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MATHEMATICS

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On One Variant of Richardson's Cyclic Iterative Method

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ABSTRACT. **For the corresponding iterative scheme, in the Richardson's cyclic iterative method instead of zeros of the raised in power *lk* Chebyshev polynomial $T_n^{lk}(x)$ we suggest to take *l*- and *l*(2*k*-1)-multiple zeros of the special raised in power *l* polynomial $R_{kn}^l(x)$. We achieve acceleration of the convergence in the Richardson's method, but the cycle itself elongates, although in both cases number *n* of different zeros remains as fixed as the general order of power *lkn* of the polynomials.**

Key words: modulus-maximum, cyclic iteration, superpositional substitution, *k*-multiple zero, normalized Chebyshev polynomial.

Consider the linear equation

$$A\varphi = f \tag{1}$$

with the symmetrical matrix A, whose eigen numbers $\{\lambda_i\}$ are on the segment

$$[m, M], \quad M > m > 0. \tag{2}$$

To solve equation (1) approximately, we can apply the Richardson's cyclic iterative method [1] which allows one to construct on the segment (2) the raised in power k normalized Chebyshev polynomial

$$T_n^k(x) = \prod_{i=1}^{n}\left(1-\frac{x}{\gamma_i}\right)^k \tag{3}$$

with γ_s=(s=1,...,n) -multiple zeros k and to apply these zeros in the iterative scheme

$$\varphi_s = \varphi_{s-1} - \frac{1}{\gamma_s}\left(A\varphi_{s-1} - f\right), \quad \varphi_0 = 0 \tag{4}$$

satisfying the condition $\gamma_{n-s}=\gamma_s$ (s=1,...,n). After k iterative cycles are completed (see [2]), for the φ_{kn} approximation to the exact solution we obtain by scheme (4) the following inequality:

$$\left\|\varphi_{kn} - \varphi\right\| \le \frac{\|f\|}{m}\max\left|T_n^k(x)\right|. \tag{5}$$

The maximum on the right-hand side of inequality (5) is taken (just as below in all analogous cases) on the segment (2).

Instead of the polynomial (3) with k-multiple zeros we take the polynomial considered in [3] in the particular case with single and $2k$-1-multiple zeros. In case n is even, we can write this polynomial in terms of

$$R_{kn}(x)=\frac{\left(\cos\frac{n}{2}\arccos(\frac{2x-M-m}{M-m}-u\right)\left(\cos\frac{n}{2}\arccos\frac{2x-M-m}{M-m}-\vartheta\right)^{2k-1}}{\left(t_{\frac{n}{2}}-u\right)\left(t_{\frac{n}{2}}-\vartheta\right)^{2k-1}}. \quad (6)$$

According to the notation introduced in [3], the value $t_{\frac{n}{2}}$ appearing in formula (6) is equal to the abscissa of that point which corresponds to the normalization point x=0 upon transformation of the segment (2) into the segment [-1, 1] using the linear substitution

$$t=\frac{2x-M-m}{M-m}.$$

As for the values u and y, they are chosen in such a way that all n+1 modulus-maxima of the polynomial (6) are equal among themselves, and this ensures maximality of the above-mentioned modulus-maxima in the condition under consideration.

The following statement holds.

Statement 1. If for the real numbers a and b (b>a>0) and for the natural number k>1 the inequality

$$\frac{b}{a}>\left(\frac{2k-1}{k^2}\right)\left(2-\frac{1}{k}\right)^2 \quad (7)$$

is fulfilled, then the inequality

$$\max_{x\in[a,b]}\left|\left(1-\frac{x}{a}\right)\left(1-\frac{x}{b}\right)^{2k-1}\right|<\max_{x\in[a,b]}\left|\left(1-\frac{x}{a}\right)^k\left(1-\frac{x}{b}\right)^k\right| \quad (8)$$

will likewise be fulfilled. If the inequality opposite to (7) is fulfilled, then the inequality opposite to (8) will likewise be fulfilled. In case $[a,b]\subset[m,M]$ and the conditions $R_{2k}(m)=|R_{2k}(M)|=|R_{2k}(X_0)|$, $R'_{2k}(X_0)=0$, are fulfilled, then the polynomial on the left-hand side of inequality (8) turns out to be the initial polynomial which after the corresponding superpositional substitution (see [3]) results in the polynomial (6) for n>2. Taking the above remark into account, we can extend Statement 1 to the case, when the segment $[a,b]$ is embedded into the segment (2), and then compare polynomials (3) and (6) with respect to the modulus-maxima on the entire segment (2) for n=2. (It is clear that in this case we consider two subsegments on the segment (2). The abscissas of the ends of one subsegment will be zeros of the polynomial (3), while those of the second subsegment will be zeros of the polynomial (6). In the case under consideration we have the following

