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Mathematical Modelling and Analysis of Interaction Problems for Piezoelectric Composites (***)

ABSTRACT. — We investigate three-dimensional transmission problems related to the interaction of metallic and piezoelectric ceramic bodies with regard to thermal effects. We give a mathematical formulation of the physical problem when the metallic and ceramic sub-domains are bonded along some proper parts of their boundaries. The corresponding nonclassical mixed boundary-transmission problem is reduced by potential methods to an equivalent strongly elliptic system of pseudodifferential equations on manifolds with boundary. We investigate the solvability of this system in different function spaces. On the basis of these results we prove uniqueness and existence theorems for the original boundary-transmission problem. We study also the regularity of the electrical and thermomechanical fields near the curves where the boundary conditions change or where the interfaces intersect the exterior boundary. The electrical and thermomechanical fields can be decomposed into singular and more regular terms near these curves. A power of the distance from a reference point to the corresponding edge-curves occurs in the singular terms and describes the regularity explicitly. We compute these complex-valued exponents and demonstrate their dependence on the material parameters.

1. - INTRODUCTION

The paper deals with mixed type boundary transmission problems arising in the modelling of complex composites consisting of piezoelectric matrix with metallic inclusions (electrodes) when thermal effects are taken into consideration. Modern industrial and technological processes apply widely such type composite materials. The phenomenon of piezoelectricity is essentially used in measuring and controlling devices, electro-mechanical converters (transducers) and in the so-called "smart materials"

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transforming mechanical loadings into electric effects and vice versa. In particular, stack actuators are used in injectors for common-rail engines as vaporizers and valves. Therefore investigation of the mathematical models for such composite materials and analysis of the corresponding mechanical, thermal and electric fields became very actual and important for both fundamental research and practical applications. We remark here that during last years more then 1000 scientific works have been published annually (see. e.g., [La1]).

W. Voigt [Vo1] was the first who constructed a linear mathematical model of an elastic medium taking the interaction of electric and mechanical fields into account and derived the corresponding system of differential equations. In their works R. Toupin, R. Mindlin, L. Knopoff, S. Kaliski and J. Petikiewicz suggested new, more refined models of an elastic medium, where a polarization vector and its gradient occur [To1], [Mi2], [Mi3] (see also [No1], [Pa1], [Qi1]).

In this paper we study the following problem: *Given is a three-dimensional composite consisting of a piezoelectric (ceramic) matrix with metallic inclusions (electrodes). Derive a linear model for the interaction of the elastic and electrical fields with regard to thermal effects and perform a rigorous mathematical analysis by potential methods.*

Similar problems of the classical theory of elasticity have been studied by G.Fichera in [Fi1] with the help of functional variational methods.

Here we apply the Voigt's linear model with regard to thermal effects in the piezoelectric part and the usual classical model of thermoelasticity in the metallic part to write the corresponding coupled systems of governing partial differential equations. As a result, in the piezoceramic part the unknown field is represented by a 5-component vector (three components of the displacement vector, the temperature distribution and the electric potential function), while in the metallic part the unknown field is described by a 4-component vector (three components of the displacement vector and the temperature distribution).

Therefore, the situation becomes complicated since we have to find boundary and transmission conditions for the physical fields possessing different dimensions in adjacent domains. The main difficulty in modelling was to find appropriate boundary and transmission conditions for the composed body and to formulate them in an efficient way. Mathematical theory of such a general boundary-transmission problems is far from being complete.

Note also, that crystal structures with central symmetry, in particular isotropic structures, do not reveal the piezoelectric properties in Voight's model [Vo1]. Therefore the piezoelectric problems should be investigated for anisotropic media. This also complicates the investigation. Thus, we have to take into account the composed anisotropic structure and the diversity of the fields in the ceramic and metallic parts.

In this paper we apply potential methods which lead to boundary integral (pseudo-differential) equations. The solutions will be constructed with the help of an indirect boundary integral equations method, writing them as layer potentials in the ceramic and metallic parts with unknown densities. The densities are to be determined in such a way,

that the transmission and boundar...
of the resulting boundary-integral...
Bessel potential (H_p^s), and Besov ...
from the representation of the sol...
near curves where the boundary c...
boundary there are restrictions to ...
of the eigenvalues of the princ...
differential boundary operators

The paper is organized as fol...
linear theory of thermoelasticity a...
matrix partial differential operator...
generated by the field equation...
appropriate function spaces fo...
piezoelectric ceramic parts. In ...
potential operators and prove th...
on the boundaries of the metallic...
this paper. Here the original...
pseudodifferential equations inv...
and the Dirichlet part Γ of the e...
matrices yield information on t...
particular, in Theorem 4.3, the...
$a \in \left(0, \frac{1}{2}\right)$ depending on the e...
eigenvalues depend on the ma...
singularity exponents. We comp...
their dependence on the material...

2. - FIELD EQUATIONS ...

2.0 *List of Notation*

$\mathbb{R}^k$ - k-dimensional space o...

$\mathbb{C}^k$ - k-dimensional space o...

$a \cdot b = \sum_{j=1}^{k} a_j \overline{b}_j$ - the s...
$b = (b_1, \cdots, b_k) \in \mathbb{C}^k$;

$\overline{\Omega}$ - domain occupied by a p...

$\overline{\Omega^{(m)}}$ - domain occupied by ...

$\Gamma^{(m)} = \partial\Omega^{(m)} \cap \partial\Omega$ - contact...
parts;

$n = (n_1, n_2, n_3)$ - unit outward...

$\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3)$ - unit outward...

that the transmission and boundary conditions are satisfied. The solvability and regularity of the resulting boundary-integral equations are analyzed in Sobolev-Slobodetski (W_p^s), Bessel potential (H_p^s), and Besov ($B_{p,t}^s$) spaces. The results for the original problem follow from the representation of the solution by boundary integrals. Due to stress singularities near curves where the boundary conditions change or the interfaces intersect the exterior boundary there are restrictions to s and p. These restrictions are written explicitly in terms of the eigenvalues of the principal symbol matrices of the corresponding pseudo-differential boundary operators.

The paper is organized as follows. In section 2 we collect the field equations of the linear theory of thermoelasticity and thermopiezoelasticity, introduce the corresponding matrix partial differential operators and the generalized matrix boundary stress operators generated by the field equations, and derive a boundary-transmission problem in appropriate function spaces for the composed body consisting of metallic and piezoelectric ceramic parts. In section 3 we summarize some known properties on potential operators and prove the invertibility of pseudo-differential operators acting on the boundaries of the metallic and ceramic sub-domains. Section 4 is the main part of this paper. Here the original transmission problem is reduced to the system of pseudodifferential equations involving boundary operators acting on the interface $\Gamma^{(m)}$ and the Dirichlet part Γ of the exterior boundary. Their principal homogeneous symbol matrices yield information on the existence and regularity of the solution fields. In particular, in Theorem 4.3, the global C^{α}-regularity results are shown with some $\alpha \in \left(0, \frac{1}{2}\right)$ depending on the eigenvalues of these symbol matrices. Note, that these eigenvalues depend on the material parameters and actually they define the stress singularity exponents. We compute these complex-valued exponents and demonstrate their dependence on the material parameters.

2. - Field equations. Formulation of the boundary-transmission problem

2.0 *List of Notation*

$\mathbb{R}^k$ - k-dimensional space of real numbers;

$\mathbb{C}^k$ - k-dimensional space of complex numbers;

$a \cdot b = \sum_{j=1}^{k} a_j \overline{b_j}$ - the scalar product of two vectors $a = (a_1, \cdots, a_k)$, $b = (b_1, \cdots, b_k) \in \mathbb{C}^k$;

$\overline{\Omega}$ - domain occupied by a piezoceramic material;

$\overline{\Omega^{(m)}}$ - domain occupied by a metallic material;

$\Gamma^{(m)} = \partial\Omega^{(m)} \cap \partial\Omega$ - contact interface subsurface between metallic and piezoceramic parts;

$n = (n_1, n_2, n_3)$ - unit outward normal vector to $\partial\Omega$;

$\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3)$ - unit outward normal vector to $\partial\Omega^{(m)}$;

$\partial = \partial_x = (\partial_1, \partial_2, \partial_3), \partial_j = \partial/\partial x_j$ – partial derivatives with respect to the spatial variables;
$\partial_t = \partial/\partial t$ – partial derivative with respect to the time variable;
ρ, $\rho^{(m)}$ – mass densities;
c_{ijkl}, $c^{(m)}_{ijkl}$ – elastic constants; $\lambda^{(m)}$, $\mu^{(m)}$ – Lamé constants;
e_{kij} – piezoelectric constants;
ε_{kj}, ε – dielectric (permittivity) constants;
γ_{kj}, $\gamma^{(m)}_{kj}$, $\gamma^{(m)}$ – thermal strain constants;
κ_{kj}, $\kappa^{(m)}_{kj}$, $\kappa^{(m)}$ – thermal conductivity constants;
$\tilde{c}$, $\tilde{c}^{(m)}$ – specific heat per unit mass;
T_0, $T^{(m)}_0$ – initial reference temperature, that is the temperature in the natural state in the absence of deformation and electromagnetic fields;
$\alpha := \rho\,\tilde{c}$, $\alpha^{(m)} := \rho^{(m)}\,\tilde{c}^{(m)}$ – thermal material constants;
g_i $(i = 1,2,3)$ – constants characterizing the relation between thermodynamic processes and piezoelectric effect (pyroelectric constants);
$X = (X_1, X_2, X_3)^\top$, $X^{(m)} = (X^{(m)}_1, X^{(m)}_2, X^{(m)}_3)^\top$ – mass force densities;
X_4, $X^{(m)}_4$ – heat source densities;
X_5 – charge density;
$u = (u_1, u_2, u_3)^\top$, $u^{(m)} = (u^{(m)}_1, u^{(m)}_2, u^{(m)}_3)^\top$ – mechanical displacement vectors;
ϕ – electric potential;
$E := -\operatorname{grad}\phi$ – electric field vector;
D – electric displacement vector;
$\vartheta = T - T_0$, $\vartheta^{(m)} = T^{(m)} - T^{(m)}_0$ – relative temperature (temperature increment);
$q = (q_1, q_2, q_3)$, $q^{(m)} = (q^{(m)}_1, q^{(m)}_2, q^{(m)}_3)$ – heat flux vector;
$s_{kj} = s_{kj}(u) := \frac{1}{2}(\partial_k u_j + \partial_j u_k)$, $s^{(m)} = s^{(m)}_{kj}(u^{(m)}) := \frac{1}{2}(\partial_k u^{(m)}_j + \partial_j u^{(m)}_k)$ – strain tensors;
$\sigma^{(m)}_{kj} = \sigma^{(m)}_{kj}(u^{(m)}, \vartheta^{(m)})$ – mechanical stress tensor in the theory of thermoelasticity;
$\sigma_{kj} = \sigma_{kj}(u, \vartheta, \phi)$ – mechanical stress tensor in the theory of thermoelectroelasticity;
$\mathcal{S}$, $\mathcal{S}^{(m)}$ – entropy densities;
$U^{(m)} := (u^{(m)}_1, u^{(m)}_2, u^{(m)}_3, u^{(m)}_4)^\top$ with $u^{(m)}_4 = \vartheta^{(m)}$;
$U := (u_1, u_2, u_3, u_4, u_5)^\top$ with $u_4 = \vartheta$ and $u_5 = \phi$.

2.1. *Thermoelastic field equations*

Here we collect the field equations of the linear theory of thermoelasticity and introduce the corresponding matrix partial differential operators (see [No2], [KGBB1]). We will treat the general anisotropic case.

The basic governing equations of the classical thermoelasticity read as follows (see the list of notation):

Constitutive relations:

$$\sigma^{(m)}_{ij} = \sigma^{(m)}_{ji} = c^{(m)}_{ijlk}\, s^{(m)}_{lk} - \gamma^{(m)}_{ij}\, \vartheta^{(m)} = c^{(m)}_{ijlk}\, \partial_l u^{(m)}_k - \gamma^{(m)}_{ij}\, \vartheta^{(m)}, \tag{2.1}$$

$$\mathcal{S}^{(m)} = \gamma^{(m)}_{ij}\, s^{(m)}_{ij} + \alpha^{(m)}\,[\,T^{(m)}_0\,]^{-1}\, \vartheta^{(m)}; \tag{2.2}$$

Fourier Law:

(2.3) $$q_j^{(m)} = -\kappa_{jl}^{(m)}\,\partial_l T^{(m)};$$

Equations of motion:

(2.4) $$\partial_i \sigma_{ij}^{(m)} + X_j^{(m)} = \rho^{(m)}\,\partial_t^2 u_j^{(m)};$$

Equation of the entropy balance:

(2.5) $$T^{(m)}\,\partial_t S^{(m)} = -\partial_j q_j^{(m)} + X_4^{(m)}.$$

The physical sense of the material parameters and mechanical characteristics involved in these relations are determined and specified in the list of notation. All these characteristics are expressed by means of the displacement vector $u^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)})^\top$ and the relative temperature (temperature increment) $\theta^{(m)}$. Here and throughout the paper the superscript $\top$ denotes transposition.

Constants involved in the above equations satisfy the symmetry conditions:

(2.6) $$c_{ijkl}^{(m)} = c_{jikl}^{(m)} = c_{klij}^{(m)},\ \ \gamma_{ij}^{(m)} = \gamma_{ji}^{(m)},\ \ \kappa_{ij}^{(m)} = \kappa_{ji}^{(m)},\ \ i,j,k,l = 1,2,3.$$

We assume that there are positive constants c_0 and c_1 such that

(2.7) $$c_{ijkl}^{(m)}\,\xi_{ij}\,\xi_{kl} \geq c_0\,\xi_{ij}\,\xi_{ij},\quad \kappa_{ij}^{(m)}\xi_i\xi_j \geq c_1\,\xi_i\,\xi_i$$

for all $\xi_{ij} = \xi_{ji}, \xi_j \in \mathbb{R}$, $i,j = 1,2,3$.

In particular, the first inequality implies that the density of potential energy corresponding to the displacement vector $u^{(m)}$,

$$E^{(m)}\big(u^{(m)}, u^{(m)}\big) = c_{ijlk}^{(m)}\,s_{ij}^{(m)}\,s_{lk}^{(m)}$$

is positive definite with respect to the symmetric components of the strain tensor

$$s_{lk}^{(m)} = 2^{-1}\,(\partial_l\,u_k^{(m)} + \partial_k\,u_l^{(m)}).$$

Substituting (2.1) into (2.4) leads to the equation:

(2.8) $$c_{ijlk}^{(m)}\,\partial_i\,\partial_l\,u_k^{(m)} - \gamma_{ij}^{(m)}\partial_i\theta^{(m)} + X_j^{(m)} = \rho^{(m)}\,\partial_t^2\,u_j^{(m)},\ \ j = 1,2,3.$$

Taking into account the Fourier law (2.3) and relation (2.2) from the equation of the entropy balance (2.5) after linearization we obtain the heat transfer equation

(2.9) $$\kappa_{il}^{(m)}\partial_i\,\partial_l\theta^{(m)} - a^{(m)}\,\partial_t\theta^{(m)} - T_0^{(m)}\,\gamma_{il}^{(m)}\,\partial_t\partial_l u_i^{(m)} + X_4^{(m)} = 0.$$

Simultaneous equations (2.8) and (2.9) represent the basic system of dynamics of the theory of thermoelasticity. If all the functions involved in these equations are harmonic time dependent, that is they represent a product of a function of the spatial variables (x_1, x_2, x_3) and the multiplier $\exp\{\tau t\}$, where $\tau = \sigma + i\omega$ is a complex parameter, we have the *pseudo-oscillation equations* of the theory of thermoelasticity. Note that the pseudo-oscillation equations can be obtained from the corresponding dynamical equations by the Laplace transform. If $\tau = i\omega$ is a pure imaginary number, with the so called frequency parameter $\omega \in \mathbb{R}$, we obtain the *steady state oscillation equations*. Finally, if $\tau = 0$ we get the *equations of statics*.

