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Method of Refinement by Higher Order Differences for 3D Poisson Equation with Nonlocal Boundary Conditions

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Abstract: We consider the Bitsadze−Samarskii type nonlocal boundary value problem for Poisson equation in a unit cube, which is first solved by a difference scheme of second-order accuracy. Using this approximate solution, we correct the right-hand side of the difference scheme. It is shown that the solution of the corrected scheme converges at the rate $O(h^s)$ in the discrete L_2-norm provided that the exact solution of the original problem belongs to the Sobolev space with exponent $s \in [2,4]$.

Keywords: Nonlocal BVP, Difference scheme, Method of corrections, Improvement of accuracy, Compatible estimates of convergence rate

1 Introduction

Finite difference method is a significant tool in the numerical solution of problems posed for differential equations. In order to minimize the amount of calculations it is desirable for the difference scheme to be sufficiently good on coarse grids, i.e. to have high order accuracy. In the present work we consider Bitsadze–Samarskii type nonlocal boundary value problem for the three– dimensional Poisson equation and study a two-stage finite difference method for improving the accuracy of the approximate solution.

In the first stage, we solve the difference scheme $\Delta_h \tilde{U} = \varphi$, using the standard seven- point finite difference approximation which has the second order of accuracy. In the second stage, we use the solution $\tilde{U}$ to correct the right-hand side of the difference scheme, $\Delta_h U = \varphi + R\tilde{U}$, and solve again on the same mesh.

This approach for boundary value problems for Poisson and Laplace equations has been studied in Volkov's papers (see, e.g. [1,2,3]), where the input data were chosen so as to ensure that the exact solution belongs to the Hölder class $C_{6,\lambda}(\overline{\Omega})$.

For establishing the convergence, we use the methodology of obtaining the compatible estimates of convergence rate of difference schemes. This methodology develops from the works of Samarskii, Lazarov and Makarov (see, e.g., [4,5,6]), and later in the works of other authors (see, e.g., [7,8]). For the elliptic problems such estimates have the form

$$\|U-u\|_{W_2^k(\omega)} \leq c|h|^{s-k}\|u\|_{W_2^s(\Omega)}, \quad s>k\geq 0,$$

where u is the solution of original problem, U is the approximate solution, k and s are integer and real numbers, respectively, $W_2^k(\omega)$ and $W_2^s(\Omega)$ are the Sobolev norms on the set of functions with discrete and continuous arguments. Here and below, c denotes a positive generic constant, independent of h and u.

We prove that the solution U of the corrected difference scheme converges at the rate $O(h^s)$ in the discrete L_2 -norm when the exact solution belongs to the Sobolev space W_2^s, $s \in [2,4]$.

The generalization of the Bitsadze – Samarskii problem [9] was investigated by many authors (see, e.g., [10,11,12,13]). V.A. Ilin and E.I.Moiseev [11] considered, for a Poisson equation, a difference scheme

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which converges at the rate $O(h^2)$ in the discrete W_2^2-norm to the exact solution from the class $C^4(\bar{\Omega})$. In [13], the first author considered a difference scheme for a second order elliptic equation with variable coefficients and compatible estimates of convergence rate in discrete W_2^1-norm were obtained.

Results for two-stage finite-difference method, analogous to those given in the present work, for different problems were obtained in [14, 15, 16, 17, 18].

One of the methods for obtaining compact high order approximations is the Mehrstellen method ("Mehrstellenverfahren"), defined by Collatz (see [19]). Instead of approximating only the left hand side of the differential equation, he proposes to take several values of the right hand side as well. In the case of three-dimensional problem, the differential operator is approximated on a 27-point stencil with the fourth order accuracy. The advantages of the Mehrstellen schemes over ordinary (second order) accuracy schemes on a coarse grid are obvious.

