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ITERATIVE SOLUTION OF A NONLINEAR STATIC BEAM EQUATION
ІТЕРАЦІЙНИЙ РОЗВ'ЯЗОК НЕЛІНІЙНОГО РІВНЯННЯ СТАТИЧНОЇ БАЛКИ

The paper deals with a boundary-value problem for the nonlinear integro-differential equation $u'''' - m\Big(\int_0^l u'^2\,dx\Big)u'' = f(x,u,u')$, $m(z) \geq \alpha > 0$, $0 \leq z < \infty$, modeling the static state of the Kirchhoff beam. The problem is reduced to a nonlinear integral equation, which is solved using the Picard iteration method. The convergence of the iteration process is established and the error estimate is obtained.

Розглядається крайова задача для нелінійного інтегро-диференціального рівняння $u'''' - m\Big(\int_0^l u'^2\,dx\Big)u'' = f(x,u,u')$, $m(z) \geq \alpha > 0$, $0 \leq z < \infty$, що моделює статичний стан балки Кірхгоффа. Задача зводиться до нелінійного інтегрального рівняння, яке розв'язується за допомогою ітераційного методу Пікара. Встановлено збіжність цього ітераційного процесу та отримано оцінку для похибки.

1. Statement of the problem. Let us consider the nonlinear beam equation

$$u''''(x) - m\left(\int_0^l u'^2(x)\,dx\right)u''(x) = f(x,u(x),u'(x)), \quad x \in (0,l), \tag{1}$$

with the conditions

$$u(0) = u(l) = 0, \qquad u''(0) = u''(l) = 0. \tag{2}$$

Here, $u = u(x)$ is the displacement function of length l of the beam subjected to the action of a force given by the function $f(x,u,u')$, the function $m(z)$,

$$m(z) \geq \alpha > 0, \quad 0 \leq z < \infty, \tag{3}$$

describes the type of a relation between stress and strain. Namely, if the function $m(z)$ is linear, this means that the above relation is consistent with Hooke's linear law, while otherwise we deal with material nonlinearities.

The equation (1) is the stationary problem associated with the equation

$$u_{tt} + u_{xxxx} - m\left(\int_0^l u_x^2\,dx\right)u_{xx} = f(x,t,u,u_x),$$

$$m(z) \geq \text{const} > 0,$$

which for the case where $m(z) = m_0 + m_1 z$, $m_0, m_1 > 0$, and $f(x,t,u,u_x) = 0$, was proposed by Woinowsky-Krieger [10] as a model of deflection of an extensible dynamic beam with hinged ends. The nonlinear term $\int_0^l u_x^2\,dx$ was for the first time used by Kirchhoff [3] who generalized D'Alembert's classical linear model. Therefore (1) is frequently called a Kirchhoff type equation for a static beam.

The problem of construction of numerical algorithms and estimation of their accuracy for equations of type (1) is investigated in [1, 5, 8, 9]. In [4], the existence of a solution of the problem (1), (2) is proved when the right-hand part of the equation is written in the form $q(x)f(x,u,u')$, where $f \in C([0,l]\times[0,\infty)\times\mathbb{R})$ is a nonnegative function and $q \in C[0,l]$ is a positive function.

In the present paper, in order to obtain an approximate solution of the problem (1), (2), an approach is used, which differs from those applied in the above-mentioned references. It consists in reducing the problem (1), (2) by means of Green's function to a nonlinear integral equation, to solve which we use the iterative process. The condition for the convergence of the method is established and its accuracy is estimated.

The Green's function method with a further iteration procedure has been applied by us previously also to a nonlinear problem for the axially symmetric Timoshenko plate [6].

2. Assumptions. Let us assume that besides (3) the function $m(z)$ also satisfies the Lipschitz condition

$$\big|m(z_1)-m(z_2)\big| \le l_1|z_2-z_1|, \qquad 0\le z_1,z_2<\infty, \quad l_1=\mathrm{const}>0.$$

Suppose that $f(x,u,v)\in L_2\left((0,l),\mathbb{R},\mathbb{R}\right)$ and, additionally, that the inequalities

$$\big|f(x,u,v)\big| \le \sigma_1(x)+\sigma_2(x)\,|u|+\sigma_3(x)\,|v|, \tag{4}$$

$$\big|f(x,u_2,v_2)-f(x,u_1,v_1)\big| \le l_2(x)\,|u_2-u_1|+l_3(x)\,|v_2-v_1|, \tag{5}$$

where

$$0<x<l, \quad u,v,u_i,v_i\in\mathbb{R}, \quad i=1,2, \quad \sigma_1(x)\in L_2(0,l), \quad \sigma_i(x),\, l_i(x)\in L_\infty(0,l), \quad i=2,3,$$

$$\sigma_1(x)\ge\mathrm{const}>0, \quad \sigma_i(x)\ge 0, \quad l_i(x)>0, \quad i=2,3,$$

are fulfilled.