In this paper we will mainly consider the system of pseudo-oscillations

$$(2.10)\qquad \begin{aligned} &c_{ijlk}^{(m)}\,\partial_i\partial_l\,u_k^{(m)} - \rho^{(m)}\,\tau^2\,u_j^{(m)} - \gamma_{ij}^{(m)}\,\partial_i\theta^{(m)} + X_j^{(m)} = 0, \quad j = 1,2,3,\\ &-\tau\,T_0^{(m)}\,\gamma_{il}^{(m)}\,\partial_l u_i^{(m)} + \kappa_{il}^{(m)}\,\partial_i\partial_l\theta^{(m)} - \tau\,a^{(m)}\,\theta^{(m)} + X_4^{(m)} = 0.\end{aligned}$$

In matrix form these equations can be rewritten as

$$A^{(m)}(\partial,\tau)\,U^{(m)}(x) + \tilde X^{(m)}(x) = 0,$$

where $U^{(m)} := (u^{(m)}, \theta^{(m)})^\top$ is the sought vector, $\tilde X^{(m)} = (X_1^{(m)}, X_2^{(m)}, X_3^{(m)}, X_4^{(m)})^\top$, $X^{(m)} = (X_1^{(m)}, X_2^{(m)}, X_3^{(m)})^\top$ is a given mass force density, $X_4^{(m)}$ is a given heat source density. $A^{(m)}(\partial,\tau)$ is the nonselfadjoint matrix differential operator generated by equations (2.10).

$$A^{(m)}(\partial,\tau) = [\,A_{jk}^{(m)}(\partial,\tau)\,]_{4\times4}, \quad A_{jk}^{(m)}(\partial,\tau) = c_{ijlk}^{(m)}\,\partial_i\,\partial_l - \rho^{(m)}\,\tau^2\,\delta_{jk},$$

$$A_{4k}^{(m)}(\partial,\tau) = -\tau\,T_0^{(m)}\,\gamma_{kl}^{(m)}\,\partial_l, \quad A_{j4}^{(m)}(\partial,\tau) = -\gamma_{ij}^{(m)}\,\partial_i, \quad A_{44}^{(m)}(\partial,\tau) = \kappa_{il}^{(m)}\,\partial_i\,\partial_l - a^{(m)}\,\tau.$$

where $j,k = 1,2,3$, and δ_{jk} is the Kronecker delta.

By $A^{(m)*}(\partial,\tau)$ we denote the 4×4 matrix differential operator formally adjoint to $A^{(m)}(\partial,\tau)$, that is $A^{(m)*}(\partial,\tau) := [\overline{A^{(m)}(-\partial,\tau)}]^\top$, where the over-bar denotes the complex conjugation.

Components of the mechanical thermostress vector acting on a surface element with a normal $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3)$ read as follows

$$\sigma_{ij}^{(m)}\,\upsilon_i = c_{ijlk}^{(m)}\,\upsilon_i\,\partial_l u_k^{(m)} - \gamma_{ij}^{(m)}\,\upsilon_i\,\theta^{(m)}, \quad j = 1,2,3,$$

while the normal component of the heat flux vector (with opposite sign) has the form

$$-q_i^{(m)}\,\upsilon_i = \kappa_{il}^{(m)}\,\upsilon_i\,\partial_l\theta^{(m)}.$$

We introduce the following generalized thermostress operator

$$(2.11)\qquad \mathcal{T}^{(m)}(\partial,\upsilon) = [\,\mathcal{T}_{jk}^{(m)}(\partial,\upsilon)\,]_{4\times4},$$

where (for $j,k = 1,2,3$)

$$\mathcal{T}_{jk}^{(m)}(\partial,\upsilon) = c_{ijlk}^{(m)}\,\upsilon_i\,\partial_l, \quad \mathcal{T}_{j4}^{(m)}(\partial,\upsilon) = -\gamma_{ij}^{(m)}\,\upsilon_i, \quad \mathcal{T}_{4k}^{(m)}(\partial,\upsilon) = 0, \quad \mathcal{T}_{44}^{(m)}(\partial,\upsilon) = \kappa_{il}^{(m)}\,\upsilon_i\,\partial_l.$$

For a four-vector $U^{(m)} = (u^{(m)}, \theta^{(m)})^\top$ we have

$$(2.12)\qquad \mathcal{T}^{(m)}\,U^{(m)} = (\,\sigma_{i1}^{(m)}\,\upsilon_i,\ \sigma_{i2}^{(m)}\,\upsilon_i,\ \sigma_{i3}^{(m)}\,\upsilon_i,\ -q_i^{(m)}\,\upsilon_i\,)^\top.$$

We introduce also the boundary operator associated with the adjoint operator $A^{(m)*}(\partial,\tau)$ which appears in Green's formulae,

$$\widetilde{\mathcal{T}}^{(m)}(\partial,\upsilon,\tau) = [\,\widetilde{\mathcal{T}}_{jk}^{(m)}(\partial,\upsilon,\tau)\,]_{4\times4},$$

where (for $j,k = 1,2,3$)

$$\begin{aligned} &\widetilde{\mathcal{T}}_{jk}^{(m)}(\partial,\upsilon,\tau) = c_{ijlk}^{(m)}\,\upsilon_i\,\partial_l, &&\widetilde{\mathcal{T}}_{j4}^{(m)}(\partial,\upsilon,\tau) = \overline{\tau}\,T_0^{(m)}\,\gamma_{ij}^{(m)}\,\upsilon_i,\\ &\widetilde{\mathcal{T}}_{4k}^{(m)}(\partial,\upsilon,\tau) = 0, &&\widetilde{\mathcal{T}}_{44}^{(m)}(\partial,\upsilon,\tau) = \kappa_{il}^{(m)}\,\upsilon_i\,\partial_l.\end{aligned}$$

2.2. *Thermopiezoelastic fie...*

In this subsection we co...
thermopiezoelasticity for a gene...
matrix partial differential operato...
In the thermopiezoelasticity ...
notation):

Constitutive relations:
(2.13) $\sigma_{ij} = \sigma_{ji} = c_{ijkl}\,s_{kl} - \ldots$
(2.14) $\mathcal{S} = \gamma_{ij}\,s_{ij} - \ldots$
(2.15) $D_j = e_{jkl}\,s_{kl} - \ldots$
Fourier Law:
(2.16)
Equations of motion:
(2.17)
Equation of the entropy ...
(2.18)
Equation of static electric ...
(2.19)
From the relations (2.13)–...
theory of thermopiezoelasticity ...

(2.20) $c_{ijlk}\,\partial_i\,\partial_l\,u_k - \ldots$ / $-\tau\,T_0\,\ldots$ / $-e_{ikl}\,\partial_i\partial_l u_k - \ldots$

or in matrix form

(2.21)

where $U := (u, \theta, \varphi)^\top$, ...
density, X_4 is a given heat source ...
differential operator generated ...

(2.22) $A(\partial,\tau) = \ldots$ / $A_{j4}(\partial,\tau) = -\ldots$ / $A_{44}(\partial,\tau) = \kappa\ldots$ / $A_{54}(\partial,\tau) = -\ldots$

2.2. *Thermopiezoelastic field equations*

In this subsection we collect the field equations of the linear theory of thermopiezoelasticity for a general anisotropic case and introduce the corresponding matrix partial differential operators (cf. [No1], [Qi1]).

In the thermopiezoelasticity we have the following governing equations (see the list of notation):

Constitutive relations:

(2.13) $\sigma_{ij} = \sigma_{ji} = c_{ijkl}\, s_{kl} - e_{lij} E_l - \gamma_{ij}\, \vartheta = c_{ijkl}\, \partial_l u_k + e_{lij}\, \partial_l \varphi - \gamma_{ij}\, \vartheta, \;\; i,j = 1,2,3,$

(2.14) $\mathcal{S} = \gamma_{ij}\, s_{ij} + g_l E_l + a\, [\, T_0\,]^{-1}\, \vartheta,$

(2.15) $D_j = e_{jkl}\, s_{kl} + \varepsilon_{jl}\, E_l + g_j\, \vartheta = e_{jkl}\, \partial_l u_k - \varepsilon_{jl}\, \partial_l \varphi + g_j\, \vartheta, \;\; j = 1,2,3.$

Fourier Law:

(2.16) $q_i = -\kappa_{il}\, \partial_l T, \quad i = 1,2,3.$

Equations of motion:

(2.17) $\partial_i \sigma_{ij} + X_j = \rho\, \partial_t^2 u_j, \quad j = 1,2,3.$

Equation of the entropy balance:

(2.18) $T\, \partial_t \mathcal{S} = -\partial_j q_j + X_4.$

Equation of static electric field:

(2.19) $\partial_i D_i - X_5 = 0.$

From the relations (2.13)-(2.19) we derive the linear system of pseudo-oscillations of the theory of thermopiezoelasticity:

$$
\begin{aligned}
& c_{ijlk}\, \partial_i\, \partial_l\, u_k - \rho\, \tau^2\, u_j - \gamma_{ij}\, \partial_i \vartheta + e_{lij}\, \partial_l \partial_i \varphi + X_j = 0, \quad j = 1,2,3, \\
& -\tau\, T_0\, \gamma_{il}\, \partial_l u_i + \kappa_{il}\, \partial_i\, \partial_l \vartheta - \tau\, a\, \vartheta + \tau\, T_0\, g_i\, \partial_i \varphi + X_4 = 0, \\
& -e_{ikl}\, \partial_i \partial_l u_k - g_i\, \partial_i \vartheta + \varepsilon_{il}\, \partial_i \partial_l \varphi + X_5 = 0,
\end{aligned}
\tag{2.20}
$$

or in matrix form

$$
A(\partial, \tau)\, U(x) + \widetilde{X}(x) = 0 \;\; \text{in} \;\; \Omega, \tag{2.21}
$$

where $U := (u, \vartheta, \varphi)^\top, \widetilde{X} = (X_1, X_2, X_3, X_4, X_5)^\top, X = (X_1, X_2, X_3)^\top$ is a given mass force density, X_4 is a given heat source density, X_5 is a given charge density, $A(\partial, \tau)$ is the matrix differential operator generated by equations (2.20)

$$
\begin{aligned}
& A(\partial, \tau) = \big[A_{jk}(\partial, \tau)\big]_{5\times 5}, \quad A_{jk}(\partial, \tau) = c_{ijlk}\, \partial_i\, \partial_l - \rho\, \tau^2\, \delta_{jk}, \\
& A_{j4}(\partial, \tau) = -\gamma_{ij}\, \partial_i, \quad A_{j5}(\partial, \tau) = e_{lij} \partial_l \partial_i, \quad A_{4k}(\partial, \tau) = -\tau\, T_0\, \gamma_{kl}\, \partial_l, \\
& A_{44}(\partial, \tau) = \kappa_{il}\, \partial_i\, \partial_l - a\, \tau, \quad A_{45}(\partial, \tau) = \tau\, T_0\, g_i\, \partial_i, \quad A_{5k}(\partial, \tau) = -e_{ikl}\, \partial_i \partial_l, \\
& A_{54}(\partial, \tau) = -g_i \partial_i, \quad A_{55}(\partial, \tau) = \varepsilon_{il}\, \partial_i \partial_l, \quad j,k = 1,2,3.
\end{aligned}
\tag{2.22}
$$

Clearly, from (2.20)-(2.22) we obtain the equations and operators of statics if $\tau = 0$. Constants involved in these equations satisfy the symmetry conditions:

$$c_{ijkl} = c_{jikl} = c_{klij}, \quad e_{ijk} = e_{ikj}, \quad \varepsilon_{ij} = \varepsilon_{ji}, \quad \gamma_{ij} = \gamma_{ji}, \quad \kappa_{ij} = \kappa_{ji}, \quad i, j, k, l = 1, 2, 3.$$

Moreover, from the physical considerations it follows that (see, e.g., [No1]):

$$(2.23) \qquad c_{ijkl}\, \xi_{ij}\, \xi_{kl} \geq c_0\, \xi_{ij}\, \xi_{ij} \ \text{ for all } \ \xi_{ij} = \xi_{ji} \in \mathbb{R},$$

$$(2.24) \qquad \varepsilon_{ij}\, \eta_i\, \eta_j \geq c_1\, |\eta|^2, \quad \kappa_{ij}\, \eta_i\, \eta_j \geq c_2\, |\eta|^2 \ \text{ for all } \ \eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3,$$

where c_0, c_1, and c_2 are positive constants. In addition, we require that (see, e.g., [No 1])

$$(2.25) \quad \varepsilon_{ij}\, \eta_i\, \overline{\eta}_j + \frac{a}{T_0}\, |\zeta|^2 - 2\, \Re(\zeta\, g_l\, \overline{\eta}_l) \geq c_3 (\, |\zeta|^2 + |\eta|^2\,) \ \text{ for all } \ \zeta \in \mathbb{C} \ \text{ and } \ \eta \in \mathbb{C}^3$$

with a positive constant c_3. A sufficient condition for (2.25) to be satisfied reads as follows $\dfrac{a\, c_1}{3\, T_0} - g^2 > 0$, where $g = \max\{|g_1|, |g_2|, |g_3|\}$ and c_1 is the constant involved in (2.24).

By $A^*(\partial, \tau)$ we denote the operator formally adjoint to $A(\partial, \tau)$, that is $A^*(\partial, \tau) := \\ := [A(-\partial, \tau)]^\top$.

In the theory of thermopiezoelasticity the components of the three-dimensional mechanical stress vector acting on a surface element with a normal $n = (n_1, n_2, n_3)$ have the form

$$\sigma_{ij}\, n_i = c_{ijlk}\, n_i\, \partial_l u_k + e_{lij}\, n_i\, \partial_l \varphi - \gamma_{ij}\, n_i\, \theta \ \text{ for } \ j = 1, 2, 3,$$

while the normal components of the electric displacement vector and the heat flux vector (with opposite sign) read as

$$-D_i\, n_i = -e_{ikl}\, n_i\, \partial_l u_k + \varepsilon_{il}\, n_i\, \partial_l \varphi - g_i\, n_i\, \theta, \qquad -q_i\, n_i = \kappa_{il}\, n_i\, \partial_l \theta.$$

Let us introduce the following matrix differential operator

$$(2.26) \qquad T(\partial, n) = [\, T_{jk}(\partial, n)\,]_{5\times 5},$$

where (for $j, k = 1, 2, 3$)

$$(2.27) \qquad \begin{aligned} &T_{jk}(\partial, n) = c_{ijlk}\, n_i\, \partial_l, \quad T_{j4}(\partial, n) = -\gamma_{ij}\, n_i, \quad T_{j5}(\partial, n) = e_{lij}\, n_i\, \partial_l, \\ &T_{4k}(\partial, n) = 0, \quad T_{44}(\partial, n) = \kappa_{il}\, n_i\, \partial_l, \quad T_{45}(\partial, n) = 0, \\ &T_{5k}(\partial, n) = -e_{ikl}\, n_i\, \partial_l, \quad T_{54}(\partial, n) = -g_i\, n_i, \quad T_{55}(\partial, n) = \varepsilon_{il}\, n_i\, \partial_l. \end{aligned}$$

For a vector $U = (u, \varphi, \theta)^\top$ we have

$$(2.28) \qquad T(\partial, n)\, U = (\, \sigma_{i1}\, n_i, \ \sigma_{i2}\, n_i, \ \sigma_{i3}\, n_i, \ -q_i\, n_i, \ -D_i\, n_i\,)^\top.$$

In Green's formulas there appear also the following boundary operator associated with the differential operator $A^*(\partial, \tau)$,

$$\widetilde{T}(\partial, n, \tau) = [\, \widetilde{T}_{jk}(\partial, n, \tau)\,]_{5\times 5},$$

where (for $j, k = 1, 2, 3$)

$$\widetilde{T}_{jk}(\partial, n, \tau) = c_{ijlk}\, n_i\, \partial_l, \quad \widetilde{T}_{j4}(\partial, n, \tau) = \overline{\tau}\, T_0\, \gamma_{ij}\, n_i, \quad \widetilde{T}_{j5}(\partial, n, \tau) = -e_{lij}\, n_i\, \partial_l,$$
$$\widetilde{T}_{4k}(\partial, n, \tau) = 0, \quad \widetilde{T}_{44}(\partial, n, \tau) = \kappa_{il}\, n_i\, \partial_l, \quad \widetilde{T}_{45}(\partial, n, \tau) = 0,$$
$$\widetilde{T}_{5k}(\partial, n, \tau) = e_{ikl}\, n_i\, \partial_l, \quad \widetilde{T}_{54}(\partial, n, \tau) = -\overline{\tau}\, T_0\, g_i\, n_i, \quad \widetilde{T}_{55}(\partial, n, \tau) = \varepsilon_{il}\, n_i\, \partial_l.$$

2.3. *Mathematical model of the physical problem: Formulation of the boundary-transmission problem*

Let $\Omega^{(m)}$ and Ω be bounded non-intersecting domains of the three-dimensional Euclidean space $\mathbb{R}^3$ with C^∞-smooth boundaries $\partial\Omega$ and $\partial\Omega^{(m)}$, respectively. Moreover, let $\partial\Omega$ and $\partial\Omega^{(m)}$ have a nonempty intersection $\overline{\Gamma^{(m)}}$ with a positive measure, i.e., $\partial\Omega \cap \partial\Omega^{(m)} = \overline{\Gamma^{(m)}}$, meas $\Gamma^{(m)} > 0$. From now on $\Gamma^{(m)}$ will be referred to as an interface surface. Throughout the paper n and υ stand for the outward unit normal vectors on $\partial\Omega$ and on $\partial\Omega^{(m)}$, respectively. Evidently, $n(x) = -\upsilon(x)$ for $x \in \Gamma^{(m)}$.

