^r}$ and $\mathrm{AWB^{lr}}$ are defined by

$$H^*_{\mathrm{AWB^r}}(P,M) = H^*(\operatorname{cone}\alpha'^*), \qquad H^*_{\mathrm{AWB^{lr}}}(P,M) = H^*\big(\operatorname{cone}(\alpha^*, \alpha'^*)\big).$$

From the definitions we obtain

Lemma 5.1.

(i) *For* $P \in \mathbf{NP}$, $H^0_{\mathrm{NP}}(P,M) = 0$ *and* $H^1_{\mathrm{NP}}(P,M) = \mathrm{Der}_{\mathrm{NP}}(P,M)$.

(ii) *For* $P \in \mathbf{AWB}$, $H^0_{\mathrm{AWB}}(P,M) = 0$, $H^1_{\mathrm{AWB}}(P,M) = \mathrm{Der}_{\mathrm{AWB}}(P,M)$, *and* $H^2_{\mathrm{AWB}}(P,M) \cong \mathrm{Ext}_{\mathrm{AWB}}(P,M)$.

Proof. (i) The proof follows directly from the fact that $C^0_{\mathrm{NP}}(P,M) = 0$, from the definition of ∂^1_{NP} and the definition of a derivation.

(ii) Since the zero term in the corresponding cone complex is zero, the first equality follows from the definition of the cohomology. The proofs of other two equalities of (ii) for $\mathrm{AWB^r}$ and $\mathrm{AWB^{lr}}$ are similar to the proofs given in [9] for $\mathrm{AWB^l}$. □

Recall that the *Hochschild cohomological dimension* $\mathrm{c.dim}_H P$ of an associative algebra P is defined as the greatest natural number n, for which there exists a P-P-bimodule S with $H^n_H(P,S) \neq 0$. The analogous meaning will have the *Leibniz cohomological dimension* of a Leibniz algebra P, AWB *cohomological dimension* of an algebra $P \in \mathrm{AWB}$ and NP *cohomological dimension* of an NP-algebra P, denoted as $\mathrm{c.dim}_L P$, $\mathrm{c.dim}_{\mathrm{AWB}} P$ and $\mathrm{c.dim}_{\mathrm{NP}} P$, respectively.

Theorem 5.2.

$\mathbb{H}^2_{\mathrm{NP}}(P,M) \cong \mathrm{Ext}_{\mathrm{NP}}(P,M)$.

The proof is similar to the one for NP^{l}-algebras presented in [4], and therefore omitted.

Corollary 5.3.

(i) *If P is a free* NP*-algebra, then* $\mathbb{H}^2_{\mathrm{NP}}(P,M)=0$ *for any representation M of P.*

(ii) *If P is an* NP^{l}*-algebra with* $\mathrm{c.dim}_H V^{\mathrm{l}}_A(P)\leq n$ *and* $\mathrm{c.dim}_L V^{\mathrm{l}}_L(P)\leq n$ *(resp.* NP^{r}*-algebra with* $\mathrm{c.dim}_H V^{\mathrm{r}}_A(P)\leq n$ *and* $\mathrm{c.dim}_L V^{\mathrm{r}}_L(P)\leq n$*), then for $k>n$ and any representation M,*

$$H^{k+1}_{\mathrm{NP}^{\mathrm{l}}}(P,M)=0 \qquad \big(\mathit{resp.}\ \ H^{k+1}_{\mathrm{NP}^{\mathrm{r}}}(P,M)=0\big).$$

Proof. (i) Since for a free NP-algebra P every extension $0\to M\xrightarrow{i}Q\xrightarrow{j}P\to 0$ splits, the fact follows from Theorem 5.2.

(ii) From the facts that α^* and β^* (resp. α'^* and β'^*) are homomorphisms of cochain complexes, by diagram chasing we obtain that $C^*_{\mathrm{NP}^{\mathrm{l}}}(P,M)$ (resp. $C^*_{\mathrm{NP}^{\mathrm{r}}}(P,M)$) is exact in dimensions $\geq k+1$, where $k>n$, from which the result follows. ☐

Lemma 5.4.

If P is a free $\mathrm{AWB}^{\mathrm{r}}$*, then* $H^n_{\mathrm{AWB}^{\mathrm{r}}}(P,M)=0$*, for $n\geq 2$ (according to the notation in [9], $n\geq 1$) and any representation M of P.*

The demonstration is analogous to the proof of this fact for $\mathrm{AWB}^{\mathrm{l}}$ given in [9] and therefore it is omitted.

