Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 170 (2016) 287–296
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

Method of corrections by higher order differences for Poisson equation with nonlocal boundary conditions

Givi Berikelashvili[a,b,*], Bidzina Midodashvili[c,d]

[a] A. Razmadze Mathematical Institute, I. Javakhishvili Tbilisi State University 6, Tamarashvili st., Tbilisi 0177, Georgia
[b] Department of Mathematics, Georgian Technical University, 77, M. Kostava Str., 0175 Tbilisi, Georgia
[c] Faculty of Exact and Natural Sciences, I. Javakhishvili Tbilisi State University, 2, University Str., Tbilisi 0186, Georgia
[d] Faculty of Education, Exact and Natural Sciences, Gori Teaching University, 5, I. Chavchavadze Str., Gori, Georgia

Available online 26 April 2016

Abstract

We consider the Bitsadze–Samarskii type nonlocal boundary value problem for Poisson equation in a unit square, which is solved by a difference scheme of second-order accuracy. Using this approximate solution, we correct the right-hand side of the difference scheme. It is shown that the solution of the corrected scheme converges at the rate $O(|h|^s)$ in the discrete L_2-norm provided that the solution of the original problem belongs to the Sobolev space with exponent $s \in [2, 4]$.

Keywords: Nonlocal BVP; Difference scheme; Method of corrections; Improvement of accuracy; Compatible estimates of convergence rate

1. Introduction

Finite difference method is a significant tool in the numerical solution of problems posed for differential equations. In order to minimize the amount of calculations it is desirable for the difference scheme to be sufficiently good on coarse meshes, i.e. to have high order accuracy. In the present work, for improving the accuracy of the approximate solution, we study two-stage finite difference method. We consider Bitsadze–Samarskii type nonlocal boundary value problem for Poisson's equation.

At the first stage we solve the difference scheme $\Delta_h \tilde{U} = \varphi$, which has the second order of approximation. Using the solution $\tilde{U}$ the right-hand side of the difference scheme is corrected, $\Delta_h U = \varphi + R\tilde{U}$, and solved again on the same mesh.

This approach for some boundary value problems posed for Poisson and Laplace equations has been studied in Volkov's papers (see, e.g. [1–3]), where the input data were chosen so as to ensure that the exact solution belongs to the Hölder class $C_{6,\lambda}(\bar{\Omega})$.

* Corresponding author at: A. Razmadze Mathematical Institute, I. Javakhishvili Tbilisi State University 6, Tamarashvili st., Tbilisi 0177, Georgia.
E-mail addresses: bergi@rmi.ge (G. Berikelashvili), bidmid@hotmail.com (B. Midodashvili).
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2016.04.002
2346-8092/

For establishing the convergence we use the methodology of obtaining the compatible estimates of convergence rate of difference schemes. This methodology develops from the works of Samarskii, Lazarov and Makarov (see, e.g., [4–6]), and later in the works of other authors (see, e.g., [7,8]). For the elliptic problems such estimates have the form

$$\|U-u\|_{W_2^k(\omega)} \le c|h|^{s-k}\|u\|_{W_2^s(\Omega)}, \quad s>k\ge 0,$$

where u is the solution of original problem, U is the approximate solution, k and s are integer and real numbers, respectively, $W_2^k(\omega)$ and $W_2^s(\Omega)$ are the Sobolev norms on the set of functions with discrete and continuous arguments. Here and below c denotes a positive generic constant, independent of h and u.

It is proved that the solution U of the corrected difference scheme converges at rate $O(h^s)$ in the discrete L_2-norm, when the exact solution belongs to the Sobolev space $W_2^s,\ s\in[2,4]$.

The generalization of the Bitsadze–Samarskii problem [9] was investigated by many authors (see, e.g., [10–13]).

In [11] for a Poisson equation it is considered a difference scheme, which converges by the rate $O(h^2)$ in the discrete W_2^2-norm to the exact solution from the class $C^4(\bar{\Omega})$.

In [13] difference scheme is considered for a second order elliptic equation with variable coefficients and the compatible estimate of convergence rate in discrete W_2^1-norm is obtained.

Results, analogous to those given in the present work, are obtained in [14] for the Dirichlet problem posed for an elliptic equation, and also in [15] for the mixed problem with third kind conditions.