We set $S^{(m)} := \partial\Omega^{(m)} \setminus \overline{\Gamma^{(m)}}$ and $S^* := \partial\Omega \setminus \overline{\Gamma^{(m)}}$. Further, we denote by Γ some open, nonempty, proper sub-manifold of S^* and let $S := S^* \setminus \overline{\Gamma}$. Thus, we have the following decomposition of the boundary surfaces

$$\partial\Omega = \overline{\Gamma^{(m)}} \cup \overline{S} \cup \overline{\Gamma}, \quad \partial\Omega^{(m)} = \overline{\Gamma^{(m)}} \cup \overline{S^{(m)}}.$$

Throughout the paper, for simplicity, we assume that

$$\partial\Omega^{(m)},\ \partial\Omega,\ \partial S^{(m)},\ \partial\Gamma^{(m)},\ \partial\Gamma,\ \partial S \in C^\infty, \quad \text{and} \quad \partial\Omega^{(m)} \cap \overline{\Gamma} = \emptyset.$$

Let Ω be filled by an anisotropic homogeneous piezoelectric medium (ceramic matrix) and $\Omega^{(m)}$ be occupied by an isotropic or anisotropic homogeneous elastic medium (metallic inclusion). These two bodies interact to each other along the interface $\Gamma^{(m)}$. In the "metallic" domain $\Omega^{(m)}$ we have a four-dimensional thermoelastic field described by the displacement vector $u^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)})^\top$ and the temperature $u_4^{(m)} := \vartheta^{(m)}$, while in the piezoelectric domain Ω we have a five-dimensional physical field described by the displacement vector $u = (u_1, u_2, u_3)^\top$, the temperature $u_4 := \vartheta$ and by the electric potential $u_5 := \phi$.

The physical interaction problem under consideration is described by strongly elliptic systems of linear partial differential equations in the corresponding elastic and piezoelastic domains with appropriate mixed type boundary and transmission conditions on $S^{(m)}$, $\Gamma^{(m)}$, S, and Γ.

Solutions to this kind mixed boundary value problems and related mechanical and electrical characteristics usually have singularities in a neighbourhood of curves across which the type of boundary conditions change (e.g., $\partial\Gamma$) or where the interface intersects the exterior boundary of the composite body (e.g., $\partial\Gamma^{(m)}$). Our goal is to study the solvability of the mixed boundary transmission problem in appropriate function spaces and analyse regularity properties of solutions. In particular, we describe dependence of the stress singularity exponents on the material parameters. As we will see below this dependence is quite nontrivial.

Throughout the paper the symbol $\{\,\cdot\,\}^+$ denotes the interior one-sided limit on $\partial\Omega$ (respectively $\partial\Omega^{(m)}$) from Ω (respectively $\Omega^{(m)}$). Similarly, $\{\,\cdot\,\}^-$ denotes the exterior one-sided limit on $\partial\Omega$ (respectively $\partial\Omega^{(m)}$) from the exterior of Ω (respectively $\Omega^{(m)}$). We will use also the notation $\{\,\cdot\,\}^{\pm}_{\partial\Omega}$ and $\{\,\cdot\,\}^{\pm}_{\partial\Omega^{(m)}}$ for the trace operators on $\partial\Omega$ and $\partial\Omega^{(m)}$.

By L_p, W^r_p, H^s_p, and $B^s_{p,q}$ (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) we denote the well-known Lebesgue, Sobolev-Slobodetski, Bessel potential, and Besov function spaces, respectively (see, e.g., [Tr1]). Recall that $H^r_2 = W^r_2 = B^r_{2,2}$, $H^s_2 = B^s_{2,2}$, $W^t_p = B^t_{p,p}$, and $H^k_p = W^k_p$, for any $r \geq 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k.

Let $\mathcal{M}_0$ be a smooth surface without boundary. For a smooth sub-manifold $\mathcal{M} \subset \mathcal{M}_0$ we denote by $\widetilde{H}^s_p(\mathcal{M})$ and $\widetilde{B}^s_{p,q}(\mathcal{M})$ the subspaces of $H^s_p(\mathcal{M}_0)$ and $B^s_{p,q}(\mathcal{M}_0)$, respectively,

$$\widetilde{H}^s_p(\mathcal{M}) = \{g : g \in H^s_p(\mathcal{M}_0),\ \operatorname{supp} g \subset \overline{\mathcal{M}}\},\ \widetilde{B}^s_{p,q}(\mathcal{M}) = \{g : g \in B^s_{p,q}(\mathcal{M}_0),\ \operatorname{supp} g \subset \overline{\mathcal{M}}\},$$

while $H^s_p(\mathcal{M})$ and $B^s_{p,q}(\mathcal{M})$ denote the spaces of restrictions on $\mathcal{M}$ of functions from $H^s_p(\mathcal{M}_0)$ and $B^s_{p,q}(\mathcal{M}_0)$, respectively,

$$H^s_p(\mathcal{M}) = \{r_{\mathcal{M}} f : f \in H^s_p(\mathcal{M}_0)\}, \quad B^s_{p,q}(\mathcal{M}) = \{r_{\mathcal{M}} f : f \in B^s_{p,q}(\mathcal{M}_0)\},$$

where $r_{\mathcal{M}}$ is the restriction operator on $\mathcal{M}$.

Now, we come back to our boundary-transmission problem, restricting the fields to the metallic and ceramic sub-domains, denoted by $U^{(m)} = (u^{(m)}, u_4^{(m)})^\top$ and $U = (u, u_4, u_5)^\top$. Moreover, we assume that the initial reference temperatures T_0 and $T_0^{(m)}$ in the adjacent domains Ω and $\Omega^{(m)}$ are the same: $T_0 = T_0^{(m)}$. The mathematical problem reads:

Find vector-functions

$$U^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)}, u_4^{(m)})^\top : \Omega^{(m)} \to \mathbb{C}^4 \quad \text{and } U = (u_1, u_2, u_3, u_4, u_5)^\top : \Omega \to \mathbb{C}^5$$

belonging to the spaces $[W^1_p(\Omega^{(m)})]^4$ and $[W^1_p(\Omega)]^5$ with $1 < p < \infty$, respectively, and satisfying

(i) *the systems of partial differential equations*:

$$\big[A^{(m)}(\partial_x, \tau)\, U^{(m)}\big]_j = X_j^{(m)} \quad \text{in } \Omega^{(m)},\ j = 1,2,3,4, \tag{2.29}$$

$$\big[A(\partial_x, \tau)\, U\big]_k = X_k \quad \text{in } \Omega,\ k = 1,2,3,4,5, \tag{2.30}$$

(ii) *the boundary conditions*:

$$r_{S^{(m)}}\big\{[\mathcal{T}^{(m)}(\partial, \upsilon)\, U^{(m)}]_j\big\}^+ = Q_j^{(m)} \quad \text{on } S^{(m)},\ j = 1,2,3,4, \tag{2.31}$$

$$r_S\{[\mathcal{T}(\partial, n)\, U]_j\}^+ = Q_j \quad \text{on } S,\ j = 1,2,3,4, \tag{2.32}$$

$$r_S\{[\mathcal{T}(\partial, n)\, U]_5\}^+ + \beta\{u_5\}^+ = Q_5 \quad \text{on } S, \tag{2.33}$$

$$r_\Gamma\{u_k\}^+ = f_k \quad \text{on } \Gamma,\ k = 1,2,3,4,5, \tag{2.34}$$

$$r_{\Gamma^{(m)}}\{u_5\}^+ = f_5^{(m)} \quad \text{on } \Gamma^{(m)}, \tag{2.35}$$

(iii) *the transmission conditions*:

$$r_{\Gamma^{(m)}}\{u_j\}^+ - r_{\Gamma^{(m)}}\{u_j^{(m)}\}^+ = f_j^{(m)} \ \text{ on } \Gamma^{(m)},\ j=\overline{1,4}, \tag{2.36}$$

$$r_{\Gamma^{(m)}}\{[\mathcal{T}(\partial,n)U]_j\}^+ + r_{\Gamma^{(m)}}\{[\mathcal{T}^{(m)}(\partial,\upsilon)U^{(m)}]_j\}^+ = F_j^{(m)} \ \text{ on } \Gamma^{(m)},\ j=\overline{1,4}, \tag{2.37}$$

where $n=-\upsilon$ on $\Gamma^{(m)}$, β is a sufficiently smooth, real valued, nonnegative function on $\partial\Omega$, and from now on throughout the paper we assume that β does not vanish identically on S, that is

$$\beta \not\equiv 0, \quad \beta \geq 0 \ \text{ on } S, \tag{2.38}$$

and

$$\begin{aligned} &X_j^{(m)} \in L_p(\Omega^{(m)}),\ j=1,2,3,4, \quad X_k \in L_p(\Omega),\ k=1,2,3,4,5,\\ &Q_k \in B_{p,p}^{-1/p}(S), \quad f_k \in B_{p,p}^{1/p'}(\Gamma), \quad f_k^{(m)} \in B_{p,p}^{1/p'}(\Gamma^{(m)}), \quad k=1,2,3,4,5,\\ &Q_j^{(m)} \in B_{p,p}^{-1/p}(S^{(m)}), \quad F_j^{(m)} \in B_{p,p}^{-1/p}(\Gamma^{(m)}), \quad j=1,2,3,4, \quad \frac{1}{p'}+\frac{1}{p}=1. \end{aligned} \tag{2.39}$$

Note that the functions $F_j^{(m)}$, Q_j, and $Q_j^{(m)}$ $(j=1,2,3,4)$ have to satisfy some compatibility conditions. Namely, for any extension $\widehat{Q}_j^{(m)} \in B_{p,p}^{-1/p}(\overline{S^{(m)}} \cup \overline{\Gamma^{(m)}})$ of $Q_j^{(m)}$ from $S^{(m)}$ onto $\overline{S^{(m)}} \cup \overline{\Gamma^{(m)}}$ and for any extension $\widehat{Q}_j \in B_{p,p}^{-1/p}(\overline{S} \cup \overline{\Gamma^{(m)}} \cup \overline{\Gamma})$ of Q_j from S onto $\overline{S} \cup \overline{\Gamma^{(m)}} \cup \overline{\Gamma}$, the following inclusions have to be fulfilled

$$F_j^{(m)} - [\, r_{\Gamma^{(m)}} \widehat{Q}_j^{(m)} + r_{\Gamma^{(m)}} \widehat{Q}_j \,] \in r_{\Gamma^{(m)}} \widetilde{B}_{p,p}^{-\frac{1}{p}}(\Gamma^{(m)}), \quad j=1,2,3,4. \tag{2.40}$$

In the classical (continuous) setting these inclusions correspond to the natural compatibility conditions $F_j^{(m)}(x) - [\,\widehat{Q}_j^{(m)}(x) + \widehat{Q}_j(x)\,] = 0$ for all $x \in \partial\Gamma^{(m)}$, $j=1,2,3,4$.

We set

$$\begin{aligned} &Q = (Q_1, Q_2, Q_3, Q_4, Q_5)^\top \in [B_{p,p}^{-1/p}(S)]^5,\\ &f = (f_1, f_2, f_3, f_4, f_5)^\top \in [B_{p,p}^{1/p'}(\Gamma)]^5,\\ &f^{(m)} = (f_1^{(m)}, f_2^{(m)}, f_3^{(m)}, f_4^{(m)}, f_5^{(m)})^\top \in [B_{p,p}^{1/p'}(\Gamma^{(m)})]^5,\\ &Q^{(m)} = (Q_1^{(m)}, Q_2^{(m)}, Q_3^{(m)}, Q_4^{(m)})^\top \in [B_{p,p}^{-1/p}(S^{(m)})]^4,\\ &F^{(m)} = (F_1^{(m)}, F_2^{(m)}, F_3^{(m)}, F_4^{(m)})^\top \in [B_{p,p}^{-1/p}(\Gamma^{(m)})]^4. \end{aligned} \tag{2.41}$$

A pair $(U^{(m)}, U) \in [W_p^1(\Omega^{(m)})]^4 \times [W_p^1(\Omega)]^5$ will be called a solution to the boundary-transmission problem (2.29)-(2.37).

The Dirichlet-type conditions (2.34), (2.35), and (2.36) involving boundary limiting values of the vectors $U^{(m)}$ and U are understood in the usual trace sense, while the Neumann-type conditions (2.31), (2.32), (2.33), and (2.37) involving boundary limiting values of the vectors $\mathcal{T}^{(m)}U^{(m)}$ and $\mathcal{T}U$ are understood in the functional sense defined by the relations related to Green's formulas

$$\langle\{\mathcal{T}^{(m)}(\partial,\upsilon)U^{(m)}\}^+,\{V^{(m)}\}^+\rangle_{\partial\Omega^{(m)}}:=\int\limits_{\Omega^{(m)}} A^{(m)}(\partial,\tau)\,U^{(m)}\cdot V^{(m)}\,dx$$

$$+\int\limits_{\Omega^{(m)}}\Big[E^{(m)}(u^{(m)},\overline{v^{(m)}})+\rho^{(m)}\,\tau^2\,u^{(m)}\cdot v^{(m)}+\kappa_{lj}^{(m)}\,\partial_j u_4^{(m)}\,\overline{\partial_l v_4^{(m)}}$$

$$+\tau\,a^{(m)}\,u_4^{(m)}\,\overline{v_4^{(m)}}+\gamma_{jl}^{(m)}\,(\tau\,T_0^{(m)}\partial_j u_l^{(m)}\,\overline{v_4^{(m)}}-u_4^{(m)}\,\overline{\partial_j v_l^{(m)}})\Big]\,dx,$$

$$\langle\{\mathcal{T}(\partial,n)U\}^+,\{V\}^+\rangle_{\partial\Omega}:=\int\limits_{\Omega}A(\partial,\tau)\,U\cdot V\,dx+\int\limits_{\Omega}\Big[E(u,\overline{v})+\rho\,\tau^2\,u\cdot v$$

$$+\gamma_{jl}\,(\tau\,T_0\,\partial_j u_l\,\overline{v_4}-u_4\,\overline{\partial_j v_l})+\kappa_{jl}\,\partial_j u_4\,\overline{\partial_l v_4}+e_{lij}\,(\partial_l u_5\,\overline{\partial_i v_j}-\partial_i u_j\,\overline{\partial_l v_5})$$

$$+\tau\,a\,u_4\,\overline{v_4}-g_l\,(\tau\,T_0\,\partial_l u_5\,\overline{v_4}+u_4\,\overline{\partial_l v_5})+\varepsilon_{jl}\,\partial_j u_5\,\overline{\partial_l v_5}\Big]\,dx,$$

where $V^{(m)}=(v^{(m)},v_4^{(m)})^\top\in[W_{p'}^1(\Omega^{(m)})]^4$ and $V=(v,v_4,v_5)^\top\in[W_{p'}^1(\Omega)]^5$ are arbitrary vector-functions, $v^{(m)}=(v_1^{(m)},v_2^{(m)},v_3^{(m)})^\top$, $v=(v_1,v_2,v_3)^\top$, $E^{(m)}(u^{(m)},\overline{v^{(m)}})=$ $=c_{ijlk}^{(m)}\,\partial_i u_j^{(m)}\,\overline{\partial_l v_k^{(m)}}$ and $E(u,\overline{v})=c_{ijlk}\,\partial_i u_j\,\overline{\partial_l v_k}$. Here $\langle\cdot\,,\,\cdot\rangle_{\partial\Omega^{(m)}}$ (respectively $\langle\cdot\,,\,\cdot\rangle_{\partial\Omega}$) denotes the duality between the function spaces.

By standard arguments it can easily be shown that the functionals, "generalized traces" $\{\mathcal{T}^{(m)}(\partial,\upsilon)U^{(m)}\}^+\in[B_{p,p}^{-1/p}(\partial\Omega^{(m)})]^4$ and $\{\mathcal{T}(\partial,n)U\}^+\in[B_{p,p}^{-1/p}(\partial\Omega)]^5$ are correctly determined by the above relations, provided that $A^{(m)}(\partial,\tau)\,U^{(m)}\in\big[L_p(\Omega^{(m)})\big]^4$ and $A(\partial,\tau)\,U\in\big[L_p(\Omega)\big]^5$.