We impose one more restriction on the beam length l and the parameters α and $\sigma_2(x)$, $\sigma_3(x)$ from the conditions (3) and (4), (5)

$$\omega=\alpha+\left(\frac{\pi}{l}\right)^2-\frac{l}{\pi}\left(\frac{2}{\pi}\big\|\sigma_2(x)\big\|_\infty+\big\|\sigma_3(x)\big\|_\infty\right)>0. \tag{6}$$

Let us assume that there exists a solution of the problem (1), (2) and $u\in W_0^{2,2}(0,l)$ [2].

3. The method. We will need the Green function for the problem

$$\begin{gathered} v''''(x)-av''(x)=\psi(x), \\ 0<x<l, \quad a=\mathrm{const}>0, \\ v(0)=v(l)=0, \qquad v''(0)=v''(l)=0. \end{gathered} \tag{7}$$

In order to obtain this function, we split the problem (7) into two problems

$$w''(x) - aw(x) = \psi(x),$$

$$w(0) = w(l) = 0,$$

and

$$v''(x) = w(x),$$

$$v(0) = v(l) = 0.$$

Calculations convince us that

$$w(x) = -\frac{1}{\sqrt{a}\sinh\left(\sqrt{a}l\right)}\left(\int\limits_0^x \cosh\left(\sqrt{a}(x-l)\right)\,\cosh\left(\sqrt{a}\xi\right)\psi(\xi)d\xi + \right.$$

$$\left. + \int\limits_x^l \cosh\left(\sqrt{a}x\right)\cosh\left(\sqrt{a}(\xi-l)\right)\psi(\xi)\,d\xi\right),$$

$$v(x) = \frac{1}{l}\left(\int\limits_0^x (x-l)\xi w(\xi)d\xi + \int\limits_x^l x(\xi-l)w(\xi)d\xi\right).$$

Substituting the first of these formulas into the second and performing integration by parts, we obtain

$$v(x) = \frac{1}{a}\left(\int\limits_0^x \left(k_1(l-x)\xi + k_2\sinh\left(\sqrt{a}(x-l)\right)\sinh\left(\sqrt{a}\xi\right)\psi(\xi)\right)d\xi + \right.$$

$$\left. + \int\limits_x^l \left(k_1x(l-\xi) + k_2\sinh\left(\sqrt{a}x\right)\sinh\left(\sqrt{a}(\xi-l)\right)\psi(\xi)\right)d\xi\right),$$

$$k_1 = \frac{1}{l}, \qquad k_2 = \frac{1}{\sqrt{a}\sinh\left(\sqrt{a}l\right)}.$$

The application of (7) to the problem (1), (2) makes it possible to replace the latter problem by the integral equation

$$u(x) = \int\limits_0^l G(x,\xi)f\left(\xi,u(\xi),u'(\xi)\right)d\xi, \quad 0 < x < l, \tag{8}$$

where

$$G(x,\xi) = \frac{1}{\tau}\begin{cases} \dfrac{1}{l}(x-l)\xi + \dfrac{1}{\sqrt{\tau}\sinh\left(\sqrt{\tau}\,l\right)}\sinh\left(\sqrt{\tau}(x-l)\right)\sinh\left(\sqrt{\tau}\xi\right), & 0 < \xi \le x < l, \\ \dfrac{1}{l}x(\xi-l) + \dfrac{1}{\sqrt{\tau}\sinh\left(\sqrt{\tau}\,l\right)}\sinh\left(\sqrt{\tau}x\right)\sinh\left(\sqrt{\tau}(\xi-l)\right), & 0 < x \le \xi < l, \end{cases}$$

$$\tau = m\left(\int\limits_0^l u'^2(x)\,dx\right).$$

The equation (8) is solved by the Picard iteration method. After choosing a function $u_0(x)$, $0 \le x \le l$, which together with its second derivative vanish for $x = 0$ and $x = l$, we find subsequent approximations by the formula

