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ON IMPROVED APPROXIMATE SOLUTION OF THE FREDHOLM INTEGRAL EQUATION

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Abstract — A method of approximate solution of the linear one-dimensional Fredholm integral equation of the second kind is constructed. With the help of the Steklov averaging operator the integral equation is approximated by a system of linear algebraic equations. On the basis of the approximation used an increased order convergence solution has been obtained.

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1. Introduction

A one-dimensional Fredholm integral equation of the second kind with solution u from the Sobolev space $W_2^1(0,1)$ is constructed in the paper. The integral equation is approximated by a system of linear algebraic equations obtained on the basis of the Steklov averaging operator P. For the solution v of the system, the estimate

$$\|v - Pu\|_{L_2(\omega)} \leqslant ch^2 \|u\|_{W_2^1(0,1)} \tag{1.1}$$

is valid; here $\|\cdot\|_{L_2(\omega)}$ represents the norm of the mesh functions defined on ω, h is a mesh step. But this is convergence to average of the exact solution and the difference between the exact solution and its average Pu is $O(h)$, if $u \in W_2^1(0,1)$. The estimate $\|v - u\|_{L_2(\omega)} = O(h^2)$ is valid only in the case $u \in W_2^2(0,1)$.

The main result of the paper is the construction of such a mesh function $\widetilde{v}$ (on the basis of the already found v) for which the estimate

$$\|\widetilde{v} - u\|_{L_2(\omega)} \leqslant ch^2 \|u\|_{W_2^1(0,1)} \tag{1.2}$$

is true.

2. Basic Mesh Scheme

Consider the one-dimensional Fredholm integral equation of the second kind

$$u(x) - \lambda \int_0^1 K(x,y)u(y)\,dy = f(x), \quad x \in [0,1], \tag{2.1}$$

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where λ is a real parameter which is not a characteristic number. It is known (see, e.g., [1]) that for $f \in W_2^\alpha(0,1)$ and $K \in W_2^\alpha(0,1)^2$, $\alpha \geqslant 0$, there exists a unique solution $u \in W_2^\alpha(0,1)$.

Consider the following mesh in $[0,1] : \omega = \{x_i = ih : \ i = 1, 2, \ldots, N\}$, where $h = 1/N$. Let $\omega^2 = \omega \times \omega$. For the mesh functions defined on ω and ω^2 we use the notation $v_i = v(ih)$, $a_{ij} = a(ih, jh)$. For the one-dimensional mesh functions let us introduce the norm

$$\|v\|_{L_2(\omega)} = \left(\sum_{i=1}^{N} h|v_i|^2 \right)^{1/2}.$$

Define the following averaging operator

$$Pu = \frac{1}{h} \int\limits_{x-h}^{x} u(t)\, dt, \quad x \in \omega.$$

In order to indicate the integration variable in operator P, sometimes we will write P_{x_i}, P_{y_i}, P_{t_i}, $\ldots$, where $x_i = y_i = t_i = \ldots = ih$.

We approximate equation (2.1) by the linear algebraic system of equations

$$v_i - \lambda \sum_{j=1}^{N} h a_{ij} v_j = \varphi_i, \quad i = 1, 2, \ldots, N, \tag{2.2}$$

where $a_{ij} = P_{x_i} P_{y_j} K$, $\varphi_i = P_{x_i} f$. This approximation was studied in [1] and estimate (1.1) was obtained for it, though under the requirement of symmetry of the kernel K. Therefore we will state here shortly the proof of this estimate with the help of an improved method.

Lemma 2.1. *If the kernel K satisfies the condition*

$$|\lambda| \, \|K\|_{L_2(0,1)^2} < 1, \tag{2.3}$$

then system (2.2) *has a unique solution.*

Proof. Multiplying both parts of (2.2) by hv_i and summing up by $i = 1, 2, \ldots, N$, we obtain

$$\|v\|^2_{L_2(\omega)} - \lambda \sum_{i,j=1}^{N} h^2 a_{ij} v_i v_j = \sum_{i=1}^{N} h \varphi_i v_i.$$