We have the following uniqueness theorem for $p=2$. The similar uniqueness theorem for $p\neq 2$ will be proved later in Section 4 (see Theorem 4.2).

THEOREM 2.1: *Let* $\tau=\sigma+i\omega$ *and either* $\sigma>0$ *or* $\tau=0$. *The homogeneous boundary-transmission problem* (2.29)-(2.37) ($X_j^{(m)}=0$, $X_k=0$, $Q_j^{(m)}=0$, $Q_k=0$, $f_k^{(m)}=0$, $f_k=0$, $F_j^{(m)}=0$, $j=\overline{1,4}$, $k=\overline{1,5}$) *has only the trivial solution in the space* $[W_2^1(\Omega^{(m)})]^4\times$ $\times[W_2^1(\Omega)]^5$, *provided* meas $\Gamma>0$.

PROOF: It follows from the corresponding Green's formulas. □

3. - PROPERTIES OF POTENTIALS AND LAYER POTENTIALS

Here, we establish basic properties of the layer potentials and certain boundary integral (pseudodifferential) operators generated by them. We recall also some necessary information concerning the theory of pseudo-differential equations on manifolds with boundary. These results are crucial to develop the potential method to the boundary-transmission problem (2.29)-(2.37) and prove the corresponding existence and regularity results for solutions in different function spaces.

3.1. *Layer potentials*

Denote by $\Psi^{(m)}(\cdot\,,\tau)=\big[\Psi^{(m)}_{kj}(\cdot\,,\tau)\big]_{4\times 4}$ and $\Psi(\cdot\,,\tau)=\big[\Psi_{kj}(\cdot\,,\tau)\big]_{5\times 5}$ the fundamental matrix-functions of the differential operators $A^{(m)}(\partial_x,\tau)$ and $A(\partial_x,\tau)$, respectively (for details, see [BCN1] and references therein),

$$A^{(m)}(\partial_x,\tau)\Psi^{(m)}(x-y,\tau)=\delta(x-y)\,I_4,\quad A(\partial_x,\tau)\Psi(x-y,\tau)=\delta(x-y)\,I_5,$$

where $\delta(\,\cdot\,)$ denotes Dirac's delta function.

Note that, if by $\Psi^{(m)*}(\,\cdot\,,\tau)$ we denote the fundamental matrix of the adjoint operator $A^{(m)*}(\partial,\tau)$, we have then the evident equalities,

$$\Psi^{(m)*}(x,\tau)=\big[\Psi^{(m)}(x,\overline{\tau})\big]^{\top},\quad \Psi^{(m)}(\,-x,\overline{\tau})=\overline{\Psi^{(m)}(x,\tau)},\quad \Psi^{(m)}(x,\tau)=\big[\overline{\Psi^{(m)*}(\,-x,\tau)}\big]^{\top}.$$

Similarly, the matrix $\Psi^{*}(x,\tau):=[\,\Psi(x,\overline{\tau})\,]^{\top}$ represents the fundamental matrix of the adjoint operator $A^{*}(\partial,\tau)$, and $\Psi(x,\tau)=\big[\overline{\Psi^{*}(\,-x,\tau)}\big]^{\top}$, since $\Psi(\,-x,\overline{\tau})=\overline{\Psi(x,\tau)}$.

Let us introduce the single and double layer potentials corresponding to the operators $A^{(m)}(\partial,\tau)$ and $A(\partial,\tau)$:

$$V^{(m)}_{\tau}(h^{(m)})(x)=\int\limits_{\partial\Omega^{(m)}}\Psi^{(m)}(x-y,\tau)\,h^{(m)}(y)\,d_yS,\quad V_{\tau}(h)(x)=\int\limits_{\partial\Omega}\Psi(x-y,\tau)\,h(y)\,d_yS,$$

$$W^{(m)}_{\tau}(h^{(m)})(x)=\int\limits_{\partial\Omega^{(m)}}\Big[\,\widetilde{T}^{(m)}(\partial_y,\nu(y),\overline{\tau})[\Psi^{(m)}(x-y,\tau)]^{\top}\Big]^{\top}h^{(m)}(y)\,d_yS,$$

$$W_{\tau}(h)(x)=\int\limits_{\partial\Omega}\Big[\,\widetilde{T}(\partial_y,n(y),\overline{\tau})[\Psi(x-y,\tau)]^{\top}\Big]^{\top}h(y)\,d_yS,$$

where $h^{(m)}=(h^{(m)}_1,h^{(m)}_2,h^{(m)}_3,h^{(m)}_4)^{\top}$ and $h=(h_1,h_2,h_3,h_4,h_5)^{\top}$ are densities of the potentials.

For the readers convenience, here we collect some results concerning these layer potentials and the corresponding boundary operators needed in subsequent analysis.

We recall that $\partial\Omega,\ \partial\Omega^{(m)}\in C^{\infty}$.

THEOREM 3.1: *Let $1<p<\infty$, $1\le t\le\infty$, and $s\in\mathbb{R}$. The operators*

$$\begin{aligned}
V^{(m)}_{\tau} &: \big[B^{s}_{p,t}(\partial\Omega^{(m)})\big]^4 \longrightarrow \big[B^{s+1+\frac{1}{p}}_{p,t}(\Omega^{(m)})\big]^4,\\
W^{(m)}_{\tau} &: \big[B^{s}_{p,t}(\partial\Omega^{(m)})\big]^4 \longrightarrow \big[B^{s+\frac{1}{p}}_{p,t}(\Omega^{(m)})\big]^4,\\
V_{\tau} &: \big[B^{s}_{p,t}(\partial\Omega)\big]^5 \longrightarrow \big[B^{s+1+\frac{1}{p}}_{p,t}(\Omega)\big]^5,\\
W_{\tau} &: \big[B^{s}_{p,t}(\partial\Omega)\big]^5 \longrightarrow \big[B^{s+\frac{1}{p}}_{p,t}(\Omega)\big]^5
\end{aligned}$$

are continuous.

Proof: For regular densities the proof for the potentials $V_\tau^{(m)}$ and $W_\tau^{(m)}$ can be found in [KGBB1], in the isotropic case, and in [JN1] in the anisotropic case, while for the potentials V_τ and W_τ the proof is given in [BG1], [BCNS1].

Note that the main ideas for generalization to the scale of Bessel potential and Besov spaces are based on the duality and interpolation technique and is described in the reference [Se1] using the theory of pseudodifferential operators on smooth manifolds without boundary.

□

For the boundary integral (pseudodifferential) operators generated by the layer potentials we will employ the following notation:

$$\mathcal{H}_\tau^{(m)}(h^{(m)})(x) := \int\limits_{\partial\Omega^{(m)}} \Psi^{(m)}(x-y,\tau)\, h^{(m)}(y)\, d_yS, \quad x \in \partial\Omega^{(m)},$$

$$\mathcal{K}_\tau^{(m)}(h^{(m)})(x) := \int\limits_{\partial\Omega^{(m)}} \big[\, \mathcal{T}^{(m)}(\partial_x, \upsilon(x))\Psi^{(m)}(x-y,\tau) \,\big]\, h^{(m)}(y)\, d_yS, \quad x \in \partial\Omega^{(m)},$$

$$\widetilde{\mathcal{K}}_\tau^{(m)*}(h^{(m)})(x) := \int\limits_{\partial\Omega_1} \big[\, \widetilde{\mathcal{T}}^{(m)}(\partial_y, \upsilon(y), \overline{\tau})[\Psi^{(m)}(x-y,\tau)]^\top \,\big]^\top h^{(m)}(y)\, d_yS, \quad x \in \partial\Omega^{(m)},$$

$$\mathcal{L}_\tau^{(m)}(h^{(m)})(x) := \big\{\, \mathcal{T}^{(m)}(\partial_x, \upsilon(x))\, W^{(m)}(h^{(m)})(x) \,\big\}^\pm, \quad x \in \partial\Omega^{(m)},$$

$$\mathcal{H}_\tau(h)(x) := \int\limits_{\partial\Omega} \Psi(x-y,\tau)\, h(y)\, d_yS, \quad x \in \partial\Omega,$$

$$\mathcal{K}_\tau(h)(x) := \int\limits_{\partial\Omega} \big[\, \mathcal{T}(\partial_x, n(x))\Psi(x-y,\tau) \,\big]\, h(y)\, d_yS, \quad x \in \partial\Omega$$

$$\widetilde{\mathcal{K}}_\tau^{*}(h)(x) := \int\limits_{\partial\Omega} \big[\, \widetilde{\mathcal{T}}(\partial_y, n(y), \overline{\tau})[\Psi(x-y,\tau)]^\top \,\big]^\top h(y)\, d_yS, \quad x \in \partial\Omega,$$

$$\mathcal{L}_\tau(h)(x) := \big\{\, \mathcal{T}(\partial_x, n(x))\, W_\tau(h)(x) \,\big\}^\pm, \quad x \in \partial\Omega.$$

The layer boundary operators $\mathcal{H}_\tau^{(m)}$, $\mathcal{H}_\tau$ and $\mathcal{L}_\tau^{(m)}$, $\mathcal{L}_\tau$ are pseudodifferential operators of order -1 and 1, respectively, while the operators $\mathcal{K}_\tau^{(m)}$, $\widetilde{\mathcal{K}}_\tau^{(m)*}$, $\mathcal{K}_\tau$ and $\widetilde{\mathcal{K}}_\tau^{*}$ are singular integral operators (pseudodifferential operators of order 0) (for details see [JN1], [BG1]).

Theorem 3.2: *Let $1 < p < \infty$, $1 \le t \le \infty$,*

$$h^{(m)} \in \big[B_{p,t}^{-\frac{1}{p}}(\partial\Omega^{(m)})\big]^4, \quad g^{(m)} \in \big[B_{p,t}^{1-\frac{1}{p}}(\partial\Omega^{(m)})\big]^4, \quad h \in \big[B_{p,t}^{-\frac{1}{p}}(\partial\Omega)\big]^5, \quad g \in \big[B_{p,t}^{1-\frac{1}{p}}(\partial\Omega)\big]^5.$$

Then

$$\{V_\tau^{(m)}(h^{(m)})\}^+ = \{V_\tau^{(m)}(h^{(m)})\}^- = \mathcal{H}_\tau^{(m)}\, h^{(m)} \;\; on \;\; \partial\Omega^{(m)},$$

$$\{\mathcal{T}^{(m)}(\partial, \upsilon)V_\tau^{(m)}(h^{(m)})\}^\pm = \big[\mp 2^{-1}I_4 + \mathcal{K}_\tau^{(m)}\big]\, h^{(m)} \;\; on \;\; \partial\Omega^{(m)},$$

$$\{W_\tau^{(m)}(g^{(m)})\}^\pm = \big[\pm 2^{-1}I_4 + \widetilde{\mathcal{K}}_\tau^{(m)*}\big]\, g^{(m)} \;\; on \;\; \partial\Omega^{(m)},$$

$$\{V_\tau(h)\}^+ = \{V_\tau(h)\}^- = \mathcal{H}_\tau\, h \;\; on \;\; \partial\Omega,$$

$$\{\mathcal{T}(\partial, n)V_\tau(h)\}^\pm = \big[\mp 2^{-1}I_5 + \mathcal{K}_\tau\big]\, h, \;\; on \;\; \partial\Omega,$$

$$\{W_\tau(g)\}^\pm = \big[\pm 2^{-1}I_5 + \widetilde{\mathcal{K}}_\tau^{*}\big]\, g \;\; on \;\; \partial\Omega,$$

where I_k stands for the $k \times k$ unit matrix.

PROOF: It can be found in [DNS1], [JN1]. □

The operators $\mathcal{L}_\tau^{(m)}$ and $\mathcal{L}_\tau$ are well defined in accordance with the following proposition.

LEMMA 3.3: *Let $1 < p < \infty$, $1 \leq t \leq \infty$, and*

$$h^{(m)} \in \big[B_{p,t}^{1-\frac{1}{p}}(\partial\Omega^{(m)})\big]^4, \quad h \in \big[B_{p,t}^{1-\frac{1}{p}}(\partial\Omega)\big]^5.$$

Then

$$\{\,\mathcal{T}^{(m)}(\partial, \upsilon)W_\tau^{(m)}(h^{(m)})\,\}^+ = \{\,\mathcal{T}^{(m)}(\partial, \upsilon)W_\tau^{(m)}(h^{(m)})\,\}^- \;\; on \;\; \partial\Omega^{(m)}$$

and

$$\{\,\mathcal{T}(\partial, n)W_\tau(h)\,\}^+ = \{\,\mathcal{T}(\partial, n)W_\tau(h)\,\}^- \;\; on \;\; \partial\Omega.$$

PROOF: We prove the second relation.

Let $W(x) := W_\tau(h)(x)$ be a double layer potential with sufficiently smooth density h. By the integral representation formulas in the domains Ω and $\mathbb{R}^3 \setminus \overline{\Omega}$ we have:

$$\int_{\partial\Omega} \big[\, \widetilde{\mathcal{T}}(\partial_y, n(y), \overline{\tau})[\Psi(x-y,\tau)]^\top \big]^\top \{W(y)\}^+ \, d_yS$$
$$- \int_{\partial\Omega} \Psi(x-y,\tau)\,\{\mathcal{T}(\partial_y, n(y))W(y)\}^+ d_yS = \begin{cases} W(x), & x \in \Omega, \\ 0, & x \in \mathbb{R}^3 \setminus \overline{\Omega}, \end{cases}$$

$$- \int_{\partial\Omega} \big[\, \widetilde{\mathcal{T}}(\partial_y, n(y), \overline{\tau})[\Psi(x-y,\tau)]^\top \big]^\top \{W(y)\}^- \, d_yS$$
$$+ \int_{\partial\Omega} \Psi(x-y,\tau)\,\{\mathcal{T}(\partial_y, n(y))W(y)\}^- d_yS = \begin{cases} 0, & x \in \Omega, \\ W(x), & x \in \mathbb{R}^3 \setminus \overline{\Omega}. \end{cases}$$

By adding termwise these equalities and applying the jump relations for the double layer potential $W(x) := W_\tau(h)(x)$ we get

$$W(x) = \int\limits_{\partial\Omega} \Psi(x-y,\tau)\,\big[\{\mathcal{T}(\partial_y, n(y))W(y)\}^- - \{\mathcal{T}(\partial_y, n(y))W(y)\}^+\big]\,dS$$

$$+ \int\limits_{\partial\Omega} \big[\,\widetilde{\mathcal{T}}(\partial_y, n(y), \overline{\tau})[\Psi^\top(x-y,\tau)]^\top\,\big]^\top h(y)\,dS, \quad x \in \Omega \cup [\,\mathbb{R}^3 \setminus \overline{\Omega}\,].$$

Since $W(x) := W_\tau(h)(x)$, we conclude that

$$\int\limits_{\partial\Omega} \Psi(x-y,\tau)\,\big[\{\mathcal{T}(\partial_\upsilon, \nu(\upsilon))\varphi(\upsilon)\}^+ - \{\mathcal{T}(\partial_\upsilon, \nu(\upsilon))\varphi(\upsilon)\}^-\big]\,\delta\Sigma = 0, \quad \chi \in \Omega \cup [\,\mathbb{R}^3 \setminus \overline{\Omega}\,],$$

which shows that the single layer potential $V_\tau(g)$ with the density $g := \{\mathcal{T}(\partial_y, n(y))W(y)\}^+ - \{\mathcal{T}(\partial_y, n(y))W(y)\}^-$ vanishes in Ω and $\mathbb{R}^3 \setminus \overline{\Omega}$. Therefore $\{\mathcal{T}V_\tau(g)\}^+ = 0$ and $\{\mathcal{T}V_\tau(g)\}^- = 0$. Then due to the jump relation for the single layer potential (see Theorem 3.2) it follows that $\{\mathcal{T}V_\tau(g)\}^+ - \{\mathcal{T}V_\tau(g)\}^- = g = 0$. Thus the theorem holds for smooth densities.

By standard limiting and duality arguments this result can be extended to the Bessel potential and Besov spaces. □

The following mapping properties of the above introduced boundary pseudodifferential operators are well known (see, e.g., [Se1], [DNS1], [JN1], [BG1], [BCNS1]).