$$u_{k+1}(x) = \int\limits_0^l G_k(x,\xi) f\big(\xi, u_k(\xi), u_k'(\xi)\big)\,d\xi, \qquad 0 < x < l, \quad k = 0,1,\ldots, \tag{9}$$

where

$$G_k(x,\xi) = \frac{1}{\tau_k}\begin{cases} \dfrac{1}{l}(x-l)\xi + \dfrac{1}{\sqrt{\tau_k}\sinh\big(\sqrt{\tau_k}\,l\big)}\sinh\big(\sqrt{\tau_k}(x-l)\big)\sinh\big(\sqrt{\tau_k}\xi\big), & 0 < \xi \le x < l, \\ \dfrac{1}{l}x(\xi-l) + \dfrac{1}{\sqrt{\tau_k}\sinh\big(\sqrt{\tau_k}\,l\big)}\sinh\big(\sqrt{\tau_k}x\big)\sinh\big(\sqrt{\tau_k}(\xi-l)\big), & 0 < x \le \xi < l, \end{cases}$$

$$\tau_k = m\left(\int\limits_0^l {u_k'}^2(x)\,dx\right),$$

and $u_k(x)$ is the kth approximation of the solution of the equation (8).

4. The problem for the error. Our aim is to estimate the error of the method, by which we understand the difference between the approximate and exact solutions

$$\delta u_k(x) = u_k(x) - u(x), \quad k = 0,1,\ldots. \tag{10}$$

For this, it is advisable to use not the formula (9), but the system of equalities

$$u_{k+1}''''(x) - m\left(\int\limits_0^l {u_k'}^2(x)\,dx\right)u_{k+1}''(x) = f\big(x, u_k(x), u_k'(x)\big), \tag{11}$$

$$u_k(0) = u_k(l) = 0, \qquad u_k''(0) = u_k''(l) = 0, \tag{12}$$

which follows from (9).

If we subtract the respective equalities in (1) and (2) from (11) and (12), then we get

$$\delta u_k''''(x) - \frac{1}{2}\left(\left[m\left(\int\limits_0^l {u'}_{k-1}^2(x)\,dx\right) + m\left(\int\limits_0^l {u'}^2(x)\,dx\right)\right]\delta u_k''(x) + \right.$$

$$\left. + \left[m\left(\int\limits_0^l {u'}_{k-1}^2(x)\,dx\right) - m\left(\int\limits_0^l {u'}^2(x)\,dx\right)\right]\left(u_k''(x) + u''(x)\right)\right) =$$

$$= f\big(x, u_{k-1}(x), u_{k-1}'(x)\big) - f\big(x, u(x), u'(x)\big), \tag{13}$$

$$\delta u_k(0) = \delta u_k(l) = 0, \qquad \delta u''(0) = \delta u''(l) = 0, \quad k = 1,2,\ldots. \tag{14}$$

We will come back to (13), (14) to estimate the error of the method (9). In meantime we have to derive several a priori estimates.

5. Auxiliary inequalities. Let

$$\|u(x)\|_p = \left(\int\limits_0^l \left(\frac{d^p u}{dx^p}(x)\right)^2 dx\right)^{1/2}, \quad p = 0, 1, 2, \qquad \|u(x)\| = \|u(x)\|_0. \tag{15}$$

The symbol $(\cdot, \cdot)$ is understood as a scalar product in $L_2(0, l)$.

Lemma 1. *The following estimates are true*:

$$\|u(x)\| \le \frac{l}{\pi} \|u(x)\|_1, \qquad \|u(x)\|_1 \le \frac{l}{\pi} \|u(x)\|_2, \tag{16}$$

respectively, for $u(x) \in W_0^{1,2}(0, l)$ *and* $u(x) \in W^{2,2}(0, l) \cap W_0^{1,2}(0, l)$.

Proof. Indeed, the first estimate of (16) is Friedrich's inequality (see, e.g., [7, p. 192]). Applying this inequality and taking into account that

$$\|u(x)\|_1^2 = u(x)u'(x)|_0^l - (u(x), u''(x)) = -(u(x), u''(x)) \le \|u(x)\| \, \|u(x)\|_2,$$

we get the second inequality of (16).

Lemma 2. *The inequality*

$$\|f(x, u(x), u'(x))\| \le \|\sigma_1(x)\| + \left(\frac{l}{\pi}\|\sigma_2(x)\|_\infty + \|\sigma_3(x)\|_\infty\right)\|u(x)\|_1 \tag{17}$$

is fulfilled for $u(x) \in W_0^{1,2}(0, l)$.

