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On increasing the convergence rate of difference solution to the third boundary value problem of elasticity theory

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Abstract

We consider the third boundary value problem of static elasticity theory (stiff contact problem) in a rectangle, for solution of which we use a difference scheme of second-order accuracy. Using this approximate solution, we correct the right-hand side of the difference scheme. It is shown that the solution of the corrected scheme is convergent at the rate $O(|h|^m)$ in the discrete L_2-norm, provided that the solution of the original problem belongs to the Sobolev space with exponent $m \in [2, 4]$.

MSC: 65N06; 74G15

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1 Introduction

The problem of accuracy is the main problem in the theory of difference schemes as well as in applications. One approach for obtaining solutions with improved accuracy is represented by the idea of refinement by differences of higher order, which was offered by Fox [1]: the right-hand side of the difference scheme is corrected by the solution obtained on the first stage and the scheme is repeatedly solved on the same grid. This empirical idea is simple, although its theoretical foundation encounters essential difficulties. This is shown by Volkov's papers [2–5], where the above-mentioned method was founded only for the Poisson and Laplace equations, besides, the input data in the problems were chosen so as to ensure that the exact solution belongs to the Hölder class $C_{6,\lambda}(\bar{\Omega})$.

In the development of the mentioned method there are two possible difficulties. The first one consists of finding the correcting addend, and the second is related to obtaining an *a priori* estimate of convergence. To overcome these difficulties we use the method for derivation of estimates for the convergence rate of difference schemes developed in the last 30 years by Samarskii and other authors (*e.g.*, see [6–10]), in which the convergence rate is compatible with the smoothness of the solution of the original differential problem. For elliptic problems, such estimates have the form

$$\|U - u\|_{W_2^s(\omega)} \le c|h|^{m-s}\|u\|_{W_2^m(\Omega)}, \quad m > s \ge 0, \tag{1}$$

where u is the solution of the original differential problem, U is an approximate solution, s and m are integer and real numbers, respectively, and $\|\cdot\|_{W_2^s(\omega)}$ and $\|\cdot\|_{W_2^s(\Omega)}$ are the Sobolev norms on the sets of functions of discrete and continuous arguments, respectively. Hereafter by c we denote positive constants that are independent of $|h|$ and u and can be distinct in distinct formulas.

For obtaining a difference solution of the Dirichlet problem posed for elliptic equations, in [11, 12] was considered the aforementioned method of correction.

In [13, 14] is considered the third boundary value problem of static elasticity theory (stiff contact problem) in a rectangle. For corresponding difference schemes, when $s = 0, 1, 2$ and $0 < m - s \leq 2$, there are accepted estimates of type (1).

In the present paper, for the same problem, we construct a correction term for the right-hand side of the difference scheme. It is proved that the corrected scheme converges at the rate $O(|h|^m)$ in the discrete $L_2(\omega)$-norm provided that the exact solution belongs to the Sobolev space $W_2^m(\Omega)$, $m \in [2, 4]$.

2 Statement of the problem

Below, everywhere we assume that index $\alpha = 1, 2$ and $\beta = 3 - \alpha$.

Let $\bar{\Omega} = \{x = (x_1, x_2) : 0 \leq x_\alpha \leq l_\alpha\}$ represent a rectangle with boundary Γ.

Consider the problem

$$
\begin{aligned}
&(\lambda + 2\mu)\frac{\partial^2 u^1}{\partial x_1^2} + (\lambda + \mu)\frac{\partial^2 u^2}{\partial x_1\,\partial x_2} + \mu\frac{\partial^2 u^1}{\partial x_2^2} + f^1(x) = 0, \\
&\mu\frac{\partial^2 u^2}{\partial x_1^2} + (\lambda + \mu)\frac{\partial^2 u^1}{\partial x_1\,\partial x_2} + (\lambda + 2\mu)\frac{\partial^2 u^2}{\partial x_2^2} + f^2(x) = 0, \quad x \in \Omega,
\end{aligned}
\tag{2}
$$

$$
u^\alpha(x) = 0, \qquad \frac{\partial u^\beta(x)}{\partial x_\alpha} = 0, \quad x \in \Gamma, x_\alpha = 0, l_\alpha. \tag{3}
$$

Here $\lambda, \mu = \mathrm{const}$, $\mu > 0$. $\lambda \geq -\mu$ are Lamé coefficients, $\mathbf{u} = (u^1, u^2)^T$ is an unknown displacement vector, and $\mathbf{f} = (f^1, f^2)^T$ is the given vector.