THEOREM 3.4: *Let $1 < p < \infty$, $1 \le t \le \infty$, $s \in \mathbb{R}$. The operators*

$$\mathcal{H}_\tau^{(m)} : [\,B^s_{p,t}(\partial\Omega^{(m)})\,]^4 \to [\,B^{s+1}_{p,t}(\partial\Omega^{(m)})\,]^4,$$

$$\mathcal{K}_\tau^{(m)}, \widetilde{\mathcal{K}}_\tau^{(m)*} : [\,B^s_{p,t}(\partial\Omega^{(m)})\,]^4 \to [\,B^s_{p,t}(\partial\Omega^{(m)})\,]^4,$$

$$\mathcal{L}_\tau^{(m)} : [\,B^{s+1}_{p,t}(\partial\Omega^{(m)})\,]^4 \to [\,B^s_{p,t}(\partial\Omega^{(m)})\,]^4,$$

$$\mathcal{H}_\tau : [\,B^s_{p,t}(\partial\Omega)\,]^5 \to [\,B^{s+1}_{p,t}(\partial\Omega)\,]^5,$$

$$\mathcal{K}_\tau, \widetilde{\mathcal{K}}^*_\tau, : [\,B^s_{p,t}(\partial\Omega)\,]^5 \to [\,B^s_{p,t}(\partial\Omega)\,]^5,$$

$$\mathcal{L}_\tau : [\,B^{s+1}_{p,t}(\partial\Omega)\,]^5 \to [\,B^s_{p,t}(\partial\Omega)\,]^5,$$

are continuous.

The operators $\mathcal{H}_\tau^{(m)}$, $-2^{-1} I_4 + \mathcal{K}_\tau^{(m)}$ and $\mathcal{H}_\tau$ possess the following invertibility properties.

THEOREM 3.5: *The operators*

$$\mathcal{H}_\tau^{(m)} : \left[B^s_{p,t}(\partial\Omega^{(m)}) \right]^4 \to \left[B^{s+1}_{p,t}(\partial\Omega^{(m)}) \right]^4,$$

$$-2^{-1} I_4 + \mathcal{K}_\tau^{(m)} : \left[B^s_{p,t}(\partial\Omega^{(m)}) \right]^4, \to \left[B^s_{p,t}(\partial\Omega^{(m)}) \right]^4,$$

$$\mathcal{H}_\tau : \left[B^s_{p,t}(\partial\Omega) \right]^5 \to \left[B^{s+1}_{p,t}(\partial\Omega) \right]^5,$$

are invertible for all $1 < p < \infty$, $1 \le t \le \infty$, *and* $s \in \mathbb{R}$.

PROOF: The proof for the operators $\mathcal{H}_\tau^{(m)}$ and $-2^{-1} I_4 + \mathcal{K}_\tau^{(m)}$ can be found in the papers [JN1] while for the operator $\mathcal{H}_\tau$ it is word for word of the proof of Theorem 3.6 in the reference [BCNS1]. □

THEOREM 3.6: *The operator* $-2^{-1} I_5 + \mathcal{K}_\tau : \left[B^s_{p,t}(\partial\Omega) \right]^5 \to \left[B^s_{p,t}(\partial\Omega) \right]^5$ *is Fredholm with zero index for all* $1 < p < \infty$, $1 \le t \le \infty$, *and* $s \in \mathbb{R}$.

PROOF: It follows from Theorems 3.6 and 3.7 in the reference [BCNS1]. □

3.2. *Auxiliary problems and representation formulae for solutions*

Here we assume that $\Re\,\tau = \sigma > 0$ and consider *two auxiliary boundary value problems* needed for our further purposes.

3.2.1. Auxiliary problem I: Find a vector function $U^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)}, u_4^{(m)})^\top : \Omega^{(m)} \to \mathbb{C}^4$ which belongs to the space $[\, W^1_2(\Omega^{(m)})\,]^4$ and satisfies the following differential equation and boundary conditions:

$$A^{(m)}(\partial, \tau)\, U^{(m)} = 0 \quad \text{in} \quad \Omega^{(m)}, \tag{3.1}$$

$$\{ \mathcal{T}^{(m)} U^{(m)} \}^+ = \chi^{(m)} \quad \text{on} \quad \partial\Omega^{(m)}, \tag{3.2}$$

where $\chi^{(m)} = (\chi_1^{(m)}, \chi_2^{(m)}, \chi_3^{(m)}, \chi_4^{(m)})^\top \in \left[H_2^{-\frac{1}{2}}(\partial\Omega^{(m)}) \right]^4$. With the help of Green's formulae it can easily be shown that the homogeneous version of this auxiliary BVP possesses only the trivial solution.

Recall that on $\partial\Omega^{(m)}$ the normal vector υ is directed outward.

From Theorem 3.5 and the above mentioned uniqueness result for the BVP (3.1)-(3.2) immediately follows

LEMMA 3.7: *Let* $\Re\,\tau = \sigma > 0$ *and* $1 < p < \infty$. *An arbitrary solution* $U^{(m)} \in [\, W^1_p(\Omega^{(m)})\,]^4$ *to the homogeneous equation* (3.1) *can be uniquely represented by the single layer potential*

$$U^{(m)}(x) = V_\tau^{(m)}\big(\big[-2^{-1} I_4 + \mathcal{K}_\tau^{(m)} \big]^{-1} \chi^{(m)} \big)(x), \quad x \in \Omega^{(m)}, \tag{3.3}$$

where the density vector $\chi^{(m)}$ *satisfies the relation* $\chi^{(m)} = \{ \mathcal{T}^{(m)} U^{(m)} \}^+ \in \left[B^{-\frac{1}{p}}_{p,p}(\partial\Omega^{(m)}) \right]^4$.

3.2.2. Auxiliary problem II: Find a vector function $U = (u_1, u_2, u_3, u_4, u_5)^\top$: $\Omega \to \mathbb{C}^5$ which belongs to the space $[\,W_2^1(\Omega)\,]^5$ and satisfies the following conditions:

$$A(\partial, \tau)U = 0 \quad \text{in} \quad \Omega, \tag{3.4}$$

$$\{\,[\,\mathcal{T}U\,]_j\,\}^+ = \chi_j \quad \text{on} \quad \partial\Omega, \quad j = \overline{1,4}, \tag{3.5}$$

$$\{\,[\,\mathcal{T}U\,]_5\,\}^+ + \beta\,\{\,U_5\,\}^+ = \chi_5 \quad \text{on} \quad \partial\Omega, \tag{3.6}$$

where $\chi_j \in H_2^{-\frac{1}{2}}(\partial\Omega)$ for $j = \overline{1,5}$. Here β is a nonnegative smooth real valued scalar function which does not vanish identically on $\partial\Omega$ (see (2.38)).

Denote $\chi := (\chi_1, \chi_2, \chi_3, \chi_4, \chi_5)^\top \in \big[H_2^{-\frac{1}{2}}(\partial\Omega)\big]^5$.

By the same arguments as in the proof of Theorem 2.1 we can easily show that the homogeneous version of this boundary value problem possesses only the trivial solution in the space $[\,W_2^1(\Omega)\,]^5$.

We look for a solution to the auxiliary BVP II as a single layer potential, $U(x) = V_\tau(f)(x)$, where $f = (f_1, f_2, f_3, f_4, f_5)^\top \in \big[H_2^{-\frac{1}{2}}(\partial\Omega)\big]^5$ is a sought density.

The boundary conditions (3.5) and (3.6) lead then to the system of equations:

$$\big[\,(\,-2^{-1}I_5 + \mathcal{K}_\tau\,)f\,\big]_j = \chi_j \qquad \text{on} \quad \partial\Omega, \quad j = \overline{1,4},$$

$$\big[\,(\,-2^{-1}I_5 + \mathcal{K}_\tau\,)f\,\big]_5 + \beta\,\big[\,\mathcal{H}_\tau f\,\big]_5 = \chi_5 \quad \text{on} \quad \partial\Omega.$$

Denote the operator generated by the left hand side expressions of these equations by $\mathcal{P}_\tau$ and rewrite the system as $\mathcal{P}_\tau f = \chi$ on $\partial\Omega$, where

$$\mathcal{P}_\tau := \begin{bmatrix} \big[\,(\,-2^{-1}I_5 + \mathcal{K}_\tau)_{jk}\,\big]_{4\times 5} \\ \big[\,(\,-2^{-1}I_5 + \mathcal{K}_\tau)_{5k}\,\big]_{1\times 5} + \beta\,\big[\,(\mathcal{H}_\tau)_{5k}\,\big]_{1\times 5} \end{bmatrix} = -2^{-1}I_5 + \mathcal{K}_\tau + \mathcal{I}(\beta)\,\mathcal{H}_\tau \tag{3.7}$$

with $\mathcal{I}(\beta) = \operatorname{diag}\{0, 0, 0, 0, \beta\}$.

Lemma 3.8: *Let $\Re\,\tau = \sigma > 0$. The operators*

$$\mathcal{P}_\tau : \big[H_p^s(\partial\Omega)\big]^5 \to \big[H_p^{s+1}(\partial\Omega)\big]^5 \quad \Big[\big[B_{p,t}^s(\partial\Omega)\big]^5 \to \big[B_{p,t}^{s+1}(\partial\Omega)\big]^5\Big], \tag{3.8}$$

are invertible for all $1 < p < \infty$, $1 \le t \le \infty$, and $s \in \mathbb{R}$.

Proof: From the uniqueness result for the auxiliary BVP II it follows that the operator (3.8) is injective for $s = -\frac{1}{2}, p = t = 2$. The operator $\mathcal{H}_\tau : \big[H_2^{-\frac{1}{2}}(\partial\Omega)\big]^5 \to \big[H_2^{-\frac{1}{2}}(\partial\Omega)\big]^5$ is compact. By Theorem 3.6 we then conclude that the index of the operator (3.8) equals to zero. Since $\mathcal{P}_\tau$ is an injective singular integral operator of normal type with zero index it follows that it is surjective. Thus the operator (3.8) is invertible.

The invertibility of the operator (3.8) for all $1 < p < \infty$, $1 \le t \le \infty$, and $s \in \mathbb{R}$ then follows by standard duality and interpolation arguments for the C^∞–regular surface $\partial\Omega$ (see, e.g., [Agl1], [Se1]). □

As a consequence we have the following

LEMMA 3.9: *Let* $\Re\,\tau = \sigma > 0$ *and* $1 < p < \infty$. *An arbitrary solution* $U \in [\,W^1_p(\Omega)\,]^5$ *to the homogeneous equation* (3.4) *can be uniquely represented by the single layer potential for* Ω: $U(x) = V_\tau\,(\mathcal{P}_\tau^{-1}\chi)(x)$, *where*

$$\chi = (\{[\mathcal{T}U]_1\}^+,\ \cdots,\ \{[\mathcal{T}U]_4\}^+,\ \{[\mathcal{T}U]_5\}^+ + \beta\,\{U_5\}^+)^\top \in \big[B^{-\frac{1}{p}}_{p,p}(\partial\Omega)\big]^5.$$

3.3. *Some results for pseudodifferential equations on manifolds with boundary*

In this subsection we shall present some principal results from the theory of strongly elliptic pseudodifferential equations on manifolds with boundary in Bessel potential and Besov spaces which are the main tools for proving existence theorems for mixed boundary-transmission problems by the potential method. In particular, in our investigation we need some results describing the Fredholm properties of pseudodifferential operators on a compact manifold with boundary. They can be found in [Esk1], [Grb1], [Sh1].

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n-dimensional, nonselfintersecting manifold with boundary $\partial\mathcal{M} \in C^\infty$ and let $\mathcal{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\upsilon \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\sigma_{\mathcal{A}}(x,\xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system ($x \in \overline{\mathcal{M}},\ \xi \in \mathbb{R}^n \setminus \{0\}$).

Let $\lambda_1(x), \cdots, \lambda_N(x)$ be the eigenvalues of the matrix

$$[\,\sigma_{\mathcal{A}}(x,0,\cdots,0,+1)\,]^{-1}[\,\sigma_{\mathcal{A}}(x,0,\cdots,0,-1)\,],\quad x \in \partial\overline{\mathcal{M}},$$

and introduce the notation $\delta_j(x) = \Re\,\big[\,(2\pi i)^{-1}\ln\lambda_j(x)\big]$, $j = 1,\cdots,N$. Here the branch in the logarithmic function $\ln\zeta$ is chosen with regard to the inequality $-\pi < \arg\zeta \le \pi$, $j = 1,\cdots,N$. Due to the strong ellipticity of $\mathcal{A}$ we have the strong inequality $-1/2 < \delta_j(x) < 1/2$ for $x \in \overline{\mathcal{M}}$. Note that the numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system. Moreover, remark that in the particular case when $\sigma_{\mathcal{A}}(x,\xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$ we have $\delta_j(x) = 0$ for $j = 1,\cdots,N$, since all the eigenvalues $\lambda_j(x)$ ($j = \overline{1,N}$) are positive numbers for any $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudo-differential operators are characterized by the following theorem.

THEOREM 3.10: *Let* $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, *and let* $\mathcal{A}$ *be a strongly elliptic pseudodifferential operator of order* $\upsilon \in \mathbb{R}$, *that is, there is a positive constant* c_0 *such that* $\Re\,\sigma_{\mathcal{A}}(x,\xi)\zeta\cdot\zeta \ge c_0\,|\zeta|^2$ *for* $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ *with* $|\xi| = 1$, *and* $\zeta \in \mathbb{C}^N$. *Then*

$$\text{(3.9)}\qquad \mathcal{A} : \big[\widetilde{H}^s_p(\mathcal{M})\big]^N \to \big[H^{s-\upsilon}_p(\mathcal{M})\big]^N \quad \Big[\big[\widetilde{B}^s_{p,q}(\mathcal{M})\big]^N \to \big[B^{s-\upsilon}_{p,q}(\mathcal{M})\big]^N\Big],$$

are Fredholm operators with index zero if

$$\text{(3.10)}\qquad \frac{1}{p} - 1 + \sup_{x\in\partial\mathcal{M},\,1\le j\le N}\delta_j(x) < s - \frac{\upsilon}{2} < \frac{1}{p} + \inf_{x\in\partial\mathcal{M},\,1\le j\le N}\delta_j(x).$$

Moreover, the null-spaces and indices of the operators (3.9) *are the same (for all values of the parameter* $q \in [1, +\infty]$*) provided* p *and* s *satisfy the inequality* (3.10).

4. - Existence and regularity results

4.1. *Reduction to boundary equations*

Let us return to the boundary-transmission problem (2.29)-(2.37) and derive the equivalent boundary integral formulation of this problem. To this end from now on without loss of generality we assume that the mass force densities, heat source densities and charge density vanish in the corresponding regions, that is,

$$X_k^{(m)} = 0 \text{ in } \Omega^{(m)} \text{ for } k = \overline{1,4}, \quad X_j = 0 \text{ in } \Omega \text{ for } j = \overline{1,5}.$$

Otherwise we can write particular solutions to the differential equations (2.29)-(2.30) explicitly in the form of volume Newtonian potentials:

$$U_0^{(m)}(x) := \int\limits_{\Omega^{(m)}} \Psi^{(m)}(x-y,\tau)\, X^{(m)}(y)\, dy, \quad x \in \Omega^{(m)},$$

$$U_0(x) := \int\limits_{\Omega} \Psi(x-y,\tau)\, X(y)\, dy, \quad x \in \Omega,$$

and introduce the new unknown fields $U^{(m)} - U_0^{(m)}$ and $U - U_0$ in order to reduce the nonhomogeneous equations (2.29)-(2.30) to the homogeneous ones.

Keeping in mind (2.41), let

$$Q_0^{(m)} = (Q_{01}^{(m)}, Q_{02}^{(m)}, Q_{03}^{(m)}, Q_{04}^{(m)})^\top \in \big[B_{p,p}^{-1/p}(\partial\Omega^{(m)})\big]^4 \tag{4.1}$$

be some fixed extension of the vector-function $Q^{(m)} = (Q_1^{(m)}, \cdots, Q_4^{(m)})^\top \in \big[B_{p,p}^{-1/p}(S^{(m)})\big]^4$ onto $\partial\Omega^{(m)}$. Note that $\partial\Omega^{(m)} = S^{(m)} \cup \overline{\Gamma}^{(m)}$. It is evident that an arbitrary extension of $Q^{(m)}$ onto $\partial\Omega^{(m)}$ has the form $\widetilde{Q}^{(m)} = Q_0^{(m)} + h^{(m)}$, where

$$h^{(m)} = (h_1^{(m)}, h_2^{(m)}, h_3^{(m)}, h_4^{(m)})^\top \in \big[\widetilde{B}_{p,p}^{-1/p}(\Gamma^{(m)})\big]^4 \tag{4.2}$$

is introduced as an unknown vector-function. Analogously, let

$$Q_0 = (Q_{01}, Q_{02}, Q_{03}, Q_{04}, Q_{05})^\top \in \big[B_{p,p}^{-1/p}(\partial\Omega)\big]^5 \tag{4.3}$$

be some fixed extension of the vector-function $Q = (Q_1, Q_2, Q_3, Q_4, Q_5)^\top \in \big[B_{p,p}^{-1/p}(S)\big]^5$ onto $\partial\Omega$. Note that $\partial\Omega = \overline{S} \cup \overline{\Gamma}^{(m)} \cup \overline{\Gamma}$. It is evident that every extension of Q onto $\partial\Omega$ can be represented then as $\widetilde{Q} = Q_0 + \psi + h$, where

$$\psi = (\psi_1, \cdots, \psi_5)^\top \in \big[\widetilde{B}_{p,p}^{-1/p}(\Gamma)\big]^5, \quad h = (h_1, \cdots, h_5)^\top \in \big[\widetilde{B}_{p,p}^{-1/p}(\Gamma^{(m)})\big]^5, \tag{4.4}$$

are introduced as unknown vector-functions.