The advantages of our method are:

a) We approximate the differential operator on a minimally acceptable stencil (7-point stencil for a three-dimensional problem). Therefore, the condition number of this operator is better as compared with the Mehrstellen schemes, which is notable on a fine grid.

b) It is a two-stage method, but it requires matrix inversion only once (on the second stage we change only the right-hand side of the equation, while the operator is kept unchanged).

c) The method of correction is useful even in the case when construction of high precision schemes is impossible.

2 Statement of the Problem and Some Auxiliary Estimates

As usual, by symbol $W_2^s(\Omega)$, $s \geq 0$ we denote the Sobolev space. For integer s, the norm in $W_2^s(\Omega)$ is given by

$$\|u\|^2_{W_2^s(\Omega)} = \sum_{j=0}^{s} |u|^2_{W_2^j(\Omega)}, \quad |u|^2_{W_2^j(\Omega)} = \sum_{|\nu|=j} \|D^\nu u\|^2_{L_2(\Omega)},$$

where $D^\nu := \partial^{|\nu|} / \left(\partial x_1^{\nu_1} \partial x_2^{\nu_2} \partial x_3^{\nu_3}\right)$, and $\nu = (\nu_1, \nu_2, \nu_3)$ is a multi-index with non-negative integer components, $|\nu| = \nu_1 + \nu_2 + \nu_3$.

If $s = \bar{s} + \varepsilon$, where $\bar{s}$ is an integer part of s and $0 < \varepsilon < 1$, then

$$\|u\|^2_{W_2^s(\Omega)} = \|u\|^2_{W_2^{\bar{s}}(\Omega)} + |u|^2_{W_2^s(\Omega)},$$

where

$$|u|_{W_2^s(\Omega)} = \sum_{|\nu|=\bar{s}} \int_\Omega \int_\Omega \frac{|D^\nu u(x) - D^\nu u(y)|^2}{|x-y|^{3+2\varepsilon}} \, dx\, dy.$$

Particularly, for $s = 0$ we have $W_2^0 = L_2$.

Let $\bar{\Omega} = \{x = (x_1, x_2, x_3) : \ 0 \leq x_k \leq 1, \ k = 1, 2, 3\}$ be the unit cube with boundary Γ; $\Gamma_0 = \Gamma \setminus \{(1, x_2, x_3) : 0 < x_k < 1, \ k = 2, 3\}$; ξ_j be fixed points in the interval $(0, 1)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$. Denote $\xi_0 = 0$, $\xi_{m+1} = 1$.

Consider the problem

$$\Delta u = f(x), \quad x \in \Omega, \quad u|_{\Gamma_0} = 0,$$

$$u(1, x_2, x_3) = \sum_{j=1}^{m} \alpha_j u(\xi_j, x_2, x_3), \quad 0 < x_2, x_3 < 1, \quad (1)$$

where the coefficients α_j are real numbers satisfying conditions

$$\varkappa := \sum_{j=1}^{m} |\alpha_j| \sqrt{\xi_j} < 1.$$

By analogy to [12] it can be proved that for $f(x) \in L_2(\Omega, \rho)$ there exists a unique strong solution of problem (1) in the weighted Sobolev space $W_2^2(\Omega, \rho)$. Throughout the following, we assume that the function $f(x)$ provides the unique solvability of problem (1) in the space $W_2^s(\Omega)$, $2 \leq s \leq 4$.

Consider the following discrete domains in $\bar{\Omega}$:

$$\bar{\omega}_k = \{x_k = i_k h : \ i_k = 0, 1, \ldots, n, \ h = 1/n\},$$

$$\bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2 \times \bar{\omega}_3,$$

$$\omega_k^+ = \bar{\omega}_k \cap (0, 1],$$

$$\omega_k = \bar{\omega}_k \cap (0, 1),$$

$$\omega = \Omega \cap \bar{\omega}, \qquad k = 1, 2, 3, \quad \gamma_0 = \Gamma_0 \cap \bar{\omega}.$$

We assume that the points ξ_j coincide with grid nodes

$$\xi_j = n_j h, \quad j = 1, 2, \ldots, m,$$

where n_j are nonnegative integers $0 < n_1 < n_2 < \cdots < n_m < n$.