One of the methods for obtaining compact high order approximations is the Mehrstellen method ("Mehrstellenverfahren"), defined by Collatz (see [16]). Instead of approximating only the left hand side of the differential equation, he proposes to take several points of the right hand side as well. In the case of two-dimensional problem, the differential operator is approximated on a 9-point stencil with the fourth order accuracy.

The advantage of the Mehrstellen schemes over ordinary (second order) accuracy schemes on a coarse grid is obvious.

The advantage of our method is:

(a) It needs to approximate the differential operator on minimally acceptable stencil (5-point stencil for a two-dimensional problem). Therefore, the condition number of this operator is better as compared with the Mehrstellen schemes, which is notable on a fine grid.

(b) It is a two-stage method, nevertheless it requires matrix inversion only once (on the second stage we change only the right-hand side of the equation, while the operator is kept unchanged).

(c) The method of correction is handy even in the case when construction of high precision schemes is impossible.

2. Statement of the problem and some auxiliary estimate

As usual, by symbol $W_2^s(\Omega),\ s\ge 0$ we denote the Sobolev space. For integer s the norm in $W_2^s(\Omega)$ is given by formula

$$\|u\|^2_{W_2^s(\Omega)} = \sum_{j=0}^{s}|u|^2_{W_2^j(\Omega)}, \qquad |u|^2_{W_2^j(\Omega)} = \sum_{|\nu|=j}\|D^\nu u\|^2_{L_2(\Omega)},$$

where $D^\nu = \partial^{|\nu|}/\left(\partial x_1^{\nu_1}\partial x_2^{\nu_2}\right)$, $\nu=(\nu_1,\nu_2)$ is multi-index with non-negative integer components, $|\nu|=\nu_1+\nu_2$.

If $s=\bar{s}+\varepsilon$, where $\bar{s}$ is an integer part of s and $0<\varepsilon<1$, then

$$\|u\|^2_{W_2^s(\Omega)} = \|u\|^2_{W_2^{\bar{s}}(\Omega)} + |u|^2_{W_2^s(\Omega)},$$

where

$$|u|_{W_2^s(\Omega)} = \sum_{|\nu|=\bar{s}}\int_\Omega\int_\Omega \frac{|D^\nu u(x)-D^\nu u(y)|^2}{|x-y|^{2+2\varepsilon}}\,dx\,dy.$$

Particularly, for $s=0$ we have $W_2^0=L_2$.

Let $\bar{\Omega} = \{(x_1, x_2): 0 \le x_\alpha \le 1, \ \alpha = 1, 2\}$ be a unit square with a boundary Γ; $\Gamma_0 = \Gamma \setminus \{(1, x_2): 0 < x_2 < 1\}$; ξ_k be fixed points from interval (0; 1), $0 < \xi_1 < \xi_2 < \cdots \xi_m < 1$. Denote $\xi_0 = 0$, $\xi_{m+1} = 1$.

Consider the problem

$$\Delta u = f(x), \quad x \in \Omega, \qquad u\big|_{\Gamma_0} = 0, \qquad u(1, x_2) = \sum_{k=1}^{m} \alpha_k u(\xi_k, x_2), \quad 0 < x_2 < 1 \tag{1}$$

where the coefficients α_k are real numbers satisfying conditions

$$\varkappa := \sum_{k=1}^{m} |\alpha_k| \sqrt{\xi_k} < 1.$$

It was shown in [12] that, for $f(x) \in L_2(\Omega, \rho)$, there exists a unique strong solution of problem (1) in the weighted Sobolev space $W_2^2(\Omega, \rho)$. Throughout the following, we assume that the function $f(x)$ provides the unique solvability of problem (1) in the $W_2^s(\Omega)$, $2 \le s \le 4$.

Consider the following grid domains in $\bar{\Omega}$:

$$\bar{\omega}_k = \{x_k = i_k h : i_k = 0, 1, \ldots, n, \ h = 1/n\}, \qquad \omega_k = \bar{\omega}_k \cap (0, 1),$$
$$\omega_k^+ = \bar{\omega}_\alpha \cap (0, 1], \quad k = 1, 2, \qquad \omega = \omega_1 \times \omega_2, \qquad \bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2, \qquad \gamma_0 = \Gamma_0 \cap \bar{\omega}.$$