Note that there should be satisfied the following compatibility conditions (see (2.31), (2.32), (2.37))

$$r_{_{\Gamma^{(m)}}} [\widetilde{Q}_j^{(m)} + \widetilde{Q}_j] - F_j^{(m)} \in r_{_{\Gamma^{(m)}}} \widetilde{B}_{p,p}^{-1/p}(\Gamma^{(m)}), \quad j = \overline{1,4}.$$

We develop here the so-called indirect boundary integral equation method. We look for a solution pair $(U^{(m)}, U)$ of the mixed boundary-transmission problem (2.29)-(2.37) with $X^{(m)} = 0$ and $X = 0$ in the form of the corresponding single layer potentials

$$(4.5) \quad U^{(m)} = (U_1^{(m)}, \cdots, U_4^{(m)})^\top = V_\tau^{(m)}([-2^{-1}I_4 + \mathcal{K}_\tau^{(m)}]^{-1}[Q_0^{(m)} + h^{(m)}]) \quad \text{in } \Omega^{(m)},$$

$$(4.6) \quad U = (U_1, \cdots, U_5)^\top = V_\tau(\mathcal{P}_\tau^{-1}[Q_0 + \psi + h]) \quad \text{in } \Omega.$$

We have to find the unknown vector-functions $h^{(m)}$, h and ψ satisfying the inclusions (4.2) and (4.4). We recall that these unknown densities $Q_0^{(m)} + h^{(m)}$ and $Q_0 + \psi + h$ have the physical meaning in accordance with Lemmas 3.7 and 3.9.

Let us remark that the homogeneous differential equations (2.29)-(2.30) ($X^{(m)} = 0$, $X = 0$) are satisfied automatically as well as the boundary conditions (2.31)-(2.33).

The remaining boundary and transmission conditions (2.34)-(2.37) lead to the equations for the unknown vector-functions ψ, h and $h^{(m)}$,

$$(4.7) \qquad r_{_\Gamma} [\mathcal{H}_\tau \mathcal{P}_\tau^{-1} [\psi + h]]_k = \widetilde{f}_k \text{ on } \Gamma, \quad k = \overline{1,5},$$

$$(4.8) \qquad r_{_{\Gamma^{(m)}}} [\mathcal{H}_\tau \mathcal{P}_\tau^{-1} [\psi + h]]_5 = \widetilde{f}_5^{(m)} \text{ on } \Gamma^{(m)},$$

$$(4.9) \qquad r_{_{\Gamma^{(m)}}} [\mathcal{H}_\tau \mathcal{P}_\tau^{-1} [\psi + h]]_j - r_{_{\Gamma^{(m)}}} \Big[\mathcal{H}_\tau^{(m)} [-2^{-1} I_4 + \mathcal{K}_\tau^{(m)}]^{-1} [h^{(m)}]\Big]_j = \widetilde{f}_j^{(m)} \text{ on } \Gamma^{(m)}, \ j = \overline{1,4}.$$

$$(4.10) \qquad r_{_{\Gamma^{(m)}}} h_j^{(m)} = \widetilde{F}_j^{(m)} - r_{_{\Gamma^{(m)}}} h_j \text{ on } \Gamma^{(m)}, \ j = \overline{1,4}.$$

where

$$(4.11) \qquad \widetilde{f}_k := f_k - r_{_\Gamma} [\mathcal{H}_\tau \mathcal{P}_\tau^{-1} Q_0]_k \in B_{p,p}^{1-1/p}(\Gamma), \quad k = \overline{1,5},$$

$$(4.12) \qquad \widetilde{f}_5^{(m)} := f_5^{(m)} - r_{_{\Gamma^{(m)}}} [\mathcal{H}_\tau \mathcal{P}_\tau^{-1} Q_0]_5 \in B_{p,p}^{1-1/p}(\Gamma^{(m)}),$$

$$(4.13) \qquad \widetilde{f}_j^{(m)} := f_j^{(m)} + r_{_{\Gamma^{(m)}}} \Big[\mathcal{H}_\tau^{(m)} [-2^{-1} I_4 + \mathcal{K}_\tau^{(m)}]^{-1} Q_0^{(m)}\Big]_j$$

$$(4.14) \qquad - r_{_{\Gamma^{(m)}}} [\mathcal{H}_\tau \mathcal{P}_\tau^{-1} Q_0]_j \in B_{p,p}^{1-1/p}(\Gamma^{(m)}), \ j = \overline{1,4},$$

$$\widetilde{F}_j^{(m)} := F_j^{(m)} - r_{_{\Gamma^{(m)}}} Q_{0j} - r_{_{\Gamma^{(m)}}} Q_{0j}^{(m)} \in \widetilde{B}_{p,p}^{-1/p}(\Gamma^{(m)}), \ j = \overline{1,4}.$$

The last inclusion follows from the compatibility condition (2.40).

Let us introduce the notation

$$(4.15) \quad \mathcal{A}_\tau := \mathcal{H}_\tau \mathcal{P}_\tau^{-1}, \qquad \mathcal{B}_\tau^{(m)} := \begin{bmatrix} [\mathcal{H}_\tau^{(m)} [-2^{-1} I_4 + \mathcal{K}_\tau^{(m)}]^{-1}]_{4\times 4} & [0]_{4\times 1} \\ [0]_{1\times 4} & [0]_{1\times 1} \end{bmatrix}_{5\times 5}.$$

We can rewrite the equations (4.7)-(4.10) as

(4.16) $$r_{\Gamma}\, \mathcal{A}_\tau \left[\psi + h\right] = \widetilde{f} \text{ on } \Gamma,$$

(4.17) $$r_{\Gamma^{(m)}}\, \mathcal{A}_\tau \left[\psi + h\right] + r_{\Gamma^{(m)}}\, \mathcal{B}_\tau^{(m)} h = \widetilde{g}^{(m)} \text{ on } \Gamma^{(m)},$$

(4.18) $$r_{\Gamma^{(m)}}\, h_j + r_{\Gamma^{(m)}}\, h_j^{(m)} = \widetilde{F}_j^{(m)} \text{ on } \Gamma^{(m)},\ j = \overline{1,4},$$

where

(4.19) $$\widetilde{f} := (\widetilde{f}_1, \cdots, \widetilde{f}_5)^\top \in [B_{p,p}^{1-1/p}(\Gamma)]^5,$$

(4.20) $$\widetilde{g}^{(m)} := (\widetilde{g}_1^{(m)}, \cdots, \widetilde{g}_5^{(m)})^\top \in [B_{p,p}^{1-1/p}(\Gamma^{(m)})]^5,$$

(4.21) $$\widetilde{F}^{(m)} := (\widetilde{F}_1^{(m)}, \cdots, \widetilde{F}_4^{(m)})^\top \in [\widetilde{B}_{p,p}^{-1/p}(\Gamma^{(m)})]^4$$

with

(4.22) $$\widetilde{g}_j^{(m)} := \widetilde{f}_j^{(m)} + r_{\Gamma^{(m)}} \Big[\mathcal{H}_\tau^{(m)} \left[-2^{-1} I_4 + \mathcal{K}_\tau^{(m)}\right]^{-1} \left[\widetilde{F}^{(m)}\right]\Big]_j,\ j = \overline{1,4}, \quad \widetilde{g}_5^{(m)} = \widetilde{f}_5^{(m)}.$$

Now, our goal is to show that the system of pseudodifferential equations (4.16)-(4.18) is uniquely solvable in appropriate function spaces.

4.2. *Existence theorems and regularity of solutions*

Let us put

$$\mathcal{N}_\tau := \begin{bmatrix} r_\Gamma\, \mathcal{A}_\tau & r_\Gamma\, \mathcal{A}_\tau & r_\Gamma\,[\,0\,]_{5\times 4} \\ r_{\Gamma^{(m)}}\, \mathcal{A}_\tau & r_{\Gamma^{(m)}} [\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}] & r_{\Gamma^{(m)}}\,[\,0\,]_{5\times 4} \\ r_{\Gamma^{(m)}}\,[\,0\,]_{4\times 5} & r_{\Gamma^{(m)}}\, I_{4\times 5} & r_{\Gamma^{(m)}}\, I_4 \end{bmatrix}_{14\times 14}, \quad I_{4\times 5} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Moreover, let

$$\mathbf{X}_p^s := [\widetilde{B}_{p,p}^s(\Gamma)]^5 \times [\widetilde{B}_{p,p}^s(\Gamma^{(m)})]^5 \times [\widetilde{B}_{p,p}^s(\Gamma^{(m)})]^4,$$

$$\mathbf{Y}_p^s := [B_{p,p}^{s+1}(\Gamma)]^5 \times [B_{p,p}^{s+1}(\Gamma^{(m)})]^5 \times [\widetilde{B}_{p,p}^s(\Gamma^{(m)})]^4,$$

$$\mathbf{X}_{p,t}^s := [\widetilde{B}_{p,t}^s(\Gamma)]^5 \times [\widetilde{B}_{p,t}^s(\Gamma^{(m)})]^5 \times [\widetilde{B}_{p,t}^s(\Gamma^{(m)})]^4,$$

$$\mathbf{Y}_{p,t}^s := [B_{p,t}^{s+1}(\Gamma)]^5 \times [B_{p,t}^{s+1}(\Gamma^{(m)})]^5 \times [\widetilde{B}_{p,t}^s(\Gamma^{(m)})]^4.$$

Evidently, we have the following mapping properties

(4.23) $$\mathcal{N}_\tau\ :\ \mathbf{X}_p^s \to \mathbf{Y}_p^s \quad [\mathbf{X}_{p,t}^s \to \mathbf{Y}_{p,t}^s],$$

for $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \leq t \leq \infty$, due to Theorems 3.4-3.6 and Lemma 3.8.

Evidently, we can rewrite the system (4.16)-(4.18) as

(4.24) $$\mathcal{N}_\tau\, \Phi = Y,$$

where $\Phi := (\psi,\ h,\ h^{(m)})^\top \in \mathbf{X}_p^s$ is an unknown vector and $Y := (\widetilde{f},\ \widetilde{g}^{(m)},\ \widetilde{F}^{(m)})^\top \in \mathbf{Y}_p^s$ is a given vector.

As we will see below the operator (4.24) is not invertible for all $s \in \mathbb{R}$. The interval $a < s < b$ of invertibility depends on p and on some parameters γ' and γ'' which are determined by the eigenvalues of special matrices constructed by means of the principal homogeneous symbol matrices of the operators $\mathcal{A}_\tau$ and $\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}$ (see (4.15)). Note that the numbers γ' and γ'' define also the smoothness exponents for the solutions to the original boundary transmission problem in a neighbourhood of the curves $\partial\Gamma^{(m)}$ and $\partial\Gamma$ (see Theorem 4.3 and Remark 4.4 below).

We start with the following theorem.

THEOREM 4.1: *Let the conditions*

$$1 < p < \infty, \quad 1 \le t \le \infty, \quad \frac{1}{p} - \frac{1}{2} + \gamma'' < s < \frac{1}{p} + \frac{1}{2} + \gamma' \tag{4.25}$$

be satisfied with γ' *and* γ'' *given by* (4.29), (4.30), *and* (4.31). *Then the operators* (4.23),

$$\mathcal{N}_\tau \; : \; \mathbf{X}_p^s \to \mathbf{Y}_p^s \quad \left[\mathbf{X}_{p,t}^s \to \mathbf{Y}_{p,t}^s\right], \tag{4.26}$$

are invertible.

PROOF: We prove the theorem in several steps. First we show that the operators (4.26) are Fredholm with zero index and afterwards we establish that the corresponding null-spaces are trivial.

Step 1. First of all let us remark that the operators

$$r_\Gamma \mathcal{A}_\tau \; : \; [\widetilde{B}_{p,t}^s(\Gamma^{(m)})]^5 \to [B_{p,t}^{s+1}(\Gamma)]^5, \quad r_{\Gamma^{(m)}} \mathcal{A}_\tau \; : \; [\widetilde{B}_{p,t}^s(\Gamma)]^5 \to [B_{p,t}^{s+1}(\Gamma^{(m)})]^5,$$

are compact.

Further we establish that the operators

$$r_\Gamma \mathcal{A}_\tau : [\widetilde{H}_2^{-1/2}(\Gamma)]^5 \to [H_2^{1/2}(\Gamma)]^5, \quad r_{\Gamma^{(m)}} [\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}] : [\widetilde{H}_2^{-1/2}(\Gamma^{(m)})]^5 \to [H_2^{1/2}(\Gamma^{(m)})]^5$$

are strongly elliptic pseudodifferential operators of order -1 with index zero. We remark that the principal homogeneous symbol matrices of these operators are strongly elliptic.

For an arbitrary solution $U \in [H_2^1(\Omega)]^5 \equiv [W_2^1(\Omega)]^5$ to the homogeneous equation $A(\partial, \tau)U = 0$ in Ω with the help of Green's formula and Korn's inequality [Fi1] and by standard manipulations we get

$$\Re \, \langle [U]^+, \, [\mathcal{T}U]^+ \rangle_{\partial\Omega} \ge c_1 \, \| \, U \, \|_{[H_2^1(\Omega)]^5}^2 - c_2 \, \| \, U \, \|_{[H_2^0(\Omega)]^5}^2 \, . \tag{4.27}$$

Substitute here $U = V_\tau(\mathcal{P}_\tau^{-1}\psi)$ with $\psi \in [H_2^{-1/2}(\partial\Omega)]^5$. Due to the equality $\psi = \mathcal{P}_\tau \mathcal{H}_\tau^{-1}\{U\}^+$ we have $\| \psi \|_{[H_2^{-1/2}(\partial\Omega)]^5}^2 \le c^* \, \| \, \{U\}^+ \, \|_{[H_2^{-1/2}(\partial\Omega)]^5}^2$ with some positive constant c^*. Therefore, by the trace theorem from (4.27) we easily obtain

$$\Re \, \langle \mathcal{H}_\tau \mathcal{P}_\tau^{-1}\psi, \, \psi \rangle_{\partial\Omega} \ge c_1' \, \| \, \psi \, \|_{[H_2^{-1/2}(\partial\Omega)]^5}^2 + \| \, \beta \, (\mathcal{H}\mathcal{P}_\tau^{-1}\psi)_5 \, \|_{H_2^0(\partial\Omega)} - c_2' \, \| \, V_\tau(\mathcal{P}_\tau^{-1}\psi) \|_{[H_2^0(\Omega)]^5}^2 .$$

In particular, in view of Theorem 3.1 for arbitrary $\psi \in [\widetilde{H}_2^{-1/2}(\Gamma)]^5$ we have

$$\Re \, \langle r_\Gamma \mathcal{H}_\tau \mathcal{P}_\tau^{-1} \psi, \, \psi \rangle_{\partial\Omega} \geq c_1'' \, \| \psi \|^2_{[\tilde{H}_2^{\,1/2}(\Gamma)]^5} - c_2'' \, \| \psi \|^2_{[\tilde{H}_2^{-3/2}(\Gamma)]^5} \, . \tag{4.28}$$

From (4.28) it follows that the operator $r_\Gamma \, \mathcal{A}_\tau = r_\Gamma \, \mathcal{H}_\tau \mathcal{P}_\tau^{-1} : [\widetilde{H}_2^{-1/2}(\Gamma)]^5 \to [H_2^{-1/2}(\Gamma)]^5$ is a strongly elliptic pseudodifferential Fredholm operator with index zero.

Then it follows that the same is true for the operator $\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}$ since the principal homogeneous symbol matrix of the operator $\mathcal{B}_\tau^{(m)}$ is nonnegative.

Therefore, the operator (4.26) is Fredholm with index zero for $s = -1/2$, $p = 2$ and $t = 2$.