For grid functions $V = V(x)$ we define difference quotients in x_k directions as follows

$$V_{x_k} = \left(V^{(+1_k)} - V\right)/h, \quad V_{\bar{x}_k} = \left(V - V^{(-1_k)}\right)/h,$$

where

$$V^{(\pm 1_1)} = V(x_1 \pm h, x_2, x_3),$$

$$V^{(\pm 1_2)} = V(x_1, x_2 \pm h, x_3),$$

$$V^{(\pm 1_3)} = V(x_1, x_2, x_3 \pm h).$$

For functions defined on Ω, we need the following averaging operators:

$$T_1 u(x) := \frac{1}{h^2} \int_{x_1 - h}^{x_1 + h} (h - |x_1 - t_1|) u(t_1, x_2, x_3) \, dt_1.$$

The operators T_2, T_3 are defined similarly. Note that these operators commute and

$$T_k \frac{\partial^2 u}{\partial x_k^2} = u_{\bar{x}_k x_k} := \Lambda_k u, \quad k = 1, 2, 3.$$

Let

$$T=\prod_{k=1}^{3}T_k,\quad T_{(\alpha)}=\prod_{\substack{k=1,\\k\neq\alpha}}^{3}T_k,\quad \Lambda_{(\alpha)}=\sum_{\substack{k=1,\\k\neq\alpha}}^{3}\Lambda_k.$$

Define the following weight functions

$$r(x_1)=1-x_1,\quad \rho(x_1)=1-x_1-\sum_{j=1}^{m}\varkappa\sigma_j\chi(\xi_j-x_1),$$

where

$$\sigma_j=\frac{|\alpha_j|}{\sqrt{\xi_j}},\quad \chi(t)=\begin{cases}t, \text{ if } t\geq 0,\\ 0, \text{ if } t<0.\end{cases}$$

Let

$$\bar{r}=\left(r+r^{(-1_1)}\right)/2,\quad \bar{\rho}=\left(\rho+\rho^{(-1_1)}\right)/2.$$

Lemma 1. *For the weight functions, the following relations hold*

$$(1-\varkappa^2)r(x_1)\leq\rho(x_1)\leq r(x_1). \tag{2}$$

$$\rho_{\bar{x}_1x_1}(x_1)=-\sum_{k=1}^{m}\frac{\varkappa\sigma_k}{h}\delta(x_1,\xi_k) \tag{3}$$

where $\delta(\cdot,\cdot)$ is the Kronecker delta.

Proof. The right-hand side of inequality (2) is obvious. The left-hand side of this inequality can be verified as follows:

$$\rho(x_1)=1-x_1-\varkappa\sum_{k=j+1}^{m}\sigma_k(\xi_k-x_1)$$
$$\geq\left(1-\varkappa\sum_{k=j+1}^{m}\sigma_k\xi_k\right)(1-x_1)$$
$$\geq(1-\varkappa^2)(1-x_1), x_1\in(\xi_j,\xi_{j+1}).$$

The validity of (3) can be obtained by direct verification.

Remark: Introduction of auxiliary (equivalent to r) weight function ρ allows us to determine the positive definiteness of the difference scheme operator.

Let $H=H(\omega)$ be the set of grid functions defined on ω with the inner product and norm

$$(U,V)_r=\sum_{x\in\omega}h^3r(x_1)U(x)V(x),$$

$$\|V\|_r=\|V\|_{L_2(\omega,r)}=(V,V)_r^{1/2}.$$

Moreover, let

$$(U,V)=\sum_{x\in\omega}h^3U(x)V(x),\quad \|V\|=(V,V)^{1/2}.$$

Inner products and norms, involving ρ index are similar to the expressions with index r.