In [9] it is proved that if P is $\mathrm{AWB}^{\mathrm{l}}$, then its cohomologies are isomorphic to Quillen cohomologies. In a similar way, applying Lemma 5.4 we have

Theorem 5.5.

$H^{n+1}_{\mathrm{AWB}^{\mathrm{r}}}(P,M)\approx H^n_{\mathrm{Q}}(P,M)$.

From the constructions of the cohomology complexes we obtain the following short exact sequences of complexes:

$$0\longrightarrow \mathrm{cone}\,\alpha^*\longrightarrow C^*_{\mathrm{NP}^{\mathrm{l}}}(P,M)\longrightarrow C^*_L(P,M)\longrightarrow 0,\qquad *\geq 3, \qquad (\mathrm{a}_1)$$

$$0\longrightarrow \mathrm{cone}\,\alpha'^*\longrightarrow C^*_{\mathrm{NP}^{\mathrm{r}}}(P,M)\longrightarrow C^*_L(P,M)\longrightarrow 0,\qquad *\geq 3, \qquad (\mathrm{a}_2)$$

$$0\longrightarrow \mathrm{cone}\,(\alpha^*,\alpha'^*)\longrightarrow C^*_{\mathrm{NP}^{\mathrm{lr}}}(P,M)\longrightarrow C^*_L(P,M)\longrightarrow 0,\qquad *\geq 3, \qquad (\mathrm{a})$$

$$0\longrightarrow \mathrm{cone}\,(-\beta^*)\longrightarrow C^*_{\mathrm{NP}^{\mathrm{l}}}(P,M)\longrightarrow C^*_H(P,M)\longrightarrow 0,\qquad *\geq 3, \qquad (\mathrm{b}_1)$$

$$0\longrightarrow \mathrm{cone}\,(-\beta'^*)\longrightarrow C^*_{\mathrm{NP}^{\mathrm{r}}}(P,M)\longrightarrow C^*_H(P,M)\longrightarrow 0,\qquad *\geq 3, \qquad (\mathrm{b}_2)$$

$$0\longrightarrow \mathrm{cone}\,(-\beta^*,-\beta'^*)\longrightarrow C^*_{\mathrm{NP}^{\mathrm{lr}}}(P,M)\longrightarrow C^*_H(P,M)\longrightarrow 0,\qquad *\geq 1, \qquad (\mathrm{b})$$

$$0\longrightarrow C^{*-1}_H(P,M^e)\longrightarrow C^*_{\mathrm{AWB}^{\mathrm{l}}}(P,M)\longrightarrow C^*_H(P,M)\longrightarrow 0,\qquad *\geq 1, \qquad (\mathrm{c}_1)$$

$$0\longrightarrow C^{*-1}_H(P,M^e)\longrightarrow C^*_{\mathrm{AWB}^{\mathrm{r}}}(P,M)\longrightarrow C^*_H(P,M)\longrightarrow 0,\qquad *\geq 1, \qquad (\mathrm{c}_2)$$

$$0\longrightarrow C^{*-1}_H(P,M^e)\xrightarrow{i_3} C^*_{\mathrm{AWB}^{\mathrm{lr}}}(P,M)\longrightarrow C^*_{\mathrm{AWB}^{\mathrm{l}}}(P,M)\longrightarrow 0,\qquad *\geq 1, \qquad (\mathrm{c})$$

$$0\longrightarrow C^{*-1}_H(P,M^e)\xrightarrow{i_2} C^*_{\mathrm{AWB}^{\mathrm{lr}}}(P,M)\longrightarrow C^*_{\mathrm{AWB}^{\mathrm{r}}}(P,M)\longrightarrow 0,\qquad *\geq 1, \qquad (\mathrm{c}')$$

$$0\longrightarrow C^{*-1}_H(P,M^e)\longrightarrow C^*_{\mathrm{NP}^{\mathrm{l}}}(P,M)\longrightarrow C^*_H(P,M)\oplus C^*_L(P,M)\longrightarrow 0,\qquad *\geq 3, \qquad (\mathrm{d}_1)$$