Step 2. With the help of the uniqueness Theorem 2.1 via representation formulas (4.5) and (4.6) with $Q_0^{(m)} = 0$ and $Q_0 = 0$ we can easily show that the operator (4.26) is injective for $s = -1/2$, $p = 2$ and $t = 2$. Since its index is zero, we conclude that it is surjective. Thus the operator (4.26) is invertible for $s = -1/2$, $p = 2$ and $t = 2$.

Step 3. To complete the proof for the general case we proceed as follows. We see that the following upper triangular operator

$$\mathcal{N}_\tau^{(0)} := \begin{bmatrix} r_\Gamma \mathcal{A}_\tau & r_\Gamma \, [\,0\,]_{5\times 5} & r_\Gamma \, [\,0\,]_{5\times 4} \\ r_{\Gamma^{(m)}} \, [\,0\,]_{5\times 5} & r_{\Gamma^{(m)}} \, [\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}] & r_{\Gamma^{(m)}} \, [\,0\,]_{5\times 4} \\ r_{\Gamma^{(m)}} \, [\,0\,]_{4\times 5} & r_{\Gamma^{(m)}} \, I_{4\times 5} & r_{\Gamma^{(m)}} \, I_4 \end{bmatrix}_{14\times 14}$$

is a compact perturbation of the operator $\mathcal{N}_\tau$. Therefore we have to investigate Fredholm properties of the operators

$$r_\Gamma \mathcal{A}_\tau \; : \; [\widetilde{B}^s_{p,t}(\Gamma)]^5 \to [B^{s+1}_{p,t}(\Gamma)]^5, \qquad r_{\Gamma^{(m)}} [\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}] \; : \; [\widetilde{B}^s_{p,t}(\Gamma^{(m)})]^5 \to [B^{s+1}_{p,t}(\Gamma^{(m)})]^5.$$

Let $\widetilde{\sigma}_1(x, \xi_1, \xi_2) := \sigma(\mathcal{A}_\tau)(x, \xi_1, \xi_2)$ be the principal symbol matrix of the operator $\mathcal{A}_\tau$ and $\lambda_j^{(1)}(x)$ $(j = \overline{1,5})$ be the eigenvalues of the matrix $[\widetilde{\sigma}_1(x, 0, +1)]^{-1} \, \widetilde{\sigma}_1(x, 0, -1)$ for $x \in \partial\Gamma$ (for details see [BCN1]).

Similarly, let $\widetilde{\sigma}_2(x, \xi_1, \xi_2) = \sigma(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})(x, \xi_1, \xi_2)$ be the principal symbol matrix of the operator $\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}$ and $\lambda_j^{(2)}(x)$ $(j = \overline{1,5})$ be the eigenvalues of the corresponding matrix $[\widetilde{\sigma}_2(x, 0, +1)]^{-1} \, \widetilde{\sigma}_2(x, 0, -1)$ for $x \in \partial\Gamma^{(m)}$.

Further, we set

$$\gamma_1' := \inf_{x \in \partial\Gamma, \, 1 \leq j \leq 5} \frac{1}{2\pi} \arg \lambda_j^{(1)}(x), \quad \gamma_1'' := \sup_{x \in \partial\Gamma, \, 1 \leq j \leq 5} \frac{1}{2\pi} \arg \lambda_j^{(1)}(x), \tag{4.29}$$

$$\gamma_2' := \inf_{x \in \partial\Gamma^{(m)}, \, 1 \leq j \leq 5} \frac{1}{2\pi} \arg \lambda_j^{(2)}(x), \quad \gamma_2'' := \sup_{x \in \partial\Gamma^{(m)}, \, 1 \leq j \leq 5} \frac{1}{2\pi} \arg \lambda_j^{(2)}(x). \tag{4.30}$$

Note that γ_j' and γ_j'' $(j = 1, 2)$ depend on the material parameters, in general, and belong to the interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$. We put

$$\gamma' := \min\{\gamma_1', \, \gamma_2'\}, \qquad \gamma'' := \max\{\gamma_1'', \, \gamma_2''\}. \tag{4.31}$$

From Theorem 3.10 we conclude that if the parameters $s, r \in \mathbb{R}$, $1 < p < \infty$,

---

$1 \leq t \leq \infty$, satisfy the con…

$+\gamma_2'' < s < \frac{1}{p} + \frac{1}{2} + \gamma_2'$, then the…

$r_\Gamma \mathcal{A}_\tau \; : \; [\widetilde{H}_p^{r-1}(\Gamma)]^5 \to$ …

$r_{\Gamma^{(m)}} [\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}] \; : \; [\widetilde{H}$ …

are Fredholm operators with …

Therefore, if the conditions …

with zero index. Consequently …

invertible due to the results …

Now we are in the position to …

the boundary-transmission prob…

THEOREM 4.2: *Let the …*

(4.32)

Then the boundary-trans… represented by formulas

(4.33) $U = V_\tau(\mathcal{P}$ …

(4.34) $U^{(m)} =$ …

where the densities ψ, … *the system* (4.16)-(4.18) …

PROOF: The existence of a … satisfying (4.32) follows from … $-\frac{1}{2} < \gamma' \leq \gamma'' < \frac{1}{2}$ we have … solvability for $p = 2$ is a conseq…

To show the uniqueness … proceed as follows. Let a pair …

(4.35)

with p satisfying (4.32) be a … problem.

Then, it is evident that there…

(4.36) $\{U^{(m)}\}$ … $\{\mathcal{T}^{(m)} U$ …

$1 \leq t \leq \infty$, satisfy the conditions $\frac{1}{p} - \frac{1}{2} + \gamma_1'' < r < \frac{1}{p} + \frac{1}{2} + \gamma_1'$ and $\frac{1}{p} - \frac{1}{2} + - \gamma_2{}'' < s < \frac{1}{p} + \frac{1}{2} + \gamma_2'$, then the operators

$$r_{\Gamma} \mathcal{A}_\tau \; : \; [\widetilde{H}_p^{r-1}(\Gamma)]^5 \to [H_p^r(\Gamma)]^5 \quad \left[[\widetilde{B}_{p,t}^{r-1}(\Gamma)]^5 \to [B_{p,t}^r(\Gamma)]^5 \right],$$

$$r_{\Gamma^{(m)}} [\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}] \; : \; [\widetilde{H}_p^{s-1}(\Gamma^{(m)})]^5 \to [H_p^s(\Gamma^{(m)})]^5 \quad \left[[\widetilde{B}_{p,t}^{s-1}(\Gamma^{(m)})]^5 \to [B_{p,t}^s(\Gamma^{(m)})]^5 \right]$$

are Fredholm operators with index zero.

Therefore, if the conditions (4.25) are satisfied then the above operators are Fredholm with zero index. Consequently, the operators (4.26) are Fredholm with zero index and are invertible due to the results obtained in Step 2. □

Now we are in the position to formulate the basic existence and uniqueness results for the boundary-transmission problem under consideration.

THEOREM 4.2: *Let the inclusions* (2.39) *and compatibility condition* (2.40) *hold and let*

$$\frac{4}{3 - 2\gamma''} < p < \frac{4}{1 - 2\gamma'} \,. \tag{4.32}$$

Then the boundary-transmission problem (2.29)-(2.37) *has a unique solution which can be represented by formulas*

$$U = V_\tau \big(\mathcal{P}_\tau^{-1} \left[Q_0 + \psi + h \right] \big) \quad \text{in } \Omega, \tag{4.33}$$

$$U^{(m)} = V_\tau^{(m)} \big(\left[-2^{-1} I_4 + \mathcal{K}_\tau^{(m)} \right]^{-1} \left[Q_0^{(m)} + h^{(m)} \right] \big) \quad \text{in } \Omega^{(m)}, \tag{4.34}$$

where the densities ψ, h, and $h^{(m)}$ are to be determined from the system (4.7)-(4.10) (*or from the system* (4.16)-(4.18)).

PROOF: The existence of a solution pair $(U^{(m)}, U) \in [W_p^1(\Omega^{(m)})]^4 \times [W_p^1(\Omega)]^5$ with p satisfying (4.32) follows from Theorem 4.1 with $s = 1 - p^{-1}$. Due to the inequalities $-\frac{1}{2} < \gamma' \leq \gamma'' < \frac{1}{2}$ we have $p = 2 \in \left(\frac{4}{3 - 2\gamma''} \, , \, \frac{4}{1 - 2\gamma'} \right)$. Therefore the unique solvability for $p = 2$ is a consequence of Theorem 2.1.

To show the uniqueness result for all other values of p from the interval (4.32) we proceed as follows. Let a pair

$$(U^{(m)}, U) \in [W_p^1(\Omega^{(m)})]^4 \times [W_p^1(\Omega)]^5 \tag{4.35}$$

with p satisfying (4.32) be a solution to the homogeneous boundary-transmission problem.

Then, it is evident that there exist the traces

$$\begin{aligned} &\{ U^{(m)} \}^+ \in [B_{p,p}^{1-\frac{1}{p}}(\partial\Omega^{(m)})]^4, && \{ U \}^- \in [B_{p,p}^{1-\frac{1}{p}}(\partial\Omega)]^5, \\ &\{ \mathcal{T}^{(m)} U^{(m)} \}^+ \in [B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(m)})]^4, && \{ \mathcal{T} U \}^- \in [B_{p,p}^{-\frac{1}{p}}(\partial\Omega)]^5. \end{aligned} \tag{4.36}$$

and the vectors $U^{(m)}$ and U in $\Omega^{(m)}$ and Ω respectively are represented in the form (cf. (4.33)-(4.34), with $Q_0^{(m)}=0$ and $Q_0=0$),

$$U^{(m)} = V_\tau^{(m)}\big(\big[-2^{-1}I_4+\mathcal{K}_\tau^{(m)}\big]^{-1}h^{(m)}\big) \quad \text{in} \quad \Omega^{(m)}, \tag{4.37}$$

$$U = V_\tau\big(\mathcal{P}_\tau^{-1}\,[h+\psi]\big) \quad \text{in} \quad \Omega, \tag{4.38}$$

due to Lemmas 3.7 and 3.9.

By the same arguments as above we arrive at the homogeneous system $\mathcal{N}_\tau\,\Phi=0$. where $\Phi:=(\psi,\,h,\,h^{(m)})^\top \in \mathbf{X}_p^{-\frac{1}{p}}$. Due to Theorem 4.1, $\Phi=0$ and we conclude that $U^{(m)}=0$ in $\Omega^{(m)}$ and $U=0$ in Ω. □

Finally, we can prove the following regularity result for the solution of the boundary-transmission problem.

THEOREM 4.3: *Let the inclusions* (2.39) *and compatibility condition* (2.40) *hold and let*

$$\frac{4}{3-2\gamma''}<p<\frac{4}{1-2\gamma'},\ 1<r<\infty,\ 1\le t\le\infty,\ \frac{1}{r}-\frac{1}{2}+\gamma''<s<\frac{1}{r}+\frac{1}{2}+\gamma'. \tag{4.39}$$

Further, let $U^{(m)}\in[W_p^1(\Omega^{(m)})]^4$ *and* $U\in[W_p^1(\Omega)]^5$ *be a unique solution pair of the boundary-transmission problem* (2.29)-(2.37) *with* $X_j^{(m)}=0$, $j=\overline{1,4}$ *and* $X_k=0$, $k=\overline{1,5}$.

Then the following hold:

i) *if*

$$Q_k\in B_{r,r}^{s-1}(S),\ f_k\in B_{r,r}^{s}(\Gamma),\ f_k^{(m)}\in B_{r,r}^{s}(\Gamma^{(m)}),\ Q_j^{(m)}\in B_{r,r}^{s-1}(S^{(m)}),\ F_j^{(m)}\in B_{r,r}^{s-1}(\Gamma^{(m)}),$$

and the compatibility condition $F_j^{(m)}-[r_{_{\Gamma^{(m)}}}\widehat{Q}_j^{(m)}+r_{_{\Gamma^{(m)}}}\widehat{Q}_j]\in r_{_{\Gamma^{(m)}}}\widetilde{B}_{r,r}^{s-1}(\Gamma^{(m)})$ *is satisfied, then* $U^{(m)}\in[H_r^{s+\frac{1}{r}}(\Omega^{(m)})]^4$ *and* $U\in[H_r^{s+\frac{1}{r}}(\Omega)]^5$;

ii) *if*

$$Q_k\in B_{r,t}^{s-1}(S),\ f_k\in B_{r,t}^{s}(\Gamma),\ f_k^{(m)}\in B_{r,t}^{s}(\Gamma^{(m)}),\ Q_j^{(m)}\in B_{r,t}^{s-1}(S^{(m)}),\ F_j^{(m)}\in B_{r,t}^{s-1}(\Gamma^{(m)}),$$

and the compatibility condition $F_j^{(m)}-[r_{_{\Gamma^{(m)}}}\widehat{Q}_j^{(m)}+r_{_{\Gamma^{(m)}}}\widehat{Q}_j]\in r_{_{\Gamma^{(m)}}}\widetilde{B}_{r,t}^{s-1}(\Gamma^{(m)})$ *is satisfied, then*

$$U^{(m)}\in[B_{r,t}^{s+\frac{1}{r}}(\Omega^{(m)})]^4,\quad U\in[B_{r,t}^{s+\frac{1}{r}}(\Omega)]^5; \tag{4.40}$$

iii) *if* $\alpha>0$ *is not integer and*

$$Q_k\in B_{\infty,\infty}^{\alpha-1}(S),\ f_k\in C^\alpha(\overline{\Gamma}),\ f_k^{(m)}\in C^\alpha(\overline{\Gamma^{(m)}}),\ Q_j^{(m)}\in B_{\infty,\infty}^{\alpha-1}(S^{(m)}),\ F_j^{(m)}\in B_{\infty,\infty}^{\alpha-1}(\Gamma^{(m)}),$$

and the compatibility condition $F_j^{(m)}-[r_{_{\Gamma^{(m)}}}\widehat{Q}_j^{(m)}+r_{_{\Gamma^{(m)}}}\widehat{Q}_j]\in r_{_{\Gamma^{(m)}}}\widetilde{B}_{\infty,\infty}^{s-1}(\Gamma^{(m)})$, *is satisfied, then*

$$U^{(m)}\in\bigcap_{\alpha'<\kappa}[C^{\alpha'}(\overline{\Omega^{(m)}})]^4,\quad U\in\bigcap_{\alpha'<\kappa}[C^{\alpha'}(\overline{\Omega})]^5,\ \text{where}\ \kappa=\min\Big\{\alpha,\ \frac{1}{2}+\gamma'\Big\}.$$

Proof: The proof of items i) and ii) easily follows from Theorems 4.1, 4.2 and 3.1.

To prove the item iii) we use the following embedding relations (see, e.g., [Tr1])

$$C^a(\mathcal{M}) = B^a_{\infty,\infty}(\mathcal{M}) \subset B^{a-\varepsilon}_{\infty,1}(\mathcal{M}) \subset B^{a-\varepsilon}_{\infty,t}(\mathcal{M}) \subset B^{a-\varepsilon}_{r,t}(\mathcal{M}) \subset C^{a-\varepsilon-k/r}(\mathcal{M}), \tag{4.41}$$

where ε is an arbitrary small positive number, $\mathcal{M} \subset \mathbb{R}^3$ is a compact k-dimensional $(k = 2, 3)$ smooth manifold with smooth boundary, $1 \leq t \leq \infty$, $1 < r < \infty$, $a - \varepsilon - k/r > 0$, and $a - \varepsilon - k/r$ are not integers.

From iii) and the embedding (4.41) the condition (4.40) follows with any $s \leq a - \varepsilon$.