We assume that the points ξ_k coincide with grid nodes

$$\xi_k = n_k h, \quad k = 1, 2, \ldots, m,$$

where n_k are nonnegative integers $0 < n_1 < n_2 < \cdots < n_m < n$. We suppose also that

$$h/2 \le 1 - \xi_m - \nu, \quad \nu = \text{const} > 0.$$

For grid functions we define difference quotients in x_k directions as follows

$$V_{x_k} = \big(V^{(+1_k)} - V\big)/h, \qquad V_{\bar{x}_k} = \big(V - V^{(-1_k)}\big)/h$$

where

$$V = V(x), \quad V^{(\pm 1_1)} = V(x_1 \pm h, x_2), \quad V^{(\pm 1_2)} = V(x_1, x_2 \pm h).$$

For functions, defined on Ω, we need the following averaging operators:

$$T_1 u(x) := \frac{1}{h^2} \int_{x_1 - h_1}^{x_1 + h_1} (h_1 - |x_1 - t_1|) u(t_1, x_2)\, dt_1.$$

Analogously is defined operator T_2. Note that these operators commute and

$$T_k \frac{\partial^2 u}{\partial x_k^2} = u_{\bar{x}_k x_k}, \quad k = 1, 2.$$

Define the following weight functions

$$r(x_1) = 1 - x_1, \qquad \rho(x_1) = 1 - x_1 - \sum_{k=1}^{m} \varkappa \sigma_k \chi(\xi_k - x_1),$$

where

$$\sigma_k = \frac{|\alpha_k|}{\sqrt{\xi_k}}, \qquad \chi(t) = \begin{cases} t, & \text{if } t \ge 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Let

$$\bar{r} = \big(r + r^{(-1_1)}\big)/2, \qquad \overline{\rho} = \big(\rho + \rho^{(-1_1)}\big)/2.$$

Notice that the following inequality

$$(1-\varkappa^2)r(x_1) \le \rho(x_1) \le r(x_1) \tag{2}$$

holds.

Indeed, the right-hand side inequality is obvious. The left-hand side inequality can be verified as follows:

$$\begin{aligned}\rho(x_1) &= 1 - x_1 - \varkappa \sum_{k=j+1}^{m} \sigma_k(\xi_k - x_1) \ge \Big(1 - \varkappa \sum_{k=j+1}^{m} \sigma_k \xi_k\Big)(1-x_1)\\ &\ge (1-\varkappa^2)(1-x_1), \quad x_1 \in (\xi_j, \xi_{j+1}).\end{aligned}$$

Remark. Introduction of auxiliary (equivalent to r) weight function ρ gives possibility to state the positive definiteness of the difference scheme operator.

Let $H = H(\omega)$ be the set of grid functions defined on ω with the inner product and norm

$$(U, V)_r = \sum_{x\in\omega} h^2 r(x_1)U(x)V(x), \qquad \|V\|_r = \|V\|_{L_2(\omega,r)} = (V, V)_r^{1/2}.$$

Moreover, let

$$(U, V) = \sum_{x\in\omega} h^2 U(x)V(x), \qquad \|V\| = (V, V)^{1/2}.$$

Inner product and norm, involving ρ in index will make similar to the expression with index r sense.

Denote by $\mathring{H} = \mathring{H}(\bar\omega)$ the set of grid functions $V(x)$, given on $\bar\omega$ and satisfying conditions

$$V(x) = 0, \quad x \in \gamma_0, \qquad V(1, x_2) = \sum_{k=1}^{m} \alpha_k V(\xi_k, x_2), \quad x_2 \in \omega_2. \tag{3}$$

Lemma 1. *For each function, defined on mesh $\bar\omega$, which equals zero on $x_1 = 0$ and satisfies the nonlocal condition from (3), the following inequalities*

$$-\sum_{\omega_1} h\rho Y_{\bar{x}_1x_1} Y \ge \sum_{\omega_1^+} h\bar\rho Y_{\bar{x}_1}^2, \tag{4}$$

$$\sum_{\omega_1} hrY^2 \le 4\sum_{\omega_1^+} h\bar r (Y_{\bar{x}_1})^2 \tag{5}$$

hold.