$$0\longrightarrow C^{*-1}_H(P,M^e)\longrightarrow C^*_{\mathrm{NP}^{\mathrm{r}}}(P,M)\longrightarrow C^*_H(P,M)\oplus C^*_L(P,M)\longrightarrow 0,\qquad *\geq 3, \qquad (\mathrm{d}_2)$$

$$0\longrightarrow C^{*-1}_H(P,M^e)\xrightarrow{i_3} C^*_{\mathrm{NP}^{\mathrm{lr}}}(P,M)\longrightarrow C^*_{\mathrm{NP}^{\mathrm{l}}}(P,M)\longrightarrow 0,\qquad *\geq 3, \qquad (\mathrm{d})$$

$$0 \longrightarrow C_H^{*-1}(P,M^e) \xrightarrow{i_2} C^*_{\mathrm{NP^{lr}}}(P,M) \longrightarrow C^*_{\mathrm{NP^{r}}}(P,M) \longrightarrow 0, \qquad * \geq 3, \qquad (\mathrm{d}')$$

$$0 \longrightarrow C_H^{*-1}(P,M^e) \oplus C_H^{*-1}(P,M^e) \longrightarrow C^*_{\mathrm{AWB^{lr}}}(P,M) \longrightarrow C^*_H(P,M) \longrightarrow 0, \qquad * \geq 3, \qquad (\mathrm{e})$$

$$0 \longrightarrow C_H^{*-1}(P,M^e) \oplus C_H^{*-1}(P,M^e) \longrightarrow C^*_{\mathrm{NP^{lr}}}(P,M) \longrightarrow C^*_H(P,M) \oplus C^*_L(P,M) \longrightarrow 0, \qquad (\mathrm{f})$$

$$0 \longrightarrow C_H^{*-1}(P,M^e) \xrightarrow{(i_2,-i_3)} \mathrm{cone}\,\alpha^* \oplus \mathrm{cone}\,(-\beta^*) \longrightarrow C^*_{\mathrm{NP^{l}}}(P,M) \longrightarrow 0, \qquad * \geq 3, \qquad (\mathrm{g}_1)$$

$$0 \longrightarrow C_H^{*-1}(P,M^e) \xrightarrow{(i_2,-i_3)} \mathrm{cone}\,\alpha'^* \oplus \mathrm{cone}\,(-\beta'^*) \longrightarrow C^*_{\mathrm{NP^{r}}}(P,M) \longrightarrow 0, \qquad * \geq 3, \qquad (\mathrm{g}_2)$$

$$0 \longrightarrow C_H^{*-1}(P,M^e) \oplus C_H^{*-1}(P,M^e) \xrightarrow{((i_2,i_3),(-i_4,-i_5))} \mathrm{cone}\,(\alpha^*,\alpha'^*) \oplus \mathrm{cone}\,(-\beta^*,-\beta'^*) \longrightarrow C^*_{\mathrm{NP^{lr}}}(P,M) \longrightarrow 0, \qquad (\mathrm{g})$$

$$0 \longrightarrow C_H^{*-1}(P,M^e) \longrightarrow \mathrm{cone}\,(-\beta^*) \longrightarrow C^*_L(P,M) \longrightarrow 0, \qquad * \geq 3, \qquad (\mathrm{h}_1)$$

$$0 \longrightarrow C_H^{*-1}(P,M^e) \longrightarrow \mathrm{cone}\,(-\beta'^*) \longrightarrow C^*_L(P,M) \longrightarrow 0, \qquad * \geq 3, \qquad (\mathrm{h}_2)$$

$$0 \longrightarrow C_H^{*-1}(P,M^e) \oplus C_H^{*-1}(P,M^e) \longrightarrow \mathrm{cone}\,(-\beta^*,-\beta'^*) \longrightarrow C^*_L(P,M) \longrightarrow 0, \qquad * \geq 3. \qquad (\mathrm{h})$$

In these sequences i_2, i_3, i_4 and i_5 denote the injections on the corresponding summands, respectively. These exact sequences are obtained directly from the constructions of the cohomology complexes of the corresponding types of algebras.

Theorem 5.6.