Proof. By (4) we write

$$\|f(x, u(x), u'(x))\| \le \|\sigma_1(x)\| + \|\sigma_2(x)\|_\infty \|u(x)\| + \|\sigma_3(x)\|_\infty \|u'(x)\|.$$

Recall also (16). The result is (17).

Lemma 3. *For the solution of the problem* (1), (2) *we have the inequality*

$$\|u(x)\|_1 \le c_1, \tag{18}$$

where

$$c_1 = \frac{l}{\omega\pi}\|\sigma_1(x)\|. \tag{19}$$

Proof. We multiply the equation (1) by $u(x)$ and integrate the resulting equality with respect to x from 0 to l. By using (2), we get

$$\|u(x)\|_2^2 + m\Big(\|u(x)\|_1^2\Big)\|u(x)\|_1^2 = \big(f(x, u(x), u'(x)), u(x)\big).$$

By (16) and (3), we obtain

$$\left(\alpha + \left(\frac{\pi}{l}\right)^2\right) \|u(x)\|_1^2 \le \frac{l}{\pi}\|f(x, u(x), u'(x))\| \, \|u(x)\|_1.$$

Therefore, by (17),

$$\left(\alpha + \left(\frac{\pi}{l}\right)^2 - \left(\frac{l}{\pi}\right)^2 \|\sigma_2(x)\|_\infty - \frac{l}{\pi}\|\sigma_3(x)\|_\infty\right)\|u(x)\|_1 \le \frac{l}{\pi}\|\sigma_1(x)\|.$$

From this relation and (6) follows (18).

Lemma 4. *Suppose we are given some numbers* $v_k \geq 0,\ k = 0, 1, \ldots,$ *for which the inequality*

$$v_k \leq a v_{k-1} + b, \quad k = 1, 2, \ldots, \tag{20}$$

where $0 \leq a < 1,\ b > 0,$ *holds. Then we have the following uniform estimate with respect to the index* k:

$$v_k \leq \frac{b}{1-a} + a \max\left(0, v_0 - \frac{b}{1-a}\right), \quad k = 1, 2, \ldots . \tag{21}$$

Proof. By virtue of (20), by the method of mathematical induction, we have $v_k \leq a^k v_0 + (a^{k-1} + a^{k-2} + \ldots + 1)b,\ k = 1, 2, \ldots,$ which implies

$$v_k \leq a^k v_0 + \frac{1-a^k}{1-a} b = \frac{b}{1-a} + a^k\left(v_0 - \frac{b}{1-a}\right). \tag{22}$$

Let us denote $\nu_k = a^k\left(v_0 - \dfrac{b}{1-a}\right)$ and consider two cases $v_0 \leq \dfrac{b}{1-a}$ and $v_0 > \dfrac{b}{1-a}$. In the first case $\nu_k \leq 0$ and by virtue of (22) $v_k \leq \dfrac{b}{1-a},\ k = 1, 2, \ldots .$ In the second case $\nu_k > 0,$ $\max \nu_k = \nu_1 = a\left(v_0 - \dfrac{b}{1-a}\right)$, which, by virtue of (22), yields $v_k = \dfrac{b}{1-a} + a\left(v_0 - \dfrac{b}{1-a}\right),$ $k = 1, 2, \ldots .$ From this conclusions the validity of the estimate (21) follows.

Lemma 5. *Approximations of the iteration method* (9) *satisfy the inequality*

$$\left\|u_k(x)\right\|_1 \leq c_2, \quad k = 1, 2, \ldots, \tag{23}$$

where

$$c_2 = \begin{cases} c_1, & \text{if} \quad \left\|\sigma_2(x)\right\|_\infty + \left\|\sigma_3(x)\right\|_\infty = 0, \\ c_1 + c_0 \max\big(0, \|u_0(x)\|_1 - c_1\big), & \text{if} \quad \left\|\sigma_2(x)\right\|_\infty + \left\|\sigma_3(x)\right\|_\infty \neq 0, \end{cases} \tag{24}$$

$$c_0 = \left(1 + \omega\left(\left(\frac{l}{\pi}\right)^2 \left\|\sigma_2(x)\right\|_\infty + \frac{l}{\pi}\left\|\sigma_3(x)\right\|_\infty\right)^{-1}\right)^{-1}.$$