Proof. After simple computations, we obtain

$$-\sum_{\omega_1} h\rho Y_{\bar{x}_1x_1} Y = \sum_{\omega_1^+} h\bar\rho Y_{\bar{x}_1}^2 - \frac{1}{2}Y^2(1, x_2) - \frac{1}{2}\sum_{\omega_1} hY^2 \rho_{\bar{x}_1x_1}.$$

Taking into account

$$\sum_{\omega_1} hY^2 \rho_{\bar{x}_1x_1} = -\sum_{\omega_1} hY^2 \sum_{k=1}^{m} \varkappa\sigma_k \frac{1}{h}\delta(x_1, \xi_k) = -\sum_{k=1}^{m} Y^2(\xi_k, x_2)\sigma_k\varkappa,$$

where $\delta(\cdot,\cdot)$ is the Kronecker delta, and

$$Y^2(1, x_2) \le \Big(\sum_{k=1}^{m} \sqrt[4]{\alpha_k^2\xi_k}\sqrt[4]{\alpha_k^2/\xi_k}\,|Y(\xi_k, x_2)|\Big)^2 \le \varkappa \sum_{k=1}^{m} \frac{|\alpha_k|}{\sqrt{\xi_k}} Y^2(\xi_k, x_2), \tag{6}$$

we obtain (4).

One can show that

$$\sum_{\omega_1^+} h\bar{r}^2 (Y^2)_{\bar{x}_1} = \sum_{\omega_1} hrY^2 + \frac{h^2}{8} Y^2(1, x_2). \tag{7}$$

On the other hand,

$$\begin{aligned}\sum_{\omega_1^+} h\bar{r}^2 (Y^2)_{\bar{x}_1} &= \sum_{\omega_1^+} h\bar{r}^2 Y_{\bar{x}_1}(Y + Y^{(-1_1)}) \\ &\le \Big(\sum_{\omega_1^+} h\bar{r}(Y_{\bar{x}_1})^2\Big)^{1/2} \Big(\sum_{\omega_1^+} h\bar{r}\big(Y + Y^{(-1_1)}\big)^2\Big)^{1/2} \\ &\le \frac{\varepsilon}{2} \sum_{\omega_1^+} h\bar{r}(Y_{\bar{x}_1})^2 + \frac{1}{2\varepsilon} \sum_{\omega_1^+} h\bar{r}(Y + Y^{(-1_1)})^2.\end{aligned}$$

Whence, choosing $\varepsilon = 4$ we obtain

$$\begin{aligned}\sum_{\omega_1^+} h\bar{r}^2 (Y^2)_{\bar{x}_1} &\le 2 \sum_{\omega_1^+} h\bar{r}(Y_{\bar{x}_1})^2 + \frac{1}{8} \sum_{\omega_1^+} h\bar{r}(Y + Y^{(-1_1)})^2 \\ &= 2 \sum_{\omega_1^+} h\bar{r}(Y_{\bar{x}_1})^2 + \frac{h^2}{8} Y^2(1, x_2) + \frac{1}{2} \sum_{\omega_1} hrY^2.\end{aligned} \tag{8}$$

(7), (8) prove the inequality (5). Lemma 1 is proved. □

3. Difference scheme, correction procedure, and main result

At the *first stage*, we approximate problem (1) by the difference scheme

$$\tilde{U}_{\bar{x}_1 x_1} + \tilde{U}_{\bar{x}_2 x_2} = \varphi(x), \quad x \in \omega, \ \tilde{U} \in \overset{\circ}{H}, \tag{9}$$

where $\varphi = T_1 T_2 f$ is the average of function f.

Define the operators

$$A := A_1 + A_2, \qquad A_k Y := -\overset{\circ}{Y}_{\bar{x}_1 x_1}, \quad k = 1, 2, \ x \in \omega,$$

where

$$Y \in H, \quad \overset{\circ}{Y} \in \overset{\circ}{H} \quad \text{and} \quad Y(x) = \overset{\circ}{Y}(x) \quad \text{for } x \in \omega.$$

The difference scheme (9) can be rewritten in the form of operator equation

$$-A\tilde{U} = \varphi(x), \quad x \in \omega, \ \tilde{U} \in H. \tag{10}$$

Operator A maps H onto H. Indeed, it suffices to show that operator A_1 on near-boundary point $(1-h, x_2)$ has the form

$$\begin{aligned}A_1 Y(1-h, x_2) &= -\overset{\circ}{Y}_{\bar{x}_1 x_1}(1-h, x_2) \\ &= -\big(\overset{\circ}{Y}(1, x_2) - 2\overset{\circ}{Y}(1-h, x_2) + \overset{\circ}{Y}(1-2h, x_2)\big)/h^2 \\ &= -\Big(\sum_{k=1}^{m} \alpha_k Y(\xi_k, x_2) - 2Y(1-h, x_2) + Y(1-2h, x_2)\Big)/h^2.\end{aligned}$$