We have the following exact sequences of cohomology vector spaces:

$$\begin{array}{l} H^2_{\mathrm{AWB^l}}(P,M) \longrightarrow H^2_{\mathrm{NP^l}}(P,M) \longrightarrow H^2_L(P,M) \longrightarrow \\ \longrightarrow H^3_{\mathrm{AWB^l}}(P,M) \longrightarrow H^3_{\mathrm{NP^l}}(P,M) \longrightarrow H^3_L(P,M) \longrightarrow \cdots \end{array} \qquad (\mathrm{A}_1)$$

where P is an $\mathrm{NP^l}$*-algebra and M a representation of P.*

$$\begin{array}{l} H^2_{\mathrm{AWB^r}}(P,M) \longrightarrow H^2_{\mathrm{NP^r}}(P,M) \longrightarrow H^2_L(P,M) \longrightarrow \\ \longrightarrow H^3_{\mathrm{AWB^r}}(P,M) \longrightarrow H^3_{\mathrm{NP^r}}(P,M) \longrightarrow H^3_L(P,M) \longrightarrow \cdots \end{array} \qquad (\mathrm{A}_2)$$

where P is an $\mathrm{NP^r}$*-algebra and M a representation of P.*

$$\begin{array}{l} H^2_{\mathrm{AWB^{lr}}}(P,M) \longrightarrow H^2_{\mathrm{NP^{lr}}}(P,M) \longrightarrow H^2_L(P,M) \longrightarrow \\ \longrightarrow H^3_{\mathrm{AWB^{lr}}}(P,M) \longrightarrow H^3_{\mathrm{NP^{lr}}}(P,M) \longrightarrow H^3_L(P,M) \longrightarrow \cdots \end{array} \qquad (\mathrm{A})$$

where P is an $\mathrm{NP^{lr}}$*-algebra and M a representation of P.*

$$\begin{array}{l} H^3\big(\mathrm{cone}\,(-\beta^*)\big) \longrightarrow H^3_{\mathrm{NP}^l}(P,M) \longrightarrow H^3_H(P,M) \longrightarrow \\ \longrightarrow H^4\big(\mathrm{cone}\,(-\beta^*)\big) \longrightarrow H^4_{\mathrm{NP}^l}(P,M) \longrightarrow H^4_H(P,M) \longrightarrow \cdots \end{array} \qquad (\mathrm{B}_1)$$

where P is an $\mathrm{NP^l}$*-algebra and M a representation of P.*

$$\begin{array}{l} H^3\big(\mathrm{cone}(-\beta'^*)\big) \longrightarrow H^3_{\mathrm{NP^r}}(P,M) \longrightarrow H^3_H(P,M) \longrightarrow \\ \longrightarrow H^4\big(\mathrm{cone}(-\beta'^*)\big) \longrightarrow H^4_{\mathrm{NP^r}}(P,M) \longrightarrow H^4_H(P,M) \longrightarrow \cdots \end{array} \tag{B$_2$}$$

where P is an $\mathrm{NP^r}$*-algebra and M a representation of P.*

$$\begin{array}{l} H^3\big(\mathrm{cone}(-\beta^*,-\beta'^*)\big) \longrightarrow H^3_{\mathrm{NP^{lr}}}(P,M) \longrightarrow H^3_H(P,M) \longrightarrow \\ \longrightarrow H^4\big(\mathrm{cone}(-\beta^*,-\beta'^*)\big) \longrightarrow H^4_{\mathrm{NP^{lr}}}(P,M) \longrightarrow H^4_H(P,M) \longrightarrow \cdots \end{array} \tag{B}$$

where P is an $\mathrm{NP^{lr}}$*-algebra and M a representation of P.*

$$\begin{array}{l} H^2_H(P,M^e) \longrightarrow H^3_{\mathrm{AWB^r}}(P,M) \longrightarrow H^3_H(P,M) \longrightarrow \\ \longrightarrow H^3_H(P,M^e) \longrightarrow H^4_{\mathrm{AWB^r}}(P,M) \longrightarrow H^4_H(P,M) \longrightarrow \cdots \end{array} \tag{C$_{1,2}$}$$

where P is an $\mathrm{AWB^r}$ *and M a representation of P. Analogous exact sequence we have for $H_{\mathrm{AWB}^l}(P,M)$.*

$$\begin{array}{l} H^2_H(P,M^e) \longrightarrow H^3_{\mathrm{AWB^{lr}}}(P,M) \longrightarrow H^3_{\mathrm{AWB^r}}(P,M) \longrightarrow \\ \longrightarrow H^3_H(P,M^e) \longrightarrow H^4_{\mathrm{AWB^{lr}}}(P,M) \longrightarrow H^4_{\mathrm{AWB^r}}(P,M) \longrightarrow \cdots \end{array} \tag{C,C$'$}$$