As usual, by the symbol $W_2^s(\Omega)$, $s \geq 0$, we denote the Sobolev space. For integer s the norm in $W_2^s(\Omega)$ is given by the formula

$$
\|u\|^2_{W_2^s(\Omega)} = \sum_{j=0}^{s} |u|^2_{W_2^j(\Omega)}, \qquad |u|^2_{W_2^j(\Omega)} = \sum_{|\nu|=j} \|D^\nu u\|^2_{L_2(\Omega)},
$$

where $D^\nu = \partial^{|\nu|}/(\partial x_1^{\nu_1}\partial x_2^{\nu_2})$, and $\nu = (\nu_1, \nu_2)$ is a multi-index with non-negative integer components, $|\nu| = \nu_1 + \nu_2$.

If $s = \bar{s} + \varepsilon$, where $\bar{s}$ is an integer part of s and $0 < \varepsilon < 1$, then

$$
\|u\|^2_{W_2^s(\Omega)} = \|u\|^2_{W_2^{\bar{s}}(\Omega)} + |u|^2_{W_2^s(\Omega)},
$$

where

$$
|u|_{W_2^s(\Omega)} = \sum_{|\nu|=\bar{s}} \int_\Omega \int_\Omega \frac{|D^\nu u(x) - D^\nu u(y)|^2}{|x - y|^{2+2\varepsilon}}\, dx\, dy.
$$

Particularly, for $s = 0$ we have $W_2^0 = L_2$.

We assume that solution of the problem (2), (3) belongs to $W_2^m(\Omega)$, $m \geq 2$.

Let us introduce the mesh domains

$$\bar{\omega}_\alpha = \{x_\alpha = i_\alpha h_\alpha : i_\alpha = 0,1,\dots,n_\alpha, n_\alpha h_\alpha = l_\alpha, n_\alpha \geq 4\},$$

$$\omega_\alpha = \bar{\omega}_\alpha \setminus \{0, l_\alpha\}, \qquad \omega_\alpha^+ = \omega_\alpha \cup \{l_\alpha\},$$

$$\gamma_{-\alpha} = \{x = (x_1,x_2) : x_\alpha = 0, x_\beta \in \omega_\beta\}, \qquad \gamma_{+\alpha} = \{x = (x_1,x_2) : x_\alpha = l_\alpha, x_\beta \in \omega_\beta\},$$

$$\gamma_\beta = \gamma \setminus (\gamma_{-\alpha} \cup \gamma_{+\alpha}), \qquad \omega = \omega_1 \times \omega_2, \qquad \bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2,$$

$$\omega^+ = \omega_1^+ \times \omega_2^+, \qquad \gamma = \bar{\omega} \setminus \omega,$$

$$\bar{\gamma}_{\pm\alpha} = \{x = (x_1,x_2) \in \gamma : x_\alpha = (l_\alpha \pm l_\alpha)/2\}, \qquad \omega_{(1)} = \omega_1 \times \bar{\omega}_2, \qquad \omega_{(2)} = \bar{\omega}_1 \times \omega_2,$$

$|h|^2 = h_1^2 + h_2^2$, $\hbar_\alpha = h_\alpha$ when $x_\alpha \in \omega_\alpha$, $\hbar_\alpha = h_\alpha/2$ when $x_\alpha = 0, l_\alpha$.

Further, we define difference quotients in the x_α direction as follows:

$$V_{x_\alpha} = \big(V^{(+1_\alpha)} - V\big)/h_\alpha, \qquad V_{\bar{x}_\alpha} = \big(V - V^{(-1_\alpha)}\big)/h_\alpha, \qquad V_{\overset{\circ}{x}_\alpha} = (V_{x_\alpha} + V_{\bar{x}_\alpha})/2,$$

where

$$V^{(\pm 1_1)}(x) = V(x_1 \pm h_1, x_2), \qquad V^{(\pm 1_2)}(x) = V(x_1, x_2 \pm h_2).$$

The notations $V^{(\pm 0.5_\alpha)}$ will have a similar meaning. Let

$$I_\alpha V(x) := \frac{V(x) + V^{(-1_\alpha)}(x)}{2}.$$

Denote

$$\Lambda_{\alpha\alpha} V := \begin{cases} \frac{2}{h_\alpha} V_{x_\alpha}, & x \in \gamma_{-\alpha}, \\ V_{\bar{x}_\alpha x_\alpha}, & x \in \bar{\omega} \setminus \gamma_\alpha, \\ -\frac{2}{h_\alpha} V_{\bar{x}_\alpha}, & x \in \gamma_{+\alpha}, \end{cases} \qquad \Lambda_{12} V := \begin{cases} V_{x_\alpha \overset{\circ}{x}_\beta}, & x \in \gamma_{-\alpha}, \\ V_{\overset{\circ}{x}_1 \overset{\circ}{x}_2}, & x \in \omega, \\ V_{\bar{x}_\alpha \overset{\circ}{x}_\beta}, & x \in \gamma_{+\alpha}. \end{cases}$$