According to the estimates (2), (4) and (5) we obtain the inequality

$$(A_1 Y, Y)_\rho \ge c\|Y\|_\rho^2, \quad Y \in H.$$

In addition, it is well known that A_2 is a self-adjoint and positive definite operator, $A_2 = A_2^*$, $(A_2 Y, Y)_\rho \ge c\|Y\|_\rho^2$. Therefore, the operator A is positive definite on the space H,

$$(AY, Y)_\rho \ge \|Y\|_\rho^2,$$

and hence the scheme (10) (i.e. (9)) is uniquely solvable.

At the *second stage*, we use the earlier-found solution of the difference scheme (10), define the correction term

$$\mathcal{R}\tilde{U} := \frac{h^2}{6}\tilde{U}_{\bar{x}_1 x_1 \bar{x}_2 x_2}$$

and solve the difference scheme

$$-AU = \varphi - \mathcal{R}\tilde{U}, \quad x \in \omega,\ U \in H \tag{11}$$

on the same grid.

The following assertion is the main result of the present paper.

Theorem 1. *Let the solution of problem (1) belong to the space $W_2^s(\Omega)$, $s \ge 2$. Then the convergence rate of the corrected difference scheme (11) in the discrete L_2-norm is defined by the estimate*

$$\|U - u\|_{L_2(\omega,r)} \le ch^s \|u\|_{W_2^s(\Omega)}, \quad 2 \le s \le 4.$$

4. A priori error estimates. Proof of Theorem 1

Let

$$\zeta_{3-k} = T_k u - u, \qquad \eta_{3-k} = T_k u - u - \frac{h^2}{12} u_{\bar{x}_k x_k}, \quad k = 1, 2.$$

By $\tilde{Z} = \tilde{U} - u$ and $Z = U - u$ we denote the errors in the solution of the schemes (10) and (11) respectively. First, notice that these functions represent solutions of the following problems:

$$-A\tilde{Z} = (\zeta_1)_{\bar{x}_1 x_1} + (\zeta_2)_{\bar{x}_2 x_2}, \quad x \in \omega,\ \tilde{Z} \in H \tag{12}$$

and

$$-AZ = (\eta_1)_{\bar{x}_1 x_1} + (\eta_2)_{\bar{x}_2 x_2} - (h^2/6)\tilde{Z}_{\bar{x}_1 x_1 \bar{x}_2 x_2}, \quad x \in \omega,\ Z \in H. \tag{13}$$

Indeed, we have

$$-AZ = -AU + Au = \varphi - \mathcal{R}\tilde{U} + Au = -\mathcal{R}\tilde{Z} + T_1 T_2 f - \mathcal{R}u + Au,$$

whence using the relation

$$T_1 T_2 \Delta u = (T_2 u)_{\bar{x}_1 x_1} + (T_1 u)_{\bar{x}_2 x_2}$$

and the expressions for the operators Au and $\mathcal{R}u$, we obtain (13). Eq. (12) is obtained analogously.

Lemma 2. *For the solutions of problems (12), (13) there hold the following a priori estimates*

$$\|\tilde{Z}_{\bar{x}_1 x_1}\|_\rho \le c\big(\|(\zeta_1)_{\bar{x}_1 x_1}\| + \|(\zeta_2)_{\bar{x}_2 x_2}\|\big), \tag{14}$$

$$\|Z\|_\rho \le c\big(\|\eta_1\| + \|\eta_2\| + h^2 \|\tilde{Z}_{\bar{x}_1 x_1}\|_\rho\big). \tag{15}$$

Proof. From (12) it follows

$$\big(\tilde{Z}_{\bar{x}_1 x_1}, \tilde{Z}_{\bar{x}_1 x_1}\big)_\rho + \big(\tilde{Z}_{\bar{x}_2 x_2}, \tilde{Z}_{\bar{x}_1 x_1}\big)_\rho = -\big((\zeta_1)_{\bar{x}_1 x_1} + (\zeta_2)_{\bar{x}_2 x_2}, \tilde{Z}_{\bar{x}_1 x_1}\big)_\rho. \tag{16}$$