where P is an $\mathrm{AWB^{lr}}$ *and M a representation of P. Analogous exact sequence we have, where $H_{\mathrm{AWB^r}}(P,M)$ is replaced by $H_{\mathrm{AWB}^l}(P,M)$.*

$$\begin{array}{l} H^2_H(P,M^e) \longrightarrow H^3_{\mathrm{NP^r}}(P,M) \longrightarrow H^3_H(P,M)\oplus H^3_L(P,M) \longrightarrow \\ \longrightarrow H^3_H(P,M^e) \longrightarrow H^4_{\mathrm{NP^r}}(P,M) \longrightarrow H^4_H(P,M)\oplus H^4_L(P,M) \longrightarrow \cdots \end{array} \tag{D$_{1,2}$}$$

where P is an $\mathrm{NP^r}$*-algebra and M a representation of P. Analogously for $H_{\mathrm{NP}^l}(P,M)$.*

$$\begin{array}{l} H^2_H(P,M^e) \longrightarrow H^3_{\mathrm{NP^{lr}}}(P,M) \longrightarrow H^3_{\mathrm{NP^r}}(P,M) \longrightarrow \\ \longrightarrow H^3_H(P,M^e) \longrightarrow H^4_{\mathrm{NP^{lr}}}(P,M) \longrightarrow H^4_{\mathrm{NP^r}}(P,M) \longrightarrow \cdots \end{array} \tag{D,D$'$}$$

where P is an $\mathrm{NP^{lr}}$*-algebra and M a representation of P. Analogous exact sequence we have when $H_{\mathrm{NP^r}}(P,M)$ is replaced by $H_{\mathrm{NP}^l}(P,M)$.*

$$\begin{array}{l} H^2_H(P,M^e)\oplus H^2_H(P,M^e) \longrightarrow H^3_{\mathrm{AWB^{lr}}}(P,M) \longrightarrow H^3_H(P,M) \longrightarrow \\ \longrightarrow H^3_H(P,M^e)\oplus H^3_H(P,M^e) \longrightarrow H^4_{\mathrm{AWB^{lr}}}(P,M) \longrightarrow H^4_H(P,M) \longrightarrow \cdots \end{array} \tag{E}$$

where P is an $\mathrm{AWB}^{\mathrm{lr}}$ *and M a representation of P.*

$$\begin{array}{l} H^2_H(P,M^e)\oplus H^2_H(P,M^e) \to H^3_{\mathrm{NP}^{\mathrm{lr}}}(P,M) \to H^3_H(P,M)\oplus H^3_L(P,M) \to \\ \to H^3_H(P,M^e)\oplus H^3_H(P,M^e) \to H^4_{\mathrm{NP}^{\mathrm{lr}}}(P,M) \to H^4_H(P,M)\oplus H^4_L(P,M) \to \cdots \end{array} \tag{F}$$

where P is an $\mathrm{NP}^{\mathrm{lr}}$*-algebra and M a representation of P.*

$$\begin{array}{l} H^2_H(P,M^e) \to H^3_{\mathrm{AWB}^l}(P,M)\oplus H^3\big(\mathrm{cone}(-\beta^*)\big) \to H^3_{\mathrm{NP}^l}(P,M) \to \\ \to H^3_H(P,M^e) \to H^4_{\mathrm{AWB}^l}(P,M)\oplus H^4\big(\mathrm{cone}(-\beta^*)\big) \to H^4_{\mathrm{NP}^l}(P,M) \to \cdots \end{array} \tag{G$_1$}$$

where P is an NP^{l}*-algebra and M a representation of P.*

$$\begin{array}{l} H^2_H(P,M^e) \to H^3_{\mathrm{AWB}^r}(P,M)\oplus H^3\big(\mathrm{cone}(-\beta'^*)\big) \to H^3_{\mathrm{NP}^r}(P,M) \to \\ \to H^3_H(P,M^e) \to H^4_{\mathrm{AWB}^r}(P,M)\oplus H^4\big(\mathrm{cone}(-\beta'^*)\big) \to H^4_{\mathrm{NP}^r}(P,M) \to \cdots \end{array} \tag{G$_2$}$$