On the set of mesh functions we define scalar products and norms:

$$(Y,V)_{\tilde{\omega}} = \sum_{x \in \tilde{\omega}} h_1 h_2 Y(x) V(x), \qquad \|Y\|_{\tilde{\omega}} = (Y,Y)_{\tilde{\omega}}^{1/2}, \quad \tilde{\omega} \subseteq \bar{\omega},$$

$$(Y,V)_{(\alpha)} = \sum_{x \in \omega_{(\alpha)}} h_\alpha \hbar_\beta Y(x) V(x), \qquad \|Y\|_{(\alpha)} = (Y,Y)_{(\alpha)}^{1/2}.$$

By $\overset{\circ}{H}_h(\omega)$ we denote a set of two-dimensional vector-functions $\mathbf{V} = (V^1, V^2)^T$, whose components are defined on $\bar{\omega}$ and equal to zero on γ_α, respectively. Let H_h be the set of two-dimensional vector-functions, whose components are defined on the meshes $\omega_{(\alpha)}$, respectively.

Define in $\overset{\circ}{H}_h(\omega)$ the inner product and the norms:

$$(\mathbf{U},\mathbf{V}) = \big(U^1, V^1\big)_{(1)} + \big(U^2, V^2\big)_{(2)}, \qquad \|\mathbf{U}\| = (\mathbf{U},\mathbf{U})^{1/2},$$

$$\|\mathbf{U}\|_{W_2^2(\omega)}^2 := \sum_{\alpha=1}^{2} \big(\|\Lambda_{11} U^\alpha\|_{(\alpha)}^2 + \|\Lambda_{22} U^\alpha\|_{(\alpha)}^2 + 2\|U^\alpha_{\bar{x}_1 \bar{x}_2}\|_{\omega^+}^2\big).$$

For functions, defined on Ω, we need the following averaging operators:

$$S_1^- u(x) := \frac{1}{h_1}\int_{x_1-h_1}^{x_1} u(t_1,x_2)\,dt_1, \qquad T_1 u(x) := \frac{1}{h_1^2}\int_{x_1-h_1}^{x_1+h_1} \bigl(h_1 - |x_1-t_1|\bigr)u(t_1,x_2)\,dt_1,$$
$$T_1 u(0,x_2) := \frac{2}{h_1^2}\int_0^{h_1} (h_1 - t_1)u(t_1,x_2)\,dt_1,$$
$$T_1 u(l_1,x_2) := \frac{2}{h_1^2}\int_{l_1-h_1}^{l_1} (h_1 - l_1 + t_1)u(t_1,x_2)\,dt_1.$$

Analogously are defined operators T_2, S_2^-. Note that these operators are commutative and

$$T_\alpha \frac{\partial^2 u}{\partial x_\alpha^2} = u_{\bar{x}_\alpha x_\alpha}, \qquad T_\alpha \frac{\partial u}{\partial x_\alpha} = \bigl(S_\alpha^- u\bigr)_{x_\alpha}.$$

3 Difference scheme, correction procedure, and main result

At the *first stage*, we approximate problem (2), (3) by the difference scheme

$$\begin{aligned}
&(\lambda+2\mu)\Lambda_{11}U^1 + (\lambda+\mu)\Lambda_{12}U^2 + \mu\Lambda_{22}U^1 + F^1 = 0, \quad x\in\omega_{(1)},\\
&\mu\Lambda_{11}U^2 + (\lambda+\mu)\Lambda_{12}U^1 + (\lambda+2\mu)\Lambda_{22}U^2 + F^2 = 0, \quad x\in\omega_{(2)},\\
&U^\alpha(x) = 0, \quad x\in\gamma_\alpha, \qquad F^\alpha(x) = T_1T_2 f^\alpha.
\end{aligned} \tag{4}$$

Let us rewrite the difference scheme (4) in the operator form

$$\mathcal{L}_h\mathbf{U} = \mathbf{F}, \qquad \mathbf{U}\in \mathring{H}_h, \qquad \mathbf{F}\in H_h, \tag{5}$$

where

$$\mathcal{L}_h = -\begin{pmatrix} (\lambda+2\mu)\Lambda_{11} + \mu\Lambda_{22} & (\lambda+\mu)\Lambda_{12} \\ (\lambda+\mu)\Lambda_{12} & \mu\Lambda_{11} + (\lambda+2\mu)\Lambda_{22} \end{pmatrix},$$
$$\mathbf{U} = \bigl(U^1,U^2\bigr)^T, \qquad \mathbf{F} = \bigl(F^1,F^2\bigr)^T.$$

Lemma 1 [13] *The operator $\mathcal{L}_h : \mathring{H}_h \to H_h$ is self-adjoint, positive definite, and the estimate*

$$\|\mathcal{L}_h V\| \geq \mu\|V\|_{W_2^2(\omega)} \tag{6}$$

is valid.