Bearing in mind (4.39) and taking r sufficiently large and ε sufficiently small, we can put

$$s = a - \varepsilon \quad \text{if} \quad \frac{1}{r} - \frac{1}{2} + \gamma'' < a - \varepsilon < \frac{1}{r} + \frac{1}{2} + \gamma', \tag{4.42}$$

and

$$s \in \left(\frac{1}{r} - \frac{1}{2} + \gamma'', \frac{1}{r} + \frac{1}{2} + \gamma'\right) \quad \text{if} \quad \frac{1}{r} + \frac{1}{2} + \gamma' < a - \varepsilon. \tag{4.43}$$

By (4.40) for the solution vectors we have $U^{(m)} \in [B^{s+\frac{1}{r}}_{r,t}(\Omega^{(m)})]^4$ and $U \in [B^{s+\frac{1}{r}}_{r,t}(\Omega)]^5$ with $s + \frac{1}{r} = a - \varepsilon + \frac{1}{r}$ if (4.42) holds, and with $s + \dfrac{1}{r} \in \left(\dfrac{2}{r} - \dfrac{1}{2} + \gamma'', \dfrac{2}{r} + \dfrac{1}{2} + \gamma'\right)$ if (4.43) holds. In the last case we can take $s + \dfrac{1}{r} = \dfrac{2}{r} + \dfrac{1}{2} + \gamma' - \varepsilon$. Therefore, we have either

$$U^{(m)} \in \left[B^{a-\varepsilon+\frac{1}{r}}_{r,t}(\Omega^{(m)})\right]^4, \quad U \in \left[B^{a-\varepsilon+\frac{1}{r}}_{r,t}(\Omega)\right]^5,$$

or

$$U^{(m)} \in \left[B^{\frac{1}{2}+\frac{2}{r}+\gamma'-\varepsilon}_{r,t}(\Omega^{(m)})\right]^4, \quad U \in \left[B^{\frac{1}{2}+\frac{2}{r}+\gamma'-\varepsilon}_{r,t}(\Omega)\right]^5,$$

in accordance with the inequalities (4.42) and (4.43). The last embedding in (4.41) (with $k = 3$) yields then that either

$$U^{(m)} \in [C^{a-\varepsilon-\frac{2}{r}}(\overline{\Omega^{(m)}})]^4, \quad U \in [C^{a-\varepsilon-\frac{2}{r}}(\overline{\Omega})]^5,$$

or

$$U^{(m)} \in [C^{\frac{1}{2}-\varepsilon+\gamma'-\frac{1}{r}}(\overline{\Omega}_1)]^4, \quad U \in [C^{\frac{1}{2}-\varepsilon+\gamma'-\frac{1}{r}}(\overline{\Omega})]^5,$$

implying $U^{(m)} \in [C^{\kappa-\varepsilon-\frac{2}{r}}(\overline{\Omega^{(m)}})]^4$, $U \in [C^{\kappa-\varepsilon-\frac{2}{r}}(\overline{\Omega})]^5$, where $\kappa = \min\left\{a, \dfrac{1}{2} + \gamma'\right\}$. Since r is sufficiently large and ε is sufficiently small, these inclusions complete the proof. □

Remark 4.4: More detailed analysis based on the asymptotic expansions of solutions (see [CD1], [CD2]) shows that for sufficiently smooth boundary data (e.g., C^∞-smooth data say) the principal singular terms of the solution vectors $U^{(m)}$ and U near the curves

$\partial\Gamma^{(m)}$ and $\partial\Gamma$ can be represented as a product of a "good" vector-function and a singular factor of the form $[\,\ln\rho(x)\,]^{m_j-1}[\,\rho(x)\,]^{a_j+i\beta_j}$. Here $\rho(x)$ is the distance from a reference point x to the curves $\partial\Gamma^{(m)}$ or $\partial\Gamma$. Therefore, near these curves the dominant singular terms of the corresponding generalized stress vectors $\mathcal{T}^{(m)}U^{(m)}$ and $\mathcal{T}U$ are represented as a product of a "good" vector-function and the factor $[\,\ln\rho(x)\,]^{m_j-1}[\,\rho(x)\,]^{-1+a_j+i\beta_j}$. The numbers β_j are different from zero, in general, and describe the oscillating character of the stress singularities.

The exponents $a_j + i\beta_j$ are related to the corresponding eigenvalues by the equalities

$$a_j = \frac{1}{2} + \frac{\arg\lambda_j}{2\pi}, \quad \beta_j = -\frac{\ln|\lambda_j|}{2\pi}.$$

Here $\lambda_j \in \{\lambda_1^{(1)}(x),\cdots,\lambda_5^{(1)}(x)\}$ for $x\in\partial\Gamma$, and $\lambda_j \in \{\lambda_1^{(2)}(x),\cdots,\lambda_5^{(2)}(x)\}$ for $x\in\partial\Gamma^{(m)}$. In the above expressions the parameter m_j denotes the multiplicity of the eigenvalue λ_j.

It is evident that at the curves $\partial\Gamma^{(m)}$ and $\partial\Gamma$ the components of the generalized stress vector behave like $\mathcal{O}\big([\,\ln\rho(x)\,]^{m_0-1}[\rho(x)]^{-\frac{1}{2}+\gamma'}\big)$, where m_0 denotes the maximal multiplicity of the eigenvalues. This is a global singularity effect for the first order derivatives of the vectors $U^{(m)}$ and U. In contrast to the classical pure elasticity case (where $\gamma'=\gamma''=0$), here γ' and γ'' depend on the material parameters and are different from zero, in general (see the example below). This is related to the fact that our transmission problem and, consequently, the corresponding strongly elliptic system of pseudodifferential equations are not selfadjoint. This implies that the eigenvalues $\lambda_j^{(k)}$ are complex numbers, in general.

REMARK 4.5: It can be shown that the eigenvalues $\lambda_5^{(1)}(x)=1$ for all $x\in\partial\Gamma$ and $\lambda_5^{(2)}(x)=1$ for all $x\in\partial\Gamma^{(m)}$. Moreover, the eigenvalues $\{\lambda_j^{(1)}(x)\}_{j=1}^{4}$ for $x\in\partial\Gamma$ and $\{\lambda_j^{(2)}(x)\}_{j=1}^{4}$ for $x\in\partial\Gamma^{(m)}$ do not depend on the thermal constants. However, they depend on the elastic and piezoelectric material parameters, in general.

If $\gamma_k'<0$ and $\gamma_k''>0$, $k=1,2$, (see (4.29) and (4.30)) then the smoothness and the singularity exponents are actually defined only by the eigenvalues $\{\lambda_j^{(1)}(x)\}_{j=1}^{4}$ and $\{\lambda_j^{(2)}(x)\}_{j=1}^{4}$, since $\arg\lambda_5^{(1)}(x)=0$ for all $x\in\partial\Gamma$ and $\arg\lambda_5^{(2)}(x)=0$ for all $x\in\partial\Gamma^{(m)}$.

EXAMPLE

Here we apply our approach to practical examples to show the dependence of the characteristics γ_k' and γ_k'' $(k=1,2)$ on the material parameters.

To compute the smoothness and the singularity exponents mentioned in Theorem 4.3 and Remark 4.4, we have to find the eigenvalues $\lambda_j^{(1)}$ and $\lambda_j^{(2)}$, $j=\overline{1,5}$.

We assume that the domain $\Omega^{(m)}$ is occupied by the isotropic metallic material *silver-palladium alloy* with Lamé constants $\lambda = 1.0\cdot 10^{11}$ Pa and $\mu = 3.17\cdot 10^{10}$ Pa, whereas the domain Ω is occupied by different piezoelectric media. We consider the piezoelectric materials $BaTiO_3$ (with the crystal symmetry of the class **4mm**), PZT-4 and PZT-5A (with the crystal symmetry of the class **6mm**). Their material constants are given in the tables below:

	c_{11}(Pa)	c_{12}(Pa)	c_{13}(Pa)	c_{33}(Pa)	c_{44}(Pa)	c_{66}(Pa)
$BaTiO_3$	$2.75 \cdot 10^{11}$	$1.79 \cdot 10^{11}$	$1.52 \cdot 10^{11}$	$1.69 \cdot 10^{11}$	$5.43 \cdot 10^{10}$	$1.13 \cdot 10^{11}$
PZT-4	$1.39 \cdot 10^{11}$	$7.80 \cdot 10^{10}$	$7.40 \cdot 10^{10}$	$1.15 \cdot 10^{11}$	$2.56 \cdot 10^{10}$	$3.05 \cdot 10^{10}$
PZT-5A	$1.20 \cdot 10^{11}$	$7.52 \cdot 10^{10}$	$7.51 \cdot 10^{10}$	$1.11 \cdot 10^{11}$	$2.11 \cdot 10^{10}$	$2.26 \cdot 10^{10}$

	e_{15}(C/m^2)	e_{31}(C/m^2)	e_{33}(C/m^2)	ε_{11}(F/m)	ε_{33}(F/m)
$BaTiO_3$	21.30	−2.69	3.65	$1.75 \cdot 10^{-8}$	$9.89 \cdot 10^{-10}$
PZT-4	12.70	−5.20	15.10	$6.50 \cdot 10^{-9}$	$5.60 \cdot 10^{-9}$
PZT-5A	12.29	−5.35	15.78	$8.14 \cdot 10^{-9}$	$7.32 \cdot 10^{-9}$

We remark that the constants c_{ijkl}, e_{ikl}, and c_{pq}, e_{pq} are related by the following rule:

$$c_{f(ij)f(kl)} = c_{ijkl}, \quad e_{if(kl)} = e_{ikl},$$

where

$$f(11)=1, f(22)=2, f(33)=3, f(23)=f(32)=4, f(13)=f(31)=5, f(12)=f(21)=6.$$

Moreover, for the above piezoelectric materials there hold:

$$c_{kj} = c_{jk},\ c_{11} = c_{22},\ c_{13} = c_{23},\ c_{44} = c_{55},\ c_{ij} = 0 \text{ for } i \neq j \text{ and } i,j = 4,5,6;$$
$$e_{24} = e_{15},\quad e_{31} = e_{32},\quad e_{1i} = e_{2j} = e_{3k} = 0 \text{ for } i \neq 5,\ j \neq 4,\ k > 3;$$
$$\varepsilon_{11} = \varepsilon_{22},\quad \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0.$$

Global regularity result. Here we give the numerical results concerning the global regularity property of the solution vectors U and $U^{(m)}$. Due to Theorem 4.3 the Hölder smoothness exponent in the closed domains $\overline{\Omega}$ and $\overline{\Omega^{(m)}}$ is calculated by the number $\kappa = \min\left\{\alpha, \frac{1}{2} + \gamma'\right\}$.

The calculations have shown that arg $\lambda_j^{(1)}(x)$ and arg $\lambda_j^{(2)}(x)$ ($j = 1,2,3,4$) do not depend on the reference point x. Moreover, the computations have shown that for the above mentioned piezoelectric materials $BaTiO_3$, PZT-4, and PZT-5A two eigenvalues ($\lambda_1^{(k)}$ and $\lambda_2^{(k)}$ say) are mutually inverse complex numbers:

$$\lambda_1^{(k)} = \exp\{-i\,\theta^{(k)}\},\quad \lambda_2^{(k)} = \exp\{i\,\theta^{(k)}\},\quad \theta^{(k)} > 0,\quad k = 1,2;$$

another two eigenvalues are equal to 1: $\lambda_3^{(k)} = \lambda_4^{(k)} = 1$. Recall that $\lambda_5^{(k)} = 1$. Therefore, $\gamma_k' < 0$, $\gamma_k'' > 0$, and $\gamma_k'' = -\gamma_k'$, $k = 1,2$ (see (4.29)-(4.30)).

The computed values of γ_1' and γ_2' corresponding to the considered three cases are as follows

		$BaTiO_3$	PZT-4	PZT-5A
(4.44)	γ_1'	−0.12	−0.12	−0.13
	γ_2'	−0.06	−0.08	−0.09 .

Therefore, for $\gamma' := \min\{\gamma_1', \gamma_2'\}$ we have (see (4.31))

	$BaTiO_3$	PZT-4	PZT-5A
γ'	−0.12	−0.12	−0.13 .

Consequently, if the boundary data of the transmission problem under consideration are sufficiently smooth (e.g., $a > 0.5$, see Theorem 4.3), then for the Hölder smoothness exponent κ we derive

	$BaTiO_3$	PZT-4	PZT-5A
κ	0.38	0.38	0.37 .

Thus, in the closed domains the solution vectors have $C^{\kappa-\delta}$-smoothness , where $\delta > 0$ is an arbitrarily small number. This shows that the smoothness exponent depends on the material parameters.

Local singularity effects at different edges. Here we compare the dominant stress singularity exponents calculated for the curves $\partial\Gamma$ and $\partial\Gamma^{(m)}$. Note that the factors of type $[\ln\rho(x)]^{m_j-1}[\rho(x)]^{a_j+i\,b_j}$ appear in the singular edge terms of the stress fields (see Remark 4.4). Recall that $\rho(x)$ is the distance from a reference point x to the curves $\partial\Gamma^{(m)}$ or $\partial\Gamma$. The exponents $a_j + i\,b_j$ are related to the eigenvalues by the equalities

$$a_j = -\frac{1}{2} + \frac{\arg\lambda_j}{2\pi}, \qquad b_j = -\frac{\ln|\lambda_j|}{2\pi},$$

where $\lambda_j \in \{\lambda_1^{(1)}(x), \cdots, \lambda_5^{(1)}(x)\}$ for $x \in \partial\Gamma$, and $\lambda_j \in \{\lambda_1^{(2)}(x), \cdots, \lambda_5^{(2)}(x)\}$ for $x \in \partial\Gamma^{(m)}$. The number m_j denotes the multiplicity of the eigenvalue λ_j.

As it has been mentioned above the calculations have shown that the arguments of the complex eigenvalues, $\arg\lambda_j^{(1)}(x)$ and $\arg\lambda_j^{(2)}(x)$ $(j = 1,2,3,4,)$ do not depend on the reference point x. Keep in mind that $\lambda_5^{(1)} = \lambda_5^{(2)} = 1$ for all values of the material parameters.

Moreover, the calculations have shown that for the above mentioned piezoelectric materials $BaTiO_3$, PZT-4, and PZT-5A the parameters $b_j, j = \overline{1,4}$, (characterizing the so-called oscillating singularity effects) vanish, which means that the modules of the eigenvalues equal to 1. Moreover, two of them ($\lambda_1^{(k)}$ and $\lambda_2^{(k)}$ say) are mutually inverse complex numbers:

$$\lambda_1^{(k)} = \exp\{-i\,\theta^{(k)}\}, \quad \lambda_2^{(k)} = \exp\{i\,\theta^{(k)}\}, \quad \theta^{(k)} > 0, \quad k = 1,2;$$

another two eigenvalues are equal to 1: $\lambda_3^{(k)} = \lambda_4^{(k)} = 1$. Therefore, $\gamma_k' < 0$, $\gamma_k'' > 0$, and $\gamma_k' = -\gamma_k''$, $k = 1,2$ (see (4.29)-(4.30)). It is evident that the complex eigenvalues $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$ with the negative arguments $\theta^{(1)}$ and $\theta^{(2)}$ correspond to the dominant stress singularity terms at $\partial\Gamma$ and $\partial\Gamma^{(m)}$, respectively.

Thus we have two simple complex eigenvalues, $\lambda_1^{(k)} = \exp\{-i\,\theta^{(k)}\}$ and $\lambda_2^{(k)} = \exp\{i\,\theta^{(k)}\}$, and one eigenvalue of multiplicity 3, $\lambda_3^{(k)} = \lambda_4^{(k)} = \lambda_5^{(k)} = 1$.

Therefore, near the curves $\partial\Gamma$ and $\partial\Gamma^{(m)}$ at the edge singular terms there appear the factors of type $[\ln\rho(x)]^2[\rho(x)]^{-\frac{1}{2}}$ which correspond to the eigenvalues $\lambda_3^{(k)} = \lambda_4^{(k)} = \lambda_5^{(k)} = 1$.

Moreover, near the curve where the type of boundary conditions change (the curve $\partial\Gamma$) in the singular terms there appears the factor of type $[\rho(x)]^{-\frac{1}{2}+\gamma_1'}$ corresponding to the eigenvalue $\lambda_1^{(1)}$, while the factor of type $[\rho(x)]^{-\frac{1}{2}+\gamma_2}$, corresponding to the eigenvalue $\lambda_1^{(2)}$, appears near the curve where the interface intersects the exterior boundary (the curve $\partial\Gamma^{(m)}$).

It is easy to see that the dominant stress singularities near the curves Γ and $\Gamma^{(m)}$ is defined by the factors $[\rho(x)]^{-\frac{1}{2}+\gamma_1'}$ and $[\rho(x)]^{-\frac{1}{2}+\gamma_2'}$ respectively.

The computed values of γ_1' and γ_2' corresponding to the considered three cases are presented in table (4.44), which gives the following principal stress singularity exponents near the curves Γ and $\Gamma^{(m)}$:

	$BaTiO_3$	PZT-4	PZT-5A
Stress singularity exponent at Γ	-0.62	-0.62	-0.63
Stress singularity exponent at $\Gamma^{(m)}$	-0.56	-0.58	-0.59.

Note that the stress singularities at the curve $\partial\Gamma$ where the type of boundary conditions change are higher than near the curve $\partial\Gamma^{(m)}$ where the interface intersects the exterior boundary.

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I. TAVKH

Classi
of Geometric Figures,

ABSTRACT. — Aim of this
surfaces and lines ("Surfaces
some problems of Shell Theory
class can be used for describing
analytic representation gives

In this paper we give
surfaces, lines and traject
Listing's bodies, which are
defined.

NOTATIONS: In this

- X, Y, Z, or
- t-time value
- τ, ψ, θ - are space

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