Denote by $\overset{\circ}{H}=\overset{\circ}{H}(\bar{\omega})$ the set of grid functions $V(x)$, given on $\bar{\omega}$ and satisfying conditions

$$V(x)=0,\ x\in\gamma_0,$$

$$V(1,x_2,x_3)=\sum_{j=1}^{m}\alpha_jV(\xi_j,x_2,x_3),\quad (x_2,x_3)\in\omega_2\times\omega_3. \tag{4}$$

Lemma 2. *For each function $Y=Y(x_1)$, defined on mesh $\bar{\omega}_1$, which equals zero on $x_1=0$ and satisfies the nonlocal condition in (4), the following inequalities hold:*

$$-\sum_{\omega_1}h\rho Y_{\bar{x}_1x_1}Y\geq\sum_{\omega_1^+}h\bar{\rho}Y_{\bar{x}_1}^2, \tag{5}$$

$$\sum_{\omega_1}hrY^2\leq\sum_{\omega_1^+}h\bar{r}(Y_{\bar{x}_1})^2. \tag{6}$$

Proof. After simple computations, we obtain

$$-\sum_{\omega_1}h\rho Y_{\bar{x}_1x_1}Y=\sum_{\omega_1^+}\rho Y_{\bar{x}_1}Y-\sum_{\omega_1^+}\rho^{(-1_1)}Y_{\bar{x}_1}Y^{(-1_1)}=$$

$$=\sum_{\omega_1^+}\frac{\rho+\rho^{(-1_1)}}{2}(Y_{\bar{x}_1})^2+\sum_{\omega_1^+}\frac{\rho-\rho^{(-1_1)}}{2}Y_{\bar{x}_1}(Y+Y+^{(-1_1)})=$$

$$=\sum_{\omega_1^+}h\bar{\rho}(Y_{\bar{x}_1})^2+\sum_{\omega_1^+}\frac{\rho-\rho^{(-1_1)}}{2}\left(Y^2-(Y^{(-1_1)})^2\right),$$

i.e.

$$-\sum_{\omega_1}h\rho Y_{\bar{x}_1x_1}Y=\sum_{\omega_1^+}h\bar{\rho}(Y_{\bar{x}_1})^2-\frac{1}{2}Y^2(1)-\frac{1}{2}\sum_{\omega_1}hY^2\rho_{\bar{x}_1x_1}.$$

Due to (3), the last term on the right hand side of this equality can be rewritten as follows

$$\sum_{\omega_1}hY^2\rho_{\bar{x}_1x_1}=-\sum_{\omega_1}hY^2\sum_{k=1}^{m}\varkappa\sigma_k\frac{1}{h}\delta(x_1,\xi_k)=-\sum_{k=1}^{m}Y^2(\xi_k)\sigma_k\varkappa,$$

and taking into account the inequality

$$Y^2(1)\leq\left(\sum_{j=1}^{m}\sqrt[4]{\alpha_j^2\xi_j}\sqrt[4]{\alpha_j^2/\xi_j}\,|Y(\xi_j)|\right)^2\leq\varkappa\sum_{j=1}^{m}\frac{|\alpha_j|}{\sqrt{\xi_j}}Y^2(\xi_j), \tag{7}$$

we obtain (5).

One can show that

$$\sum_{\omega_1^+}h\bar{r}^2(Y^2)_{\bar{x}_1}=\sum_{\omega_1}\bar{r}^2Y^2-\sum_{\omega_1}(\bar{r}^{(+1_1)})^2Y^2+\frac{(r(1-h))^2}{4}Y^2(1)$$

$$=\sum_{\omega_1}\left(\bar{r}-\bar{r}^{(+1_1)}\right)\left(\bar{r}+\bar{r}^{(+1_1)}\right)Y^2+\frac{h^2}{4}Y^2(1).$$

Whence, according to the identities $\bar r-\bar r^{(+1_1)}=h$ and $\bar r+\bar r^{(+1_1)}=2r$, we have

$$\sum_{\omega_1^+} h\bar r^2 (Y^2)_{\bar x_1} = 2\sum_{\omega_1} hrY^2+\frac{h^2}{4}Y^2(1). \qquad (8)$$