From here

$$\|v\|^2_{L_2(\omega)} - |\lambda| \Bigg(\sum_{i,j=1}^{N} h^2 a_{ij}^2 \Bigg)^{1/2} \Bigg(\sum_{i=1}^{N} h v_i^2 \Bigg)^{1/2} \Bigg(\sum_{j=1}^{N} h v_j^2 \Bigg)^{1/2} \leqslant \|\varphi\|_{L_2(\omega)} \, \|v\|_{L_2(\omega)},$$

that is

$$\Bigg(1 - |\lambda| \Bigg(\sum_{i,j=1}^{N} h^2 a_{ij}^2 \Bigg)^{1/2} \Bigg) \|v\|_{L_2(\omega)} \leqslant \|\varphi\|_{L_2(\omega)}. \tag{2.4}$$

On the other hand, applying the Cauchy — Buniakovski inequality, we come to

$$\sum_{i,j=1}^{N} h^2 a_{ij}^2 = \sum_{i,j=1}^{N} h^{-2} \Bigg(\int\limits_{x_{i-1}}^{x_i} \int\limits_{y_{j-1}}^{y_j} K(x,y) dx dy \Bigg)^2 \leqslant \sum_{i,j=1}^{N} h^{-2} \int\limits_{x_{i-1}}^{x_i} \int\limits_{y_{j-1}}^{y_j} dx dy \int\limits_{x_{i-1}}^{x_i} \int\limits_{y_{j-1}}^{y_j} K^2(x,y) dx dy = \|K\|^2_{L_2(0,1)^2}.$$

Therefore, from (2.4) follows

$$\|v\|_{L_2(\omega)} \leqslant c_1\|\varphi\|_{L_2(\omega)}, \quad c_1 = |\lambda|(1-|\lambda|\,\|K\|_{L_2(0,1)^2})^{-1}. \tag{2.5}$$

Thus, if $\varphi(x)=0$, $x\in\omega$, then $v(x)=0$, $x\in\omega$ and, consequently, the solution of system (2.2) exists and is unique. This completes the proof of the lemma. □

Let u be a solution of the integral equation (2.1) and v — a solution of the mesh scheme (2.2). Then for the error $z=v-Pu$ ve obtain the problem

$$z_i - \lambda\sum_{j=1}^{N} ha_{ij}z_j = \lambda\psi_i, \quad i=1,2,\ldots,N \tag{2.6}$$

where $\psi_i = \sum_{j=1}^{N} hP_{x_i}P_{y_j}K(x,y)\,P_{t_j}u(t) - \sum_{j=1}^{N} hP_{x_i}P_{y_j}K(x,y)u(y)$, $i=1,2,\ldots,N$. It follows from (2.5) that the a priori estimate

$$\|z\|_{L_2(\omega)} \leqslant c_1|\lambda|\,\|\psi\|_{L_2(\omega)} \tag{2.7}$$

is valid for the solution of problem (2.6).

In order to estimate the convergence rate of the mesh scheme (2.2), it is enough to estimate the norm of the approximation error ψ. To this end, we apply the well-known tecnique (see e.g. [2, 3]).

It is easy to verify that the approximation error can be written in the form

$$\psi_i = 0.5\sum_{j=1}^{N} hP_{x_i}P_{y_j}P_{t_j}\big(K(x,y)-K(x,t)\big)\big(u(t)-u(y)\big).$$

This gives the following $\|\psi\|_{L_2(\omega)} \leqslant (h^2/2)\|\partial K/\partial y\|_{L_2(0,1)^2}\,\|u'\|_{L_2(0,1)}$ and therefore (2.7) implies

$$\|v-Pu\|_{L_2(\omega)} \leqslant ch^2\|u\|_{W_2^1(0,1)}. \tag{2.8}$$

3. Emprovement of the Approximate Solution

Let v be a solution of problem (2.2); let us define one more approximation:

$$\widetilde{v}_i = f(x_i) + \lambda\sum_{j=1}^{N} hP_{t_j}K(x_i,t)v_j, \quad i=1,2,\ldots,N. \tag{3.1}$$

Theorem 3.1. *If* $f\in W_2^1(0,1)$, $K\in W_2^1(0,1)^2$ *and condition* (2.3) *holds, then the function* $\widetilde{v}$ *determined from the* (3.1) *converges to the solution of equation* (2.1) *and estimate* (1.2) *is valid.*