Due to the positive definiteness of $\mathcal{L}_h$ the solution of equation (5) (or the difference scheme (4)) exists and is unique. The rate of convergence of the scheme (4) is determined by the estimate [13]

$$\|\mathbf{U}-\mathbf{u}\|_{W_2^2(\omega)} \leq c|h|^{m-2}\|\mathbf{u}\|_{W_2^m(\Omega)}, \quad 2\leq m\leq 4. \tag{7}$$

To construct and study a corrector, we use the operators

$$\mathcal{P}_\alpha V := \begin{cases} I_\beta(V_{\bar{x}_\alpha \mathring{x}_\alpha}), & x_\alpha \in \omega_\alpha \setminus \{h_\alpha\}, \\ I_\beta(V_{\bar{x}_\alpha x_\alpha} - \frac{h_\alpha}{2} V_{\bar{x}_\alpha x_\alpha x_\alpha}), & x_\alpha = h_\alpha, x_\beta \in \omega_\beta^+, \\ I_\beta(V_{\bar{x}_\alpha \bar{x}_\alpha} + \frac{h_\alpha}{2} V_{\bar{x}_\alpha \bar{x}_\alpha \bar{x}_\alpha}), & x_\alpha = l_\alpha, \end{cases}$$

$$\mathcal{B}_{12} V := \begin{cases} \frac{2}{h_\alpha} V_{x_\beta}^{(+1_\alpha)}, & x \in \gamma_{-\alpha}, \\ V_{x_1 x_2}, & x \in \omega, \\ -\frac{2}{h_\alpha} V_{x_\beta}, & x \in \gamma_{+\alpha}, \end{cases}$$

$$\mathcal{D}_\alpha V := \begin{cases} V_{\bar{x}_\alpha x_\alpha}, & x \in \omega_{(\alpha)}, \\ V_{x_\alpha x_\alpha} - \frac{4h_\alpha}{5} V_{x_\alpha x_\alpha x_\alpha}, & x \in \bar{\gamma}_{-\alpha}, \\ V_{\bar{x}_\alpha \bar{x}_\alpha} + \frac{4h_\alpha}{5} V_{\bar{x}_\alpha \bar{x}_\alpha \bar{x}_\alpha}, & x \in \bar{\gamma}_{+\alpha}. \end{cases}$$

At the *second stage*, we use the earlier-found solution of the difference scheme (4) for defining the correction term $\mathcal{R}U$ and on the same grid solve the difference scheme

$$\mathcal{L}_h \tilde{\mathbf{U}} = \mathbf{F} + \mathcal{R}\mathbf{U}, \qquad \tilde{\mathbf{U}} \in \mathring{H}_h, \qquad \mathbf{F} \in H_h, \tag{8}$$

where

$$\mathcal{R} := \begin{pmatrix} (\lambda + 2\mu)\frac{h_2^2}{12}\Lambda_{11}\mathcal{D}_2 + \mu\frac{h_2^2}{12}\Lambda_{22}\mathcal{D}_1 & -(\lambda + \mu)(\frac{h_1^2}{12}\mathcal{B}_{12}\mathcal{P}_1 + \frac{h_2^2}{12}\mathcal{B}_{12}\mathcal{P}_2) \\ -(\lambda + \mu)(\frac{h_1^2}{12}\mathcal{B}_{12}\mathcal{P}_1 + \frac{h_2^2}{12}\mathcal{B}_{12}\mathcal{P}_2) & (\lambda + 2\mu)\frac{h^2}{12}\Lambda_{22}\mathcal{D}_1 + \mu\frac{h^2}{12}\Lambda_{11}\mathcal{D}_2 \end{pmatrix}.$$

The following theorem is the main result of the present paper.

Theorem 1 *Let the solution of problem* (2), (3) *belong to the space* $W_2^m(\Omega)$, $m \geq 2$. *Then the convergence rate of the corrected difference scheme* (7) *in the discrete* L_2*-norm is given by the estimate*

$$\|\tilde{\mathbf{U}} - \mathbf{u}\|_{L_2(\omega)} \leq c|h|^m \|\mathbf{u}\|_{W_2^m(\Omega)}, \quad 2 \leq m \leq 4. \tag{9}$$

4 *A priori* estimate for the error of the corrected solution

Let

$$\zeta_{\alpha\alpha}^\alpha := T_\beta u^\alpha - u^\alpha - \frac{h_\beta^2}{12}\mathcal{D}_\beta u^\alpha, \quad x \in \bar{\omega}, \tag{10}$$

$$\zeta_{\alpha\alpha}^\beta := T_\beta u^\beta - u^\beta - \frac{h_\beta^2}{12}\mathcal{D}_\beta u^\beta, \quad x \in \omega_{(\beta)}, \tag{11}$$

$$\zeta_{12}^\alpha := S_1 S_2 u^\alpha - I_1 I_2 u^\alpha + \frac{h_1^2}{12}\mathcal{P}_1 u^\alpha + \frac{h_2^2}{12}\mathcal{P}_2 u^\alpha, \quad x \in \omega^+. \tag{12}$$

By $\tilde{\mathbf{Z}} = \mathbf{u} - \tilde{\mathbf{U}}$ we denote the error in the solution of the corrected difference scheme (8).