Proof. Replace k by the index $k-1$ in the equation (11), multiply the resulting relation by $u_k(x)$ and integrate over x from 0 to l. Taking (12) into account, we get

$$\left\|u_k(x)\right\|_2^2 + m\Big(\left\|u_{k-1}(x)\right\|_1^2\Big) \left\|u_k(x)\right\|_1^2 = \Big(f\big(x, u_{k-1}(x), u'_{k-1}(x)\big), u_k(x)\Big), \quad k = 1, 2, \ldots .$$

Applying (3) and (15), we have

$$\left(\alpha + \left(\frac{\pi}{l}\right)^2\right)\left\|u_k(x)\right\|_1^2 \leq \frac{l}{\pi}\left\|f\big(x, u_{k-1}(x), u'_{k-1}(x)\big)\right\| \left\|u_k(x)\right\|_1,$$

which implies

$$\left(\alpha + \left(\frac{\pi}{l}\right)^2\right) \|u_k(x)\|_1 \leq \frac{l}{\pi}\left\|f(x, u_{k-1}(x)\, u'_{k-1}(x))\right\|.$$

Hence, by using (17), we conclude that

$$\left\|u_k(x)\right\|_1 \leq \frac{1}{\alpha + \left(\dfrac{\pi}{l}\right)^2} \frac{l}{\pi} \left(\left\|\sigma_1(x)\right\| + \left(\left\|\sigma_2(x)\right\|_\infty \frac{l}{\pi} + \left\|\sigma_3(x)\right\|_\infty \right) \left\|u_{k-1}(x)\right\|_1 \right).$$

This relation is an inequality of type (20), where

$$v_k = \left\|u_k(x)\right\|_1,$$

$$a = \frac{1}{\alpha + \left(\dfrac{\pi}{l}\right)^2} \frac{l}{\pi} \left(\left\|\sigma_2(x)\right\|_\infty \frac{l}{\pi} + \left\|\sigma_3(x)\right\|_\infty \right), \qquad b = \frac{1}{\alpha + \left(\dfrac{\pi}{l}\right)^2} \frac{l}{\pi} \left\|\sigma_1(x)\right\|.$$

Let us apply (6), (19) to these formulas and carry out some calculations. As a result, for $\left\|\sigma_2(x)\right\|_\infty + \left\|\sigma_3(x)\right\|_\infty = 0$, we obtain $a = 0$ and $\dfrac{b}{1-a} = c_1$, while for $\left\|\sigma_2(x)\right\|_\infty + \left\|\sigma_3(x)\right\|_\infty \neq 0$ we have $a = c_0$ and $\dfrac{b}{1-a} = c_1$. By considering these two cases with the estimate (21), we get convinced that (23) is valid.

By Lemmas 3 and 5 it will be natural to require that the initial approximation $u_0(x)$ in (9) satisfy the condition

$$\left\|u_0(x)\right\|_1 \leq c_1. \tag{25}$$

Then, by virtue of (24) and (23), we have $\left\|u_k(x)\right\|_1 \leq c_1$, which, with (19) taken into account, implies

$$\left\|u_k(x)\right\|_1 \leq \frac{l}{\omega\pi} \left\|\sigma_1(x)\right\|, \quad k = 0, 1, \ldots . \tag{26}$$

6. Convergence of the method. Multiplying (13) by $\delta u_k(x)$, integrating the resulting equality with respect to x from 0 to l and using (14), we come to the relation

$$\left\|\delta u_k(x)\right\|_2^2 + \frac{1}{2} \Big(\Big(m\Big(\left\|u_{k-1}(x)\right\|_1^2\Big) + m\Big(\left\|u(x)\right\|_1^2\Big) \Big) \left\|\delta u_k(x)\right\|_1^2 +$$

$$+ \Big(m\Big(\left\|u_{k-1}(x)\right\|_1^2\Big) - m\Big(\left\|u(x)\right\|_1^2\Big) \Big) \big(u_k'(x) + u'(x), \delta u_k'(x)\big) \Big) =$$

$$= \Big(f\big(x, u_{k-1}(x), u_{k-1}'(x)\big) - f\big(x, u(x), u'(x)\big), \delta u_k(x) \Big), \quad k = 1, 2, \ldots .$$