where P is an NP^{r}*-algebra and M a representation of P.*

$$\begin{array}{l} H^2_H(P,M^e)\oplus H^2_H(P,M^e) \to H^3_{\mathrm{AWB}^{\mathrm{lr}}}(P,M)\oplus H^3\big(\mathrm{cone}(-\beta^*,-\beta'^*)\big) \to H^3_{\mathrm{NP}^{\mathrm{lr}}}(P,M) \to \\ \to H^3_H(P,M^e)\oplus H^3_H(P,M^e) \to H^4_{\mathrm{AWB}^{\mathrm{lr}}}(P,M)\oplus H^4\big(\mathrm{cone}(-\beta^*,-\beta'^*)\big) \to H^4_{\mathrm{NP}^{\mathrm{lr}}}(P,M) \to \cdots \end{array} \tag{G}$$

where P is an $\mathrm{NP}^{\mathrm{lr}}$*-algebra and M a representation of P.*

$$\begin{array}{l} H^2_H(P,M^e) \longrightarrow H^3\big(\mathrm{cone}(-\beta^*)\big) \longrightarrow H^3_L(P,M) \to \\ \to H^3_H(P,M^e) \longrightarrow H^4\big(\mathrm{cone}(-\beta^*)\big) \longrightarrow H^4_L(P,M) \longrightarrow \cdots \end{array} \tag{H$_{1,2}$}$$

where P is an NP^{l}*-algebra and M a representation of P. Analogous exact sequence we have for the cohomologies of the* $\mathrm{cone}(-\beta'^*)$ *and for an* NP^{r}*-algebra P.*

$$\begin{array}{l} H^2_H(P,M^e)\oplus H^2_H(P,M^e) \longrightarrow H^3\big(\mathrm{cone}(-\beta^*,-\beta'^*)\big) \longrightarrow H^3_L(P,M) \to \\ \to H^3_H(P,M^e)\oplus H^3_H(P,M^e) \longrightarrow H^4\big(\mathrm{cone}(-\beta^*,-\beta'^*)\big) \longrightarrow H^4_L(P,M) \longrightarrow \cdots \end{array} \tag{H}$$

for any $\mathrm{NP}^{\mathrm{lr}}$*-algebra P and a representation M of P.*

These exact sequences are obtained directly from the corresponding short exact sequences of the cohomology complexes.

Corollary 5.7.

Let P be an $\mathrm{NP^r}$-algebra with $\mathrm{c.dim}_H\, V_A^{\mathrm{r}}(P) \leq n$ and $\mathrm{c.dim}_L\, V_L^{\mathrm{r}}(P) \leq n$ (resp. an $\mathrm{NP^l}$-algebra with $\mathrm{c.dim}_H\, V_A^{\mathrm{l}}(P) \leq n$ and $\mathrm{c.dim}_L\, V_L^{\mathrm{l}}(P) \leq n$), $n \geq 2$, and M be a representation of P. Then we have:

(i) $H^{k+1}_{\mathrm{AWB^r}}(P, M) = 0,\ k > n$ $\big($*resp.* $H^{k+1}_{\mathrm{AWB}^l}(P, M) = 0,\ k > n\big)$, *where P is the underlying $\mathrm{AWB^r}$ (resp. $\mathrm{AWB^l}$) of the given algebra P;*

(ii) $H^{k+1}\big(\mathrm{cone}(-\beta'^*)\big) = 0$ $\big($*resp.* $H^{k+1}\big(\mathrm{cone}(-\beta^*)\big) = 0\big)$, $k > n$.

Proof. (i) By Corollary 5.3 (ii), $H^{k+1}_{\mathrm{NP^r}}(P, M) = 0,\ k > n$. Since M is a representation of P in the category of $\mathrm{NP^r}$-algebras, it follows that it is a representation of $V_L^{\mathrm{r}}(P)$ in **Leib** as well, i.e., P considered as the underlying Leibniz algebra. Now applying the condition $\mathrm{c.dim}_L\, V_L^{\mathrm{r}}(P) \leq n$ the result follows from long exact sequence (A_2) in Theorem 5.6. For $P \in \mathrm{NP^l}$ by the same Corollary 5.3 (ii), $H^{k+1}_{\mathrm{NP^l}}(P, M) = 0,\ k > n$. Now it is sufficient to apply the condition on cohomological dimension and (A_1).