Lemma 2 *The error* $\tilde{\mathbf{Z}}$ *is a solution of the problem*

$$\mathcal{L}_h \tilde{\mathbf{Z}} = \mathcal{R}\mathbf{Z} + \mathbf{\Psi}, \tag{13}$$

where $\mathbf{Z} = \mathbf{u} - \mathbf{U}$ *is the error in the solution of the scheme* (4) *and*

$$\boldsymbol{\Psi} = (\lambda + 2\mu)\begin{pmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{pmatrix}\begin{pmatrix} \zeta_{11}^1 \\ \zeta_{22}^2 \end{pmatrix} + \mu\begin{pmatrix} \Lambda_{22} & 0 \\ 0 & \Lambda_{11} \end{pmatrix}\begin{pmatrix} \zeta_{22}^1 \\ \zeta_{11}^2 \end{pmatrix} + (\lambda + \mu)\begin{pmatrix} 0 & \mathcal{B}_{12} \\ \mathcal{B}_{12} & 0 \end{pmatrix}\begin{pmatrix} \zeta_{12}^1 \\ \zeta_{12}^2 \end{pmatrix}.$$

Proof From equalities (10), (11) we have

$$T_1T_2\left(\frac{\partial^2 u^\alpha}{\partial x_\alpha^2}\right) = \Lambda_{\alpha\alpha}u^\alpha + \frac{h_\beta^2}{12}\Lambda_{\alpha\alpha}\left(\mathcal{D}_\beta u^\alpha\right) + \Lambda_{\alpha\alpha}\zeta_{\alpha\alpha}^\alpha, \quad x \in \omega_{(\alpha)}, \tag{14}$$

$$T_1T_2\left(\frac{\partial^2 u^\beta}{\partial x_\alpha^2}\right) = \Lambda_{\alpha\alpha}u^\beta + \frac{h_\beta^2}{12}\Lambda_{\alpha\alpha}\left(\mathcal{D}_\beta u^\beta\right) + \Lambda_{\alpha\alpha}\zeta_{\alpha\alpha}^\beta, \quad x \in \omega_{(\beta)}. \tag{15}$$

It is not hard to verify that

$$T_1T_2\left(\frac{\partial^2 u^\alpha}{\partial x_1\,\partial x_2}\right) = B_{12}\left(S_1^- S_2^- u^\alpha\right), \quad x \in \omega_{(\beta)}. \tag{16}$$

Besides,

$$B_{12}\left(I_1 I_2 u^\alpha\right) = \Lambda_{12}u^\alpha. \tag{17}$$

Indeed, in the case $x_2 = 0$ we have

$$\begin{aligned} B_{12}\left(I_1I_2u^2\right) &= \frac{2}{h_2}\left(I_1I_2u^2\right)_{x_1}^{(+1_2)} = \frac{2}{h_2}\left(I_2u^2\right)_{\mathring{x}_1}^{(+1_2)} = \frac{2}{h_2}\left(\frac{(u^2)^{(+1_2)} + u^2}{2}\right)_{\mathring{x}_1} \\ &= \frac{2}{h_2}\left(\frac{(u^2)^{(+1_2)} - u^2}{2}\right)_{\mathring{x}_1} = u^2_{\mathring{x}_1 x_2} = \Lambda_{12}u^2. \end{aligned}$$

It is easy to verify also the other cases of (17). As a result from (12) we obtain

$$T_1T_2\left(\frac{\partial^2 u^\alpha}{\partial x_1\,\partial x_2}\right) = \Lambda_{12}u^\alpha - \frac{h_1^2}{12}B_{12}\mathcal{P}_1 u^\alpha - \frac{h_2^2}{12}B_{12}\mathcal{P}_2 u^\alpha + B_{12}\zeta_{12}^\alpha, \quad x \in \omega_{(\beta)}. \tag{18}$$

Let us apply the operator T_1T_2 on the equations of system (2) on the grid points $\omega_{(1)}$, $\omega_{(2)}$, respectively, and then use equations (14), (15), (18). The obtained results may be compactly written as follows:

$$\mathcal{L}_h\mathbf{u} = \mathbf{F} + \mathcal{R}\mathbf{u} + \boldsymbol{\Psi}. \tag{19}$$

Subtracting (8) from equality (19) we obtain the required equality (13). □

Theorem 2 *For the solution of problem* (13) *the following a priori estimate is valid*:

$$\|\tilde{\mathbf{Z}}\|_{L_2(\omega)} \le c\left(|h|^2\|\mathbf{Z}\|_{W_2^2(\omega)} + \sum_{\alpha=1}^{2}\left(\left\|\zeta_{11}^\alpha\right\|_{(\alpha)} + \left\|\zeta_{22}^\alpha\right\|_{(\alpha)} + \left\|\zeta_{12}^\alpha\right\|_{\omega^+}\right)\right). \tag{20}$$