Statement 2. For the segment $[m, M]$ (M>m>0) and for natural numbers k>1 there exists a function $\psi(k)$ such that if the inequality

$$\frac{M}{m}>\psi(k) \quad (9)$$

is fulfilled, the inequality

$$\max|R_{2k}(x)| < \max|T_2^k(x)| \tag{10}$$

will likewise be fulfilled, and if the inequality opposite to (9) is fulfilled, the inequality opposite to (10) will be fulfilled as well.

Remark to Statement 2. Using the corresponding superpositional substitution, we can extend inequality (10) to the case $n>2$ (see [3]). It should be noted that unlike condition (7) we have not managed to establish the function $\psi(k)$ in condition (9) explicitly, we have succeeded only in establishing the exact lower bounds for every fixed k using the "exhaustive" method (here, under "exhaustive" is meant variation of the value M/m and checking the validity of inequality (10) for the fixed k).

In the Table below, for natural numbers $k>1$ and $n=2$ we present values of the function $\psi(k)$ which appears in condition (9). For every separately taken k and for the segments of type (2), these values show the exact lower bound of the Todd number values starting from which inequality (10) is fulfilled.

Table 1

k	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\psi(k)$	3.1	5.1	7.1	9	11	13	14.9	16.8	19	21	23	24.4	26.3	28.2	30.1

Inequality (10) with regard for inequality (5) gives us all grounds to prefer as iterative parameters the zeros of the polynomial (6) to those of the polynomial (3), if iterations are performed by the scheme (4) for a number of cycles multiple to k, i.e., equal to lk. In the latter case instead of inequality (10) there takes place the inequality $\max|R_{2k}(x)|^l < \max|T_2^k(x)|^l$, for which for sufficiently large l one can expect essential decrease of modulus-maxima in case we take the polynomial (6) instead of the polynomial (3) and, respectively, essential practical gain decreasing the volume of calculations. Along with the above said, it should be taken into account that if in scheme (4) we use zeros of the polynomial (6), the length of the cycle increases k times taking kn instead of n (the number n of different zeros remains fixed), and therefore in constructing the polynomial (6) it is not desirable to take large k.

Table 2

l	50	70	125	180	220	300	380	420	450	492
$\max\|T_n^{kl}(x)\|$	$2x10^{-3}$	$2x10^{-4}$	$2x10^{-7}$	$2x10^{-10}$	10^{-12}	$4x10^{-17}$	$2x10^{-21}$	10^{-23}	$3x10^{-25}$	$2x10^{-27}$
$\max\|R_{kn}^l(x)\|$	$2x10^{-4}$	$7x10^{-6}$	$7x10^{-10}$	$6x10^{-14}$	$7x10^{-17}$	$9x10^{-23}$	10^{-28}	10^{-31}	$9x10^{-34}$	$7x10^{-37}$

Table 3

l	4	10	13	18	24	29	32	40	45	50
$\max\|T_n^{kl}(x)\|$	$2x10^{-2}$	$4x10^{-5}$	$2x10^{-6}$	10^{-18}	$3x10^{-11}$	$2x10^{-13}$	10^{-14}	$4x10^{-18}$	$2x10^{-20}$	$2x10^{-22}$
$\max\|R_{kn}^l(x)\|$	$3x10^{-3}$	$5x10^{-7}$	$7x10^{-9}$	$5x10^{-12}$	$9x10^{-16}$	$7x10^{-19}$	$9x10^{-21}$	$9x10^{-26}$	$7x10^{-29}$	$5x10^{-32}$

The second and the third lines of Table 2 give for fixed M/m=500, n=4 and k=2 the values of modulus-maxima, respectively, of polynomials (3) and (6) raised in power l whose values are given in the first line.

The conditions in Table 3 are the same as those in Table 2 with the only exception that instead of the condition k=2 we take k=16.

Introduce into our consideration the value [4]

$$\sqrt{\sum_i P_n^2(\lambda_i)}, \tag{11}$$

where $\{\lambda_i\}$ are assumed to be uniformly distributed on the segment (2). By (11) one can judge to what extent the polynomial $P_n(\lambda_i)$ is good for the Richardson's method, since the value (11) is by itself the error norm of the n-th approximation of equation (1) to the exact solution, provided the relation

$$f = A\sum_i v_i = \sum_i \lambda_i v_i$$

is fulfilled.