(ii) The result follows from the statement (i) of this corollary and the exact sequence (G_2). Analogously we obtain the equality $H^n\big(\mathrm{cone}(-\beta^*)\big) = 0$, where we apply the exact sequence (G_1) in Theorem 5.6. □

Corollary 5.8.

Let P be an AWB. If $\mathrm{c.dim}_H\, P \leq n$, $n \geq 1$, where P is the corresponding underlying associative algebra, then

$$\mathrm{c.dim}_{\mathrm{AWB}}\, P \leq n + 1.$$

Proof. Let P be an $\mathrm{AWB^r}$ or an $\mathrm{AWB^l}$. The results follow from the exact sequences ($C_{1,2}$) in Theorem 5.6. Let P be a left-right AWB. Applying the result for $\mathrm{AWB^r}$ (or $\mathrm{AWB^l}$) for the underlying algebra P as an $\mathrm{AWB^r}$ (resp. as an $\mathrm{AWB^l}$), the result follows from the exact sequences (C,C′) in Theorem 5.6. □

Corollary 5.9.

Let P be an $\mathrm{NP^{lr}}$-algebra and $\mathrm{c.dim}_H\, V_A^{\mathrm{lr}}(P) \leq n$, $n \geq 2$. If M is a representation of P, then we have:

(i) $H^{k+1}_{\mathrm{NP^{lr}}}(P, M) \approx H^{k+1}\big(\mathrm{cone}(-\beta^*, -\beta'^*)\big)$, $H^{k+1}_{\mathrm{NP}^l}(P, M) \approx H^{k+1}\big(\mathrm{cone}(-\beta^*)\big)$, $H^{k+1}_{\mathrm{NP^r}}(P, M) \approx H^{k+1}\big(\mathrm{cone}(-\beta'^*)\big)$, $k > n$, *where in the last two isomorphisms P denotes the underlying $\mathrm{NP^l}$ and $\mathrm{NP^r}$-algebras of the given $\mathrm{NP^{lr}}$-algebra P, respectively;*

(ii) $H^{k+1}_{\mathrm{NP^{lr}}}(P, M) \approx H^{k+1}_{\mathrm{NP^r}}(P, M) \approx H^{k+1}_{\mathrm{NP}^l}(P, M)$, $k > n$, *where P in the last two right terms denotes the underlying $\mathrm{NP^l}$ and $\mathrm{NP^r}$-algebras of the given algebra P, respectively;*

(iii) $H^{k+1}_{\mathrm{NP^{lr}}}(P, M) \approx H^{k+1}_L(P, M)$, $k > n$, *where on the right side P denotes the underlying Leibniz algebra of the given algebra P.*

Proof. (i) follows from exact sequences (B_1), (B_2) and (B) in Theorem 5.6. Analogously, for the proofs of (ii) and (iii) we apply exact sequences (D,D′) and (F), respectively. Note that (iii) can be obtained as well by application of statement (i) of this corollary and the exact sequence (H). □

The below stated corollaries are proved by analogous arguments, therefore the proofs are left to the reader.

Corollary 5.10.

Let P be an NP-algebra and M be a representation of P. If $\mathrm{c.dim}_H\, P \leq n$ and $\mathrm{c.dim}_L\, P \leq n$, $n \geq 2$, where P denotes the underlying associative and Leibniz algebras, respectively, then we have:

(i) $\mathrm{c.dim}_{\mathrm{NP}}\, P \leq n + 1$;

(ii) $H^{k+1}\big(\mathrm{cone}(-\beta^*)\big) = H^{k+1}\big(\mathrm{cone}(-\beta'^*)\big) = H^{k+1}\big(\mathrm{cone}(-\beta^*, -\beta'^*)\big) = 0$, $k > n$.

Corollary 5.11.
Let P be an NP-algebra and M be a representation of P. If $\mathrm{c.dim}_L P \leq n$, where P is the underlying Leibniz algebra, then we have:

(i) $H^{k+1}_{\mathrm{NP}}(P, M) \approx H^{k+1}_{\mathrm{AWB}}(P, M)$, $k > n$;

(ii) $H^{k+1}\big(\mathrm{cone}(-\beta^*)\big) \approx H^{k+1}\big(\mathrm{cone}(-\beta'^*)\big) \approx H^{k+1}\big(\mathrm{cone}(-\beta^*, -\beta'^*)\big) \approx H^k_H(P, M^e)$, $k > n$.

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