Proof First, note that based on the inequalities

$$\left\| \mathcal{L}_h^{-1} \begin{pmatrix} \Lambda_{\alpha\alpha} & 0 \\ 0 & \lambda_{\beta\beta} \end{pmatrix} \right\| \le \frac{1}{\mu}, \qquad \left\| \mathcal{L}_h^{-1} \begin{pmatrix} 0 & \mathcal{B}_{12} \\ \mathcal{B}_{12} & 0 \end{pmatrix} \right\| \le \frac{1}{\mu},$$

implied by (6), we have

$$\left\| \mathcal{L}_h^{-1} \boldsymbol{\Psi} \right\| \le c \sum_{\alpha=1}^{2} \left(\left\| \zeta_{11}^{\alpha} \right\|_{(\alpha)} + \left\| \zeta_{22}^{\alpha} \right\|_{(\alpha)} + \left\| \zeta_{12}^{\alpha} \right\|_{\omega^+} \right). \tag{21}$$

Further, the operator $\mathcal{L}_h$ is positive definite and self-adjoint, therefore,

$$\left\| \mathcal{L}_h^{-1} \mathcal{R} \mathbf{Z} \right\| = \sup_{\|\mathbf{V}\| \ne 0} \frac{|(\mathcal{R}\mathbf{Z}, \mathbf{V})|}{\|\mathcal{L}_h \mathbf{V}\|}. \tag{22}$$

Using the equalities obtained by partial summation,

$$\begin{cases} (\Lambda_{\alpha\alpha} \mathcal{D}_\beta Z^\alpha, V^\alpha)_{(\alpha)} = (\mathcal{D}_\beta Z^\alpha, \Lambda_{\alpha\alpha} V^\alpha)_{(\alpha)}, & \text{since } Z^\alpha, V^\alpha = 0 \text{ for } x \in \gamma_\alpha, \\ (\Lambda_{\beta\beta} \mathcal{D}_\alpha Z^\alpha, V^\alpha)_{(\alpha)} = (\mathcal{D}_\alpha Z^\alpha, \Lambda_{\beta\beta} V^\alpha)_{(\alpha)}, & \text{by definition of } \Lambda_{\beta\beta}, \\ (\mathcal{B}_{12} \mathcal{P}_\alpha Z^\beta, V^\alpha)_{(\alpha)} = (\mathcal{P}_\alpha Z^\beta, V^\alpha_{\bar{x}_1\bar{x}_2})_{\omega^+}, & \text{since } V^\alpha = 0 \text{ for } x \in \gamma_\alpha, \\ (\mathcal{B}_{12} \mathcal{P}_\beta Z^\beta, V^\alpha)_{(\alpha)} = (\mathcal{P}_\beta Z^\beta, V^\alpha_{\bar{x}_1\bar{x}_2})_{\omega^+}, & \text{since } V^\alpha = 0 \text{ for } x \in \gamma_\alpha \end{cases}$$

and after considering the Cauchy-Schwarz inequality, for an estimate of the numerator of (22) we have

$$\left| (\mathcal{R}\mathbf{Z}, \mathbf{V}) \right| \le c|h|^2 \|\mathbf{Z}\| \|\mathbf{V}\|_{W_2^2(\omega)}. \tag{23}$$

Taking into account (8), (22) from (21) we obtain

$$\left\| \mathcal{L}_h^{-1} \mathcal{R} \mathbf{Z} \right\| \le c|h|^2 \|\mathbf{Z}\|_{W_2^2(\omega)}. \tag{24}$$

Inequalities (21) and (24) together with

$$\|\tilde{\mathbf{Z}}\| \le \left\| \mathcal{L}_h^{-1} \mathcal{R} \mathbf{Z} \right\| + \left\| \mathcal{L}_h^{-1} \boldsymbol{\Psi} \right\|$$

complete the proof of Theorem 2. □

5 Proof of Theorem 1

To determine the rate of convergence of the corrected finite difference scheme (8) with the help of Theorem 2, it is sufficient to estimate the norms of ζ_{11}^{α}, ζ_{22}^{α}, ζ_{12}^{α} on the right-hand part of (20). To this aim, we need the next lemma.

Lemma 3 *Let the linear functional $l(u)$ be bounded in $W_2^k(E)$, where $k = \bar{k} + \epsilon$, $\bar{k}$ is an integer number, $0 < \epsilon \le 1$. If $l(u)$ equals zero on polynomials of two variables with order equal to $\bar{k}$, then there exists a constant c, dependent on E, but not dependent on $u(x)$, such that $|l(u)| \le c|u|_{W_2^k(E)}$.*

This lemma is a particular case of the Dupont-Scott approximation theorem [15], and it represents a generalization of the Bramble-Hilbert lemma [16] (see, *e.g.*, [8]).