On the other hand,

$$\sum_{\omega_1^+} h\bar r^2 (Y^2)_{\bar x_1} = \sum_{\omega_1^+} h\bar r^2 Y_{\bar x_1}(Y+Y^{(-1_1)})$$

$$\leq \Big(\sum_{\omega_1^+} h\bar r (Y_{\bar x_1})^2\Big)^{1/2}\Big(\sum_{\omega_1^+} h\bar r\big(Y+Y^{(-1_1)}\big)^2\Big)^{1/2}$$

$$\leq \frac{\varepsilon}{2}\sum_{\omega_1^+} h\bar r(Y_{\bar x_1})^2+\frac{1}{2\varepsilon}\sum_{\omega_1^+} h\bar r(Y+Y^{(-1_1)})^2.$$

Whence, choosing $\varepsilon=2$, we obtain

$$\sum_{\omega_1^+} h\bar r^2 (Y^2)_{\bar x_1} \leq \sum_{\omega_1^+} h\bar r(Y_{\bar x_1})^2+\frac{1}{4}\sum_{\omega_1^+} h\bar r(Y+Y^{(-1_1)})^2$$

$$\leq \sum_{\omega_1^+} h\bar r(Y_{\bar x_1})^2+\frac{h^2}{4}Y^2(1)+\sum_{\omega_1} hrY^2. \qquad (9)$$

The results (8), (9) prove the inequality (6) and thus Lemma 2.2 is proved.

Corollary 1. *For any function $V\in\overset{\circ}{H}$ the following estimate holds*

$$\sum_{\omega_1}\sum_{\omega_k} h^2\rho V_{\bar x_1x_1}V_{\bar x_kx_k} \geq \sum_{\omega_1^+}\sum_{\omega_k^+} h^2\bar\rho (V_{\bar x_1\bar x_k})^2, \quad k=2,3. \quad (10)$$

Indeed, if $V\in\overset{\circ}{H}$, then functions $V_{\bar x_k}, k=2,3$, satisfy the conditions of Lemma 2

$$V_{\bar x_k}(0,x_2,x_3)=0, \quad V_{\bar x_k}(1,x_2,x_3)=\sum_{j=1}^{m}\alpha_j V_{\bar x_k}(\xi_j,x_2,x_3),$$

and the validity of (10) follows from the identity

$$\sum_{\omega_1}\sum_{\omega_k} h^2\rho V_{\bar x_1x_1}V_{\bar x_kx_k} = -\sum_{\omega_1}\sum_{\omega_k^+} h^2\rho\left(V_{\bar x_k}\right)_{\bar x_1x_1}V_{\bar x_k}.$$

3 Difference Scheme, Correction Procedure, and Main Result

At the *first stage*, we approximate problem (1) by the difference scheme

$$\Delta_h\tilde U=\varphi(x), \quad x\in\omega, \quad \tilde U\in\overset{\circ}{H}, \qquad (11)$$

where $\Delta_hY := Y_{\bar x_1x_1}+Y_{\bar x_2x_2}+Y_{\bar x_3x_3}$ and $\varphi=Tf$ is the average of function f.

Lemma 3. *Finite difference scheme (11) is uniquely solvable.*

Proof. Define the operators

$$A:=A_1+A_2+A_3, \quad A_kY:=-\overset{\circ}{Y}_{\bar x_kx_k}, \quad k=1,2,3, \quad x\in\omega,$$

where

$$Y\in H, \quad \overset{\circ}{Y}\in\overset{\circ}{H} \quad \text{and} \quad Y(x)=\overset{\circ}{Y}(x) \quad \text{for} \quad x\in\omega.$$