Table 4 reproduces the values of (11) for $R_{kn}^l(\lambda)$ and, respectively, for $T_n^{kl}(\lambda)$ for different values of n, k, l, M/m and also of the subintegral k upon the uniform partitioning of the entire segment of the spectrum $\left[1, \frac{M}{m}\right]$.

Table 4

n	k	l	M/m	h	$\sqrt{\sum_i R_{kn}^{2l}(\lambda_i)}$	$\sqrt{\sum_i T_n^{2kl}(\lambda_i)}$
6	6	30	1101	1	$9\text{x}10^{-8}$	$7\text{x}10^{-5}$
10	6	30	1101	1	$9\text{x}10^{-18}$	$3\text{x}10^{-13}$
2	6	30	1101	1	0.40908	1.68052
10	8	16	1101	1	$4\text{x}10^{-12}$	$2\text{x}10^{-9}$
6	8	20	1101	1	$8\text{x}10^{-7}$	$3\text{x}10^{-4}$
8	8	16	1101	2	$9\text{x}10^{-9}$	$3\text{x}10^{-6}$
4	16	20	1025	2	$3\text{x}10^{-7}$	$2\text{x}10^{-4}$
8	2	70	497	1	$4\text{x}10^{-18}$	$2\text{x}10^{-14}$

The advantage of the polynomials of type (6) over the polynomials of type (3) in achieving small modulus-maxima is so appreciable that under additional strengthened conditions we may have the inequality

$$\max\left|R_{kn}(x)\right| \le \max\left|T_n^{2k-1}(x)\right| \tag{12}$$

in which one have to pay special attention to the fact that on the left-hand side there is the polynomial of order kn, while on the right-hand side the polynomial of order n raised in power $2k$-1, and the difference in power order for the above-mentioned polynomials is rather essential, i.e., equal to $(k-1)n$.

The following statement is valid.

Statement 3. If for real numbers a and b ($b>a>0$) and for the natural number $k>1$ the inequality

$$\frac{b}{a} > \left[\frac{(4k)^k}{(k+1)^{k+1}}\right]^{\frac{1}{k-1}} + 1, \tag{13}$$

is fulfilled, then the inequality

$$\max_{x\in[a,b]}\left|(1-\frac{x}{a})(1-\frac{x}{b})^k\right| \le \max_{x\in[a,b]}\left|(1-\frac{x}{a})^k(1-\frac{x}{b})^k\right|, \tag{14}$$

will be fulfilled likewise, and if the inequality oppose to (13) is fulfilled, then the inequality opposite to (14) will be fulfilled as well.

Just as above (see Statement 2), we can compare for $n=2$ the polynomial of type (3) of order $2(2k-1)$ with that of type (6) of order $2k$ with respect to the modulus-maxima on the entire segment (2), and then we shall have the following

Statement 4. For the segment $[m, M]$ ($M>m>0$) and for natural numbers $k>5$ there exists the function $\omega(k)$ such that if the inequality

$$\frac{M}{m} > \omega(k) \tag{15}$$

is fulfilled, then the inequality

$$\max\left|R_{2k}(x)\right| \le \max\left|T_2^{2k-1}(x)\right|, \tag{16}$$

will be fulfilled likewise, and if the inequality opposite to (15) is fulfilled, then the inequality opposite to (16) will be fulfilled as well.

Remark to Statement 4. Using the corresponding superpositional substitution, we can extend inequality (16) to the case $n>2$ (see [3]).

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მათემატიკა

ი. ბუხნიკაშვილი

რიჩარდსონის ციკლური იტერაციული მეთოდის ერთი ვარიანტის შესახებ

რეზიუმე. რიჩარდსონის ციკლურ იტერაციულ მეთოდში lk ხარისხში აყვანილი ჩებიშევის პოლინომის $T_n^{lk}(x)$ ნულების ნაცვლად სათანადო იტერაციული სქემისათვის შემოღებულია l ხარისხში აყვანილი სპეციალური სახის $R_{kn}^{l}(x)$ პოლინომის l-ჯერადი და $l(2k-1)$-ჯერადი ნულები. ამასთან ხდება რიჩარდსონის მეთოდის კრებადობის დაჩქარება, თუმცა ამავე დროს იზრდება ციკლის სიგრძე, მაგრამ ორივე შემთხვევაში განსხვავებული ნულების რაოდენობა n უცვლელი რჩება ისევე, როგორც პოლინომების საერთო რიგი lkn.