Lemma 4 *Let* $\mathbf{u} \in W_2^m(\Omega)$, $m \in [2,4]$. *Then for the expressions defined by equations* (10), (11), (12) *the following inequalities hold*:

$$\|\zeta_{\alpha\alpha}^{\alpha}\|_{(\alpha)} \le c|h|^m \|u^\alpha\|_{W_2^m(\Omega)}, \tag{25}$$

$$\|\zeta_{\alpha\alpha}^{\beta}\|_{(\beta)} \le c|h|^m \|u^\beta\|_{W_2^m(\Omega)}, \tag{26}$$

$$\|\zeta_{12}^{\alpha}\|_{\omega^+} \le c|h|^m \|u^\alpha\|_{W_2^m(\Omega)}. \tag{27}$$

Proof Let us denote by π_3 the set of third degree polynomials. In the cases $x \in \gamma_{\pm\beta}$, taking into account boundary condition, let us represent the values $\zeta_{\alpha\alpha}^{\alpha}$ in the form

$$\zeta_{\alpha\alpha}^{\alpha} = \zeta_{\alpha\alpha}^{\alpha} \pm \frac{h_\beta}{3}\frac{\partial u^\alpha}{\partial x_\beta}.$$

Particularly,

$$\zeta_{11}^1 = T_2u^1 - u^1 - \frac{h_2}{3}\frac{\partial u^1}{\partial x_2} - \frac{h_2^2}{12}u^1_{x_2x_2} + \frac{h_2^3}{15}u^1_{x_2x_2x_2}, \quad x \in \gamma_{-2}.$$

Let $e = \{\xi = (\xi_1,\xi_2) : |\xi_1 - x_1| \le h_1, 0 \le \xi_2 \le 3h_2\}$. By $\bar{u}(t)$ we denote a function obtained from $u^1(\xi)$ by changing the variables $\xi_\alpha + t_\alpha h_\alpha$, and mapping the domain e onto $E = \{t = (t_1,t_2) : |t_1| \le 1, 0 \le t_2 \le 3\}$. Since $u^1(\xi) = u^1(x_1 + t_1h_1, x_2 + t_2h_2) = \bar{u}(t)$, the expression ζ_{11}^1 turns into

$$\begin{aligned}\zeta_{11}^1 = {}& 2\int_0^1 (1-t_2)\bar{u}(0,t_2)\,dt_2 - \bar{u}(0,0) - \frac{1}{3}\frac{\partial\bar{u}(0,0)}{\partial x_2} \\ & - \frac{1}{12}\big(\bar{u}(0,2) - 2\bar{u}(0,1) + \bar{u}(0,0)\big) + \frac{1}{15}\big(\bar{u}(0,3) - 3\bar{u}(0,2) + 3\bar{u}(0,1) - \bar{u}(0,0)\big).\end{aligned}$$

Consequently, $|\zeta_{11}^1| \le c\|\bar{u}\|_{C^1(E)} \le c\|\bar{u}\|_{W_2^m(E)}$ as $W_2^m \subset C^1$ when $m > 2$. Utilizing the fact that ζ_{11}^1, as a functional of $\bar{u}$, vanishes on π_3 (which can be verified directly) and using Lemma 3, we obtain $|\zeta_{11}^1| \le c|\bar{u}|_{W_2^m(E)}$, $m \in (2,4]$. Reverting to the old variables, this yields

$$|\zeta_{11}^1| \le c|h|^m(h_1h_2)^{-1/2}|u^1|_{W_2^m(e)}, \quad m \in (2,4]. \tag{28}$$

In the case $x \in \gamma_{+2}$ the estimate can be obtained analogously.

When $x \in \omega$, we have $\zeta_{11}^1 = T_2u^1 - u^1 - (h_2^2/12)u^1_{\bar{x}_2x_2}$. In this case the elementary domain $e = \{\xi = (\xi_1,\xi_2) : |\xi_\alpha - x_\alpha| \le h_\alpha\}$, and we again obtain the estimate of the form (27) (see, *e.g.*, [17]). Consequently, we have

$$\|\zeta_{11}^1\|_{(1)}^2 = \sum_{\omega_{(1)}} h_1\hbar_2|\zeta_{11}^1|^2 \le c|h|^{2m}\sum_{\omega_{(1)}}|u^1|^2_{W_2^m(e)} \le c|h|^{2m}|u^1|^2_{W_2^m(\Omega)}.$$

The case $\alpha = 2$ in (25) can be proved in the same way.

The proof of the inequality (26) is also clear, so let us obtain the inequality (27).