The difference scheme (11) can be rewritten in the form of operator equation

$$-A\tilde U=\varphi(x), \; x\in\omega, \; \tilde U\in H. \qquad (12)$$

Operator A maps H onto H. Indeed, it suffices to notice, that operator A_1 on near-boundary point $(1-h,x_2,x_3)$ has the form

$$A_1Y(1-h,x_2,x_3)=-\overset{o}{Y}_{\bar x_1x_1}(1-h,x_2,x_3)$$

$$=-\big(\overset{o}{Y}(1,x_2,x_3)-2\overset{o}{Y}(1-h,x_2,x_3)+\overset{o}{Y}(1-2h,x_2,x_3)\big)/h^2$$

$$=-\left(\sum_{j=1}^{m}\alpha_jY(\xi_j,x_2,x_3)-2Y(1-h,x_2,x_3)+Y(1-2h,x_2,x_3)\right)/h^2.$$

From the estimates (2), (5) and (6), we obtain the inequality

$$(A_1Y,Y)_\rho \geq c\|Y\|_\rho^2, \quad Y\in H.$$

In addition, since weighted function ρ does not depend on variables x_2,x_3, the operators A_2, A_3 are self-adjoint and positive definite, $A_k=A_k^*$, and $(A_kY,Y)_\rho\geq c\|Y\|_\rho^2$, $k=2,3$. Therefore, the operator A is positive definite on the space H,

$$(AY,Y)_\rho\geq c\|Y\|_\rho^2,$$

and hence the scheme (12) (i.e. (11)) is uniquely solvable.

At the *second stage*, we use the earlier-found solution of the difference scheme (12), define the correction term

$$\mathscr{R}\tilde U := \frac{h^2}{6}\left(\tilde U_{\bar x_1x_1\bar x_2x_2}+\tilde U_{\bar x_1x_1\bar x_3x_3}+\tilde U_{\bar x_2x_2\bar x_3x_3}\right)$$

and on the same grid solve the difference scheme

$$\Delta_hU=\varphi-\mathscr{R}\tilde U, \quad x\in\omega, \quad U\in\overset{\circ}{H}$$

or

$$-AU=\varphi-\mathscr{R}\tilde U, \quad x\in\omega, \quad U\in H. \qquad (13)$$

The following assertion is the main result of the present paper.

Theorem 1. *Let the solution of problem (1) belong to the space $W_2^s(\Omega)$, $s\geq 2$. Then the convergence rate of the corrected difference scheme (13) in the discrete L_2-norm is defined by the estimate*

$$\|U-u\|_{L_2(\omega,r)}\leq ch^s\|u\|_{W_2^s(\Omega)}, \quad 2\leq s\leq 4.$$

4 A Priori Error Estimates. Proof of Theorem 1

Let

$$\zeta_k = T_{(k)}u - u, \quad \eta_k = T_{(k)}u - u - \frac{h^2}{12}\Lambda_{(k)}u, \quad k = 1,2,3.$$

By $\tilde{Z} = \tilde{U} - u$ and $Z = U - u$ we denote the errors in the solution of the schemes (12) and (13), respectively. First, notice that these functions represent solutions of the following problems:

$$-A\tilde{Z} = \Lambda_1\zeta_1 + \Lambda_2\zeta_2 + \Lambda_3\zeta_3, \quad x \in \omega, \quad \tilde{Z} \in H \quad (14)$$

and

$$-AZ = \sum_{\alpha=1}^{3} \Lambda_\alpha \eta_\alpha - (h^2/6)(\Lambda_1\Lambda_2 + \Lambda_1\Lambda_3 + \Lambda_2\Lambda_3)\tilde{Z}, \quad x \in \omega, \quad Z \in H. \quad (15)$$

Indeed, we have

$$-AZ = -AU + Au = \varphi - \mathscr{R}\tilde{U} + Au = -\mathscr{R}\tilde{Z} + Tf - \mathscr{R}u - \Delta_h u,$$

whence using the relation

$$T\Delta u = \Lambda_1(T_{(1)}u) + \Lambda_2(T_{(2)}u) + \Lambda_3(T_{(3)}u)$$

and the expressions for the operators Au and $\mathscr{R}u$, we obtain (15). Equation (14) is obtained analogously.