Summing up by parts, we get

$$\big(\tilde Z_{\bar x_2 x_2}, \tilde Z_{\bar x_1 x_1}\big)_\rho = \sum_{\omega^+} h^2\bar\rho(\tilde Z_{\bar x_1\bar x_2})^2 - \sum_{\omega_2^+}\frac{h}{2}(\tilde Z_{\bar x_2}(1,x_2))^2 - \frac{1}{2}\sum_{\omega_1\times\omega_2^+} h^2\rho_{\bar x_1 x_1}(\tilde Z_{\bar x_2})^2$$
$$= \sum_{\omega^+} h^2\bar\rho(\tilde Z_{\bar x_1\bar x_2})^2 - \sum_{\omega_2^+}\frac{h}{2}\left[(\tilde Z_{\bar x_2}(1,x_2))^2 - \sum_{k=1}^m \frac{\varkappa|\alpha_k|}{\sqrt{\xi_k}}(\tilde Z_{\bar x_2}(\xi_k,x_2))^2\right].$$

Using analogous to the estimate (6), written for $\tilde Z_{\bar x_2}$, we obtain

$$\big(\tilde Z_{\bar x_2 x_2}, \tilde Z_{\bar x_1 x_1}\big)_\rho \ge \sum_{\omega^+} h^2\bar\rho(\tilde Z_{\bar x_1\bar x_2})^2 \ge 0.$$

Therefore, from (16) we obtain the validity of (14).

Now, represent the solution of the problem (13) in the form of sum

$$Z = Z^{(1)} + Z^{(2)},$$

where $Z^{(k)}$, $k = 1, 2$, are the solutions of the following problems

$$-AZ^{(1)} = (\eta_1)_{\bar x_1 x_1}, \quad x\in\omega,\ Z^{(1)}\in H, \tag{17}$$

$$-AZ^{(2)} = (\eta_2)_{\bar x_2 x_2} - \frac{h^2}{6}\tilde Z_{\bar x_1 x_1\bar x_2 x_2}, \quad x\in\omega,\ Z^{(2)}\in H. \tag{18}$$

From (17) we have

$$Z^{(1)} + A_1^{-1}A_2 Z^{(1)} = -\eta_1,$$
$$\|Z^{(1)}\|_\rho^2 + \big(A_1^{-1}A_2Z^{(1)}, Z^{(1)}\big)_\rho = -\big(\eta_1, Z^{(1)}\big)_\rho.$$

The operator A_2 is self-adjoint and positive definite, therefore, there exists quadratic root $A_2^{1/2}$, which is self-adjoint and commutable with A_1^{-1}. Thus

$$\big(A_1^{-1}A_2Z^{(1)}, Z^{(1)}\big)_\rho = \big(A_1^{-1}(A_2^{1/2}Z^{(1)}), (A_2^{1/2}Z^{(1)})\big)_\rho \ge 0$$

and

$$\|Z^{(1)}\|_\rho \le \|\eta_1\|. \tag{19}$$

From (18) it follows

$$A_2^{-1}A_1Z^{(2)} + Z^{(2)} = -\eta_2 + (h^2/6)\tilde Z_{\bar x_1 x_1},$$

and since

$$\big(A_2^{-1}A_1Z^{(2)}, Z^{(2)}\big)_\rho = \big(A_1(A_2^{-1/2}Z^{(2)}), (A_2^{-1/2}Z^{(2)})\big)_\rho \ge 0,$$

we obtain

$$\|Z^{(2)}\|_\rho \le \|\eta_2\| + (h^2/6)\|\tilde Z_{\bar x_1 x_1}\|_\rho. \tag{20}$$

(19) and (20) prove (15). □

To determine the rate of convergence of the two-stage finite difference method with the help of Lemma 2, it is sufficient to estimate the terms on the right-hand sides of (18), (19). For that purpose we use the following lemma.