Using the Taylor expansion we can show that

$$S_1^- S_2^- u - I_1 I_2 u = -\left(\frac{h_1^2}{12}\frac{\partial^2 u}{\partial x_1^2} + \frac{h_2^2}{12}\frac{\partial^2 u}{\partial x_2^2}\right)^{(-0.5_1,-0.5_2)} \quad \text{for } u \in \pi_3. \tag{29}$$

On the other hand, each of the following expressions:

$$u_{\bar{x}_\alpha \mathring{x}_\alpha}, \qquad u_{\bar{x}_\alpha x_\alpha} - \frac{h_\alpha}{2} u_{\bar{x}_\alpha x_\alpha x_\alpha}, \quad \text{and} \quad u_{\bar{x}_\alpha \bar{x}_\alpha} + \frac{h_\alpha}{2} u_{\bar{x}_\alpha \bar{x}_\alpha \bar{x}_\alpha}$$

equals $\frac{\partial^2 u^{(-0.5_\alpha)}}{\partial x_\alpha^2}$ for $u \in \pi_3$. Therefore,

$$\mathcal{P}_\alpha u = I_\beta \frac{\partial^2 u^{(-0.5_\alpha)}}{\partial x_\alpha^2} = \frac{\partial^2 u^{(-0.5_1,-0.5_2)}}{\partial x_\alpha^2} \quad \text{when } u \in \pi_3. \tag{30}$$

From (29), (30) it is easy to see that the expressions ζ_{11}^α, as linear functionals with respect to u^α, turn into zero on the polynomials of third order and are bounded when $u^\alpha \in W_2^m(\Omega)$, $m \geq 2$. Therefore, for them we have again the estimates similar to (26), whence follows the validity of (27).

Lemma 4 is proved. □

In view of (20) and the inequality (7), taking into account the Lemma 4, the validity of Theorem 1 is obvious.

6 Numerical experiments

Now, we present some numerical results to demonstrate the convergence order of the proposed method. The experimental order of convergence in the discrete L_2-norm is computed by the formula

$$\operatorname{Ord}(Y) = \log_2 \frac{\|Y_h - u\|}{\|Y_{h/2} - u\|},$$

where u is the exact solution of original problem, while Y_h denotes the solution of the difference scheme on the grid with step h.

Below, the symbols U, $\tilde{U}$ denote solutions of the difference schemes (5), (8), respectively.

Example 1 Consider the problem (2), (3), where $l_1, l_2 = 1$, $\lambda = 5$, $\mu = 1$, and

$$f^1(x) = 4c_1 \sin(\pi x_1)\cos(\pi x_2), \qquad f^2(x) = 4c_1 \cos(\pi x_1)\sin(\pi x_2), \qquad c_1 = \pi^2(2\lambda + 4\mu).$$

For calculation of the right-hand side of the difference scheme note that

$$T_\alpha\bigl(\sin(\pi x_\alpha)\bigr) = c_2^2 \sin(\pi x_\alpha), \qquad T_\alpha\bigl(\cos(\pi x_\alpha)\bigr) = c_2^2 \cos(\pi x_\alpha), \qquad c_2 = \frac{2}{\pi h}\sin\frac{\pi h}{2}.$$

Therefore,

$$F^\alpha = T_1 T_2 f^\alpha = c_2^4 f^\alpha, \quad \alpha = 1, 2.$$

Table 1 Experimental order of convergence

h	$\|U_h - u\|$	$\|\tilde{U}_h - u\|$	Ord(U)	Ord($\tilde{U}$)
1/4	3.743333e–02	1.865509e–02	1.8843	4.1817
1/8	1.013944e–02	1.027991e–03	1.9739	3.9656
1/16	2.581113e–03	6.579804e–05	1.9936	4.1623
1/32	6.481292e–04	3.674768e–06	1.9984	3.9866
1/64	1.622098e–04	2.318145e–07		

To solve the difference schemes $\mathcal{L}_h Y = \varphi$ we use the iterative method of minimal residuals

$$\frac{Y^{(k+1)} - Y^{(k)}}{\tau_k} + \mathcal{L}_h Y^{(k)} = \varphi, \quad k = 0, 1, 2, \ldots, \forall Y^{(0)} \in \mathring{H}_h,$$

where

$$\tau_k = \frac{(\mathcal{L}_h \operatorname{Res}^{(k)}, \operatorname{Res}^{(k)})}{(\mathcal{L}_h \operatorname{Res}^{(k)}, \mathcal{L}_h \operatorname{Res}^{(k)})}, \qquad \operatorname{Res}^{(k)} = \mathcal{L}_h Y^{(k)} - \varphi.$$

It is clear that the exact solution

$$u^1(x) = 4\sin(\pi x_1)\cos(\pi x_2), \qquad u^2(x) = 4\cos(\pi x_1)\sin(\pi x_2)$$

belongs to $W_2^4(\Omega)$, therefore, for the refined scheme the fourth order of convergence is expected.

The results of the calculations are given by Table 1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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