Lemma 4. *For the solutions of problems (14),(15) the following a priori estimates hold:*

$$\|\tilde{Z}_{\bar{x}_k x_k}\|_\rho \le c\big(\|(\zeta_1)_{\bar{x}_1x_1}\| + \|(\zeta_2)_{\bar{x}_2x_2}\| + \|(\zeta_3)_{\bar{x}_3x_3}\|\big), \quad k = 1,2, \quad (16)$$

$$\|Z\|_\rho \le c\big(\|\eta_1\| + \|\eta_2\| + \|\eta_3\| + h^2\|\tilde{Z}_{\bar{x}_1x_1}\|_\rho + h^2\|\tilde{Z}_{\bar{x}_2x_2}\|_\rho\big). \quad (17)$$

Proof. From (14) it follows

$$(\Lambda_1\tilde{Z}, \Lambda_k\tilde{Z})_\rho + (\Lambda_2\tilde{Z}, \Lambda_k\tilde{Z})_\rho + (\Lambda_3\tilde{Z}, \Lambda_k\tilde{Z})_\rho = (\Lambda_1\zeta_1 + \Lambda_2\zeta_2 + \Lambda_3\zeta_3, \Lambda_k\tilde{Z})_\rho. \quad (18)$$

From (10) we obtain

$$(\tilde{Z}_{\bar{x}_k x_k}, \tilde{Z}_{\bar{x}_1 x_1})_\rho \ge 0, \quad k = 2,3.$$

It is easy to see that

$$(\tilde{Z}_{\bar{x}_2x_2}, \tilde{Z}_{\bar{x}_3x_3})_\rho \ge 0.$$

Therefore, from (18) we obtain the validity of (16).

Now, represent the solution of the problem (15) in the form of sum

$$Z = Z^{(1)} + Z^{(2)} + Z^{(3)},$$

where $Z^{(k)}$, $k = 1,2,3$, are the solutions of the following problems

$$-AZ^{(1)} = \Lambda_1\eta_1, \quad x \in \omega, \quad Z^{(1)} \in H, \quad (19)$$

$$-AZ^{(2)} = \Lambda_2\eta_2 - \frac{h^2}{6}\Lambda_1\Lambda_2\tilde{Z}, \quad x \in \omega, \quad Z^{(2)} \in H. \quad (20)$$

$$-AZ^{(3)} = \Lambda_3\eta_3 - \frac{h^2}{6}(\Lambda_1\Lambda_3 + \Lambda_2\Lambda_3)\tilde{Z}, \quad x \in \omega, \quad Z^{(3)} \in H. \quad (21)$$

From (19) we have

$$Z^{(1)} + A_1^{-1}(A_2 + A_3)Z^{(1)} = \eta_1,$$

$$\|Z^{(1)}\|_\rho^2 + \big(A_1^{-1}(A_2 + A_3)Z^{(1)}, Z^{(1)}\big)_\rho = \big(\eta_1, Z^{(1)}\big)_\rho.$$

The operator A_k, $k = 2,3$ is self-adjoint and positive definite, therefore, there exists quadratic root $A_k^{1/2}$, which is self-adjoint, positive definite and commutable with A_1^{-1}. Thus

$$\big(A_1^{-1}A_kZ^{(1)}, Z^{(1)}\big)_\rho = \big(A_1^{-1}(A_k^{1/2}Z^{(1)}), (A_k^{1/2}Z^{(1)})\big)_\rho \ge 0$$

and, therefore

$$\|Z^{(1)}\|_\rho \le \|\eta_1\|. \quad (22)$$

From (20) it follows

$$A_2^{-1}(A_1 + A_3)Z^{(2)} + Z^{(2)} = \eta_2 - (h^2/6)\tilde{Z}_{\bar{x}_1x_1},$$

and since

$$\big(A_2^{-1}(A_1 + A_3)Z^{(2)}, Z^{(2)}\big)_\rho = \big((A_1 + A_3)(A_2^{-1/2}Z^{(2)}), (A_2^{-1/2}Z^{(2)})\big)_\rho \ge 0,$$

we obtain

$$\|Z^{(2)}\|_\rho \le \|\eta_2\| + (h^2/6)\|\tilde{Z}_{\bar{x}_1x_1}\|_\rho. \quad (23)$$