Proof. Taking into account (2.1), (2.2), (3.1), we can write for the error $\widetilde{v}-u$ as follows:

$$\widetilde{v}_i - u(x_i) = \lambda\sum_{j=1}^{N} hP_{y_j}K(x_i,y)v_j - \lambda\int_0^1 K(x_i,y)u(y)\,dy =$$

$$\lambda\sum_{j=1}^{N} hP_{y_j}K(x_i,y)v_j - \lambda\sum_{j=1}^{N} hP_{y_j}K(x_i,y)u(y) = \lambda(A_i+0.5B_i), \tag{3.2}$$

where

$$A_i = \sum_{j=1}^{N} hP_{y_j}K(x_i, y)(v_j - P_{x_j}u(x)), \quad B_i = \sum_{j=1}^{N} hP_{y_j}P_{t_j}\big(K(x_i, y) - K(x_i, t)\big)\big(u(t) - u(y)\big).$$

It follows from (3.2) that

$$\|\widetilde{v} - u\|_{L_2(\omega)} \leqslant |\lambda|\left(\sum_{i=1}^{N} hA_i^2\right)^{1/2} + 0.5|\lambda|\left(\sum_{i=1}^{N} hB_i^2\right)^{1/2}. \tag{3.3}$$

By virtue of the Cauchy — Buniakovski inequality we have

$$\sum_{i=1}^{N} hA_i^2 \leqslant \sum_{i=1}^{N} h\sum_{j=1}^{N} h\big(P_{y_j}K(x_i, y)\big)^2 \sum_{j=1}^{N} h\big(v_j - P_{y_j}u(y)\big)^2 \leqslant$$

$$\sum_{i=1}^{N} h\sum_{j=1}^{N} \int\limits_{y_{j-1}}^{y_j} (K(x_i, y))^2\, dy\, \|v - Pu\|^2_{L_2(\omega)} = \sum_{i=1}^{N} h\int\limits_0^1 (K(x_i, y))^2\, dy\, \|v - Pu\|^2_{L_2(\omega)}. \tag{3.4}$$

Further, it is easy to see that

$$K(x_i, y) = \frac{1}{h}\int\limits_{x_{i-1}}^{x_i}\int\limits_{x}^{x_i} \frac{\partial K(\xi, y)}{\partial \xi}\, d\xi dx + \frac{1}{h}\int\limits_{x_{i-1}}^{x_i} K(x, y)\, dx,$$

Therefore,

$$|K(x_i, y)|^2 \leqslant h\int\limits_{x_{i-1}}^{x_i} \left|\frac{\partial K(x, y)}{\partial x}\right|^2 dx + \frac{2}{h}\int\limits_{x_{i-1}}^{x_i} |K(x, y)|^2\, dx$$

and we obtain

$$\sum_{i=1}^{N} h(K(x_i, y))^2 \leqslant \sum_{i=1}^{N} h^2\int\limits_{x_{i-1}}^{x_i} \left|\frac{\partial K(x, y)}{\partial x}\right|^2 dx +$$

$$\sum_{i=1}^{N} 2\int\limits_{x_{i-1}}^{x_i} |K(x, y)|^2 dx \leqslant 2\int\limits_0^1 \left(\left|\frac{\partial K(x, y)}{\partial x}\right|^2 + |K(x, y)|^2\right) dx. \tag{3.5}$$

Substituting (3.5) into (3.4), we get

$$\sum_{i=1}^{N} hA_i^2 \leqslant 2\|K\|^2_{W_2^1(0,1)^2}\, \|v - Pu\|^2_{L_2(\omega)}. \tag{3.6}$$