Lemma 3. *Assume that the linear functional $l(u)$ is bounded in $W_2^s(E)$, where $s = \bar s + \varepsilon$, $\bar s$ is an integer, $0 < \varepsilon \le 1$, and $l(P) = 0$ for every polynomial P of degree $\le \bar s$ in two variables. Then, there exists a constant c, independent of u, such that $|l(u)| \le c|u|_{W_2^s(E)}$.*

Table 1
Experimental order of convergence in $L_2(\omega, r)$-norm.

h	$\|\tilde{U}_h - u\|_r$	$\|U_h - u\|_r$	$Ord(\tilde{U})$	$Ord(U)$
1/8	2.53376e−03	3.39828e−05		
			1.9949	3.9896
1/16	6.35699e−04	2.13925e−06		
			1.9987	3.9974
1/32	1.59065e−04	1.33943e−07		
			1.9997	3.9994
1/64	3.977507e−05	8.37520e−09		
			1.9999	3.9998
1/128	9.94431e−06	5.23507e−10		

This lemma is a particular case of the Dupont–Scott approximation theorem [17] and represents a generalization of the Bramble–Hilbert lemma [18].

Quantities $(\zeta_k)_{\bar{x}_k x_k}$, as a linear functionals with respect to u, vanish on the third order polynomials and are bounded in $W_2^s(\Omega)$, $s \geq 2$. Using the well known methodology (see, e.g., [6, Ch. 4, §1]), based on Lemma 3, for them we obtain the estimates

$$\|(\zeta_k)_{\bar{x}_k x_k}\| \leq ch^{s-2}\|u\|_{W_2^s(\Omega)}, \quad k = 1, 2, \tag{21}$$

$$\|\eta_k\| \leq ch^s\|u\|_{W_2^s(\Omega)}, \quad k = 1, 2. \tag{22}$$

Due to Lemma 2

$$\|Z\|_\rho \leq c\left(\|\eta_1\| + \|\eta_2\| + h^2\|(\zeta_1)_{\bar{x}_1 x_1}\| + h^2\|(\zeta_2)_{\bar{x}_2 x_2}\|\right),$$

which together with the estimates (21), (22) accomplishes the proof of Theorem 1.

5. Numerical experiments

Now, we present some numerical results to demonstrate the convergence order of the proposed method. The experimental order of convergence in the discrete $L_2(\omega, r)$ and $L_2(\omega)$ norms is computed by formulas

$$Ord(Y) = \log_2 \frac{\|Y_h - u\|_r}{\|Y_{h/2} - u\|_r}, \qquad Ord(Y) = \log_2 \frac{\|Y_h - u\|}{\|Y_{h/2} - u\|},$$

where u is the exact solution of original problem, while Y_h denotes the solution of the difference scheme on the grid with step h.

Below, in the examples the symbols $\tilde{U}$, U denote solutions of the difference schemes (10), (11), respectively.

The results of calculations are given by Tables 1, 2.

Consider the following problem

$$\Delta u = f, \quad x \in (0,1)^2, \; u\big|_{\Gamma_0} = 0, \qquad u(1, x_2) = u(0.5, x_2), \quad 0 < x_2 < 1, \tag{23}$$

where

$$f(x) = -\frac{13\pi^2}{9}\sin\left(\frac{2\pi x_1}{3}\right)\sin(\pi x_2).$$

The exact solution $u(x) = \sin(\frac{2\pi x_1}{3})\sin(\pi x_2)$ of the problem (23) belongs to the space W_2^4, therefore, theoretical convergence rate of the difference scheme equals 4.

The right-hand side of the scheme is calculated by the formula

$$\varphi(x) = T_1 T_2 f = -\frac{13\pi^2}{9}\lambda_1^2\lambda_2^2\sin\left(\frac{2\pi ih}{3}\right)\sin(\pi jh),$$

$$\lambda_1 = \frac{3}{\pi h}\sin\left(\frac{\pi h}{3}\right), \quad \lambda_2 = \frac{2}{\pi h}\sin\left(\frac{\pi h}{2}\right).$$

Table 2
Experimental order of convergence in $L_2(\omega)$-norm.

h	$\|\tilde{U}_h - u\|$	$\|U_h - u\|$	$Ord(\tilde{U})$	$Ord(U)$
1/8	4.15297e−03	5.56995e−05		
			1.9654	3.9601
1/16	1.06347e−03	3.57879e−06		
			1.9844	3.9831
1/32	2.68761e−04	2.26315e−07		
			1.9926	3.9923
1/64	6.75360e−05	1.42207e−08		
			1.9964	3.9963
1/128	1.69262e−06	8.91061e−10		

6. Conclusion

For solution of the Bitsadze–Samarskii type nonlocal problem posed in unit square for Poisson equation it is used a finite-difference scheme. Using the solution, obtained by the method with second order accuracy, we correct the right-hand side of the scheme and solve it again on the same grid. It is proved that if the solution of original problem belongs to the Sobolev space with fractional exponent $s \in [2; 4]$, then the corrected scheme converges with the rate $O(|h|^s)$. The theoretical results are supported by numerical experiments. The obtained results can be extended to the nonlocal problem posed for general elliptic equations, and also to three-dimensional case.