From (21) we obtain

$$\big((A_3^{-1}A_1 + A_3^{-1}A_2)Z^{(3)} + Z^{(3)}, Z^{(3)}\big)_\rho = (\eta_3, Z^{(3)})_\rho - (h^2/6)\big(\tilde{Z}_{\bar{x}_1x_1} + \tilde{Z}_{\bar{x}_2x_2}, Z^{(3)}\big)_\rho$$

and, therefore,

$$\|Z^{(3)}\|_\rho^2 \le \|\eta_3\|\,\|Z^{(3)}\|_\rho + (h^2/6)\big(\|\tilde{Z}_{\bar{x}_1x_1} + \tilde{Z}_{\bar{x}_2x_2}\|_\rho\big)\,\|Z^{(3)}\|_\rho. \quad (24)$$

(22), (23) and (24) prove (17) and thus lemma 4.1 is established.

Due to lemma 4.1

$$\|Z\|_\rho \le c\sum_{\alpha=1}^{3}\big(\|\eta_\alpha\| + h^2\|(\zeta_\alpha)_{\bar{x}_\alpha x_\alpha}\|\big). \quad (25)$$

To determine the rate of convergence of the proposed two-stage finite difference method it is sufficient to estimate the terms on the right-hand sides of (25). For that purpose we use the following lemma.

Lemma 5. *Assume that the linear functional $l(u)$ is bounded in $W_2^s(E)$, where $s = \bar{s} + \varepsilon$, $\bar{s}$ is an integer, $0 < \varepsilon \le 1$, and $l(P) = 0$ for every polynomial P of degree $\bar{s}$ in three variables. Then, there exists a constant c, independent of u, such that $|l(u)| \le c|u|_{W_2^s(E)}$.*

This lemma is a particular case of the Dupont–Scott approximation theorem [20] and represents a generalization of the Bramble–Hilbert lemma [21]. Quantities $(\zeta_k)_{\bar{x}_k x_k}$, η_k, $k = 1,2,3$, as linear functionals with respect to u, vanish on the third order polynomials and are bounded in $W_2^s(\Omega)$, $s \geq 2$. Using the well known methodology, based on lemma 3, one may obtain the estimates [22]

$$\|(\zeta_k)_{\bar{x}_k x_k}\| \leq ch^{s-2}\|u\|_{W_2^s(\Omega)}, \ \|\eta_k\| \leq ch^s\|u\|_{W_2^s(\Omega)}, \quad 2 \leq s \leq 4,$$

which together with (25) completes the proof of Theorem 1.

5 Perspective

In this paper, we consider Bitsadze-Samarskii type nonlocal boundary value problems for a three–dimensional Poisson equation and study a two-stage finite difference method for improving the accuracy of the approximate solution.The differential operator is approximated on the minimally acceptable seven- point stencil.Although this is a two-stage method, nevertheless it requires matrix inversion only once because on the second stage we change only the right-hand side of the equation while the operator is kept unchanged. We establish the convergence by obtaining the compatible estimates of convergence rates of difference schemes. The convergence of the corrected difference scheme is proved to be $O(h^s)$ in the discrete L_2 -norm, assuming that the exact solution belongs to the Sobolev spaceW_2^s, $s \in [2,4]$.

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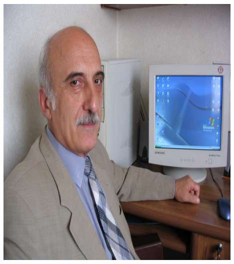

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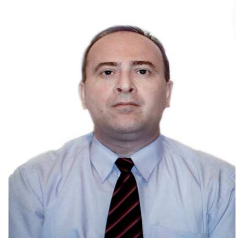

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