Estimate now the second addend in the right hand side of (3.3)

$$\sum_{i=1}^{N} hB_i^2 = \sum_{i=1}^{N} h\left[\sum_{j=1}^{N} \frac{1}{h}\int\limits_{y_{j-1}}^{y_j}\int\limits_{t_{j-1}}^{t_j} \big(K(x_i, y) - K(x_i, t)\big)\big(u(t) - u(y)\big)\, dydt\right]^2 \leqslant$$

$$\sum_{i=1}^{N} h\left[\sum_{j=1}^{N}\frac{1}{h}\Big(\int\limits_{y_{j-1}}^{y_j}\int\limits_{t_{j-1}}^{t_j}\big(K(x_i,y)-K(x_i,t)\big)^2\,dy\,dt\Big)^{1/2}\Big(\int\limits_{y_{j-1}}^{y_j}\int\limits_{t_{j-1}}^{t_j}\big(u(t)-u(y)\big)^2\,dy\,dt\Big)^{1/2}\right]^2\leqslant$$

$$\sum_{i=1}^{N}\frac{1}{h}\left[\sum_{j=1}^{N}\frac{1}{h}\int\limits_{y_{j-1}}^{y_j}\int\limits_{t_{j-1}}^{t_j}|K(x_i,y)-K(x_i,t)|^2\,dy\,dt\sum_{j=1}^{N}\int\limits_{y_{j-1}}^{y_j}\int\limits_{t_{j-1}}^{t_j}|u(t)-u(y)|^2\,dy\,dt\right]\leqslant$$

$$\sum_{i=1}^{n} h^2\sum_{j=1}^{N}\int\limits_{y_{j-1}}^{y_j}\int\limits_{t_{j-1}}^{t_j}\big(K(x_i,y)-K(x_i,t)\big)^2\,dy\,dt\|u'\|^2_{L_2(0,1)}. \tag{3.7}$$

It is easy to see that

$$K(x_i,y)-K(x_i,t)=\frac{1}{h}\int\limits_{\xi_{i-1}}^{\xi_i}\int\limits_{\xi}^{x_i}\frac{\partial K(x,y)}{\partial x}dxd\xi+\frac{1}{h}\int\limits_{\xi_{i-1}}^{\xi_i}\int\limits_{t}^{y}\frac{\partial K(\xi,\zeta)}{\partial \zeta}d\xi d\zeta+\frac{1}{h}\int\limits_{\xi_{i-1}}^{\xi_i}\int\limits_{x_i}^{\xi}\frac{\partial K(x,t)}{\partial x}dxd\xi$$

from which

$$|K(x_i,y)-K(x_i,t)|\leqslant\int\limits_{x_{i-1}}^{x_i}\left|\frac{\partial K(x,y)}{\partial x}\right|dx+\frac{1}{h}\int\limits_{x_{i-1}}^{x_i}\int\limits_{\zeta_{j-1}}^{\zeta_j}\left|\frac{\partial K(x,\zeta)}{\partial \zeta}\right|dxd\zeta+\int\limits_{x_{i-1}}^{x_i}\left|\frac{\partial K(x,t)}{\partial x}\right|dx.$$

Therefore, from (3.7) we have

$$\sum_{i=1}^{N} hB_i^2\leqslant\sum_{i=1}^{N} h^2\sum_{j=1}^{N}\int\limits_{y_{j-1}}^{y_j}\int\limits_{t_{j-1}}^{t_j}\left[3h\int\limits_{x_{i-1}}^{x_i}\left|\frac{\partial K(x,y)}{\partial x}\right|^2dx+\right.$$

$$\left.3\int\limits_{x_{i-1}}^{x_i}\int\limits_{\zeta_{j-1}}^{\zeta_j}\left|\frac{\partial K(x,\zeta)}{\partial \zeta}\right|^2dxd\zeta+3h\int\limits_{x_{i-1}}^{x_i}\left|\frac{\partial K(x,t)}{\partial x}\right|^2dx\right]dy\,dt\|u'\|^2_{L_2(0,1)}\leqslant$$

$$\left(6h^4\int\limits_0^1\int\limits_0^1\left|\frac{\partial K(x,y)}{\partial x}\right|^2dxdy+3h^4\int\limits_0^1\int\limits_0^1\left|\frac{\partial K(x,y)}{\partial y}\right|dxdy\right)\|u'\|^2_{L_2(0,1)}. \tag{3.8}$$

Due to (2.8), (3.6), (3.8) estimate (1.2) follows from (3.3). The theorem has been proved. □

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