Acknowledgments

The work was supported by the Shota Rustaveli National Science Foundation (Grant FR/406/5-106/12).
The authors sincerely thank the anonymous referee for helpful comments.

References

[1] E.A. Volkov, Solving the Dirichlet problem by a method of corrections with higher order differences, I, Differ. Uravn. 1 (7) (1965) 946–960 (in Russian).
[2] E.A. Volkov, Solving the Dirichlet problem by a method of corrections with higher order differences, II, Differ. Uravn. 1 (8) (1965) 1070–1084 (in Russian).
[3] E.A. Volkov, A two–stage difference method for solving the Dirichlet problem for the Laplace equation on a rectangular parallelepiped, Comput. Math. Math. Phys. 49 (3) (2009) 496–501.
[4] R.D. Lazarov, V.L. Makarov, Convergence of the method of nets and the method of lines for multidimensional problems of mathematical physics in classes of generalized solutions, Dokl. Akad. Nauk SSSR 259 (2) (1981) 282–286 (in Russian).
[5] R.D. Lazarov, V.L. Makarov, A.A. Samarskii, Application of exact difference schemes to the construction and study of difference schemes for generalized solutions, Mat. Sb. (N.S.) 117 (159) (4) (1982) 469–480 (in Russian).
[6] A.A. Samarskii, R.D. Lazarov, V.L. Makarov, Difference Schemes for Differential Equations with Generalized Solutions, Visshaja Shkola, Moscow, 1987 (in Russian).
[7] B.S. Jovanović, The finite difference method for boundary-value problems with weak solutions, in: Posebna izdanja, vol.16, Matematički Institut u Beogradu, Belgrade, 1993.
[8] G. Berikelashvili, Construction and analysis of difference schemes for some elliptic problems, and consistent estimates of the rate of convergence, Mem. Differential Equations Math. Phys. 38 (2006) 1–131.
[9] A.V. Bitsadze, A.A. Samarskii, On some simples generalizations of linear elliptic problems, Dokl. Akad. Nauk SSSR 185 (1969) 739–740 (in Russian).
[10] D.G. Gordeziani, On the methods of solution for one class of nonlocal boundary value problems, in: Tbil. Gos. Univ., Inst. Prikl. Mat., Tbilisi, 1981 (in Russian).
[11] V.A. Ilin, E.I. Moiseev, A two–dimensional nonlocal boundary value problem for poisson operator in the differential and the difference interpretation, Mat. Model. 2 (8) (1990) 130–156 (in Russian); Math. Model. 2 (8) (1990) 598–611 (Transl.).
[12] G. Berikelashvili, On the solvability of a nonlocal boundary value problem in the weighted Sobolev spaces, Proc. A. Razmadze Math. Inst. 119 (1999) 3–11.
[13] G. Berikelashvili, On the convergence of finite-difference scheme for a nonlocal elliptic boundary value problem, Publ. Inst. Math. (Beograd) (N.S.) 70 (84) (2001) 69–78.
[14] G.K. Berikelashvili, B.G. Midodashvili, Compatible convergence estimates in the method of refinement by higher–order differences, Differ. Uravn. 51 (1) (2015) 108–115 (in Russian); Differ. Equ. 51 (1) (2015) 107–115 (Transl.).
[15] G. Berikelashvili, B. Midodashvili, On the improvement of convergence rate of difference scheme for one mixed boundary value problem, Mem. Differential Equations Math. Phys. 65 (2015) 23–34.

[16] L. Collatz, The Numerical Treatment of Differential Equations, third ed., Springer–Verlag, Berlin, 1966.
[17] T. Dupont, R. Scott, Polinomial approximation of functions in Sobolev spaces, Math. Comp. 34 (150) (1980) 441–463.
[18] J.H. Bramble, S.R. Hilbert, Bounds for a class of linear functionals with application to Hermite interpolation, Numer. Math. 16 (1971) 362–369.