Applying (3) – (5) and (16), we first obtain

$$\left\|\delta u_k(x)\right\|_2^2 + \alpha \left\|\delta u_k(x)\right\|_1^2 \leq \frac{1}{2} l_1 \prod_{p=0}^{1} \left| \big(u_{k-p}'(x) + u', \delta u_{k-p}'(x)\big) \right| +$$

$$+ \Big(\|l_2(x)\|_\infty \, \|\delta u_{k-1}(x)\| + \|l_3(x)\|_\infty \, \|\delta u_{k-1}'(x)\| \Big) \left\|\delta u_k(x)\right\| \leq$$

$$\leq \frac{1}{2} l_1 \prod_{p=0}^{1} \Big(\|u_{k-p}(x)\|_1 + \left\|u(x)\right\|_1 \Big) \left\|\delta u_{k-p}(x)\right\|_1 +$$

$$+\frac{l}{\pi}\left(\frac{l}{\pi}\|l_2(x)\|_\infty + \|l_3(x)\|_\infty\right)\prod_{p=0}^{1}\|\delta u_{k-p}(x)\|_1,$$

and after that, by virtue of (18) and (26), we have

$$\big\|\delta u_k(x)\big\|_1 \le \left(\alpha + \left(\frac{\pi}{l}\right)^2\right)^{-1}\left(\frac{1}{2}l_1\prod_{p=0}^{1}\left(\|u_{k-p}(x)\|_1 + \|u\|_1(x)\right) + \right.$$

$$\left. + \left(\frac{l}{\pi}\right)^2\|l_2(x)\|_\infty + \frac{l}{\pi}\|l_3(x)\|_\infty\right)\|\delta u_{k-1}\|_1 \le q\|\delta u_{k-1}\|_1, \quad k = 1, 2, \ldots,$$

where

$$q = \left(\alpha + \left(\frac{\pi}{l}\right)^2\right)^{-1}\left(2c_1^2 l_1 + \|l_2(x)\|_\infty\left(\frac{l}{\pi}\right)^2 + \big\|l_3(x)\big\|_\infty\frac{l}{\pi}\right).$$

Taking (10), (19) and (16) into consideration we come to the following result.

Theorem 1. *Let assumptions* (3) – (6) *and* (25) *be fulfilled. Suppose besides*

$$q = \frac{1}{\alpha + \left(\frac{\pi}{l}\right)^2}\left(\frac{l}{\pi}\right)^2\left(2l_1\left(\frac{\big\|\sigma_1(x)\big\|}{\omega}\right)^2 + \|l_2(x)\|_\infty + \frac{\pi}{l}\|l_3(x)\|_\infty\right) < 1.$$

Then the approximations of the iteration method (9) *converge to the exact solution of the problem* (1), (2) *and for the error the estimate*

$$\big\|u_k(x) - u(x)\big\|_p \le \left(\frac{l}{\pi}\right)^{1-p} q^k\big\|u_0(x) - u(x)\big\|_1, \qquad k = 1, 2, \ldots, \quad p = 0, 1,$$

is true.

7. Numerical experiment. The results on the convergence of the iteration process (9) to the sought function (8) were confirmed by numerical experiments. For illustration, we present here the result of numerical computations of one of the test problem.

We consider a special case, where $m(z) = m_0 + m_1 z$, $m_0 = 1$, $m_1 = \frac{1}{2}$, the beam length $l = 1$, exact solution $u(x) = x(x-1)\big(x^2 - x - 1\big)$, i.e., $u(x) = x^4 - 2x^3 + x$, the right-hand side of the equation (1)

$$f\big(x, u(x), u'(x)\big) = \frac{1}{35}\Big(43.5u'^2(x) - 348x^3u'(x) -$$

$$-1566u(x) + 696x^6 - 3132x^3 + 2088x + 796.5\Big).$$

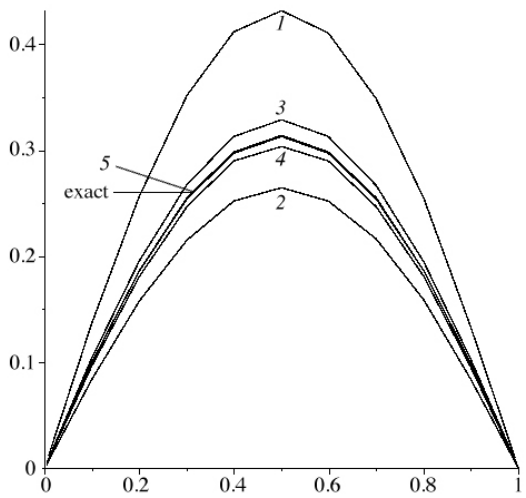

Fig. 1. Case of five iterations, $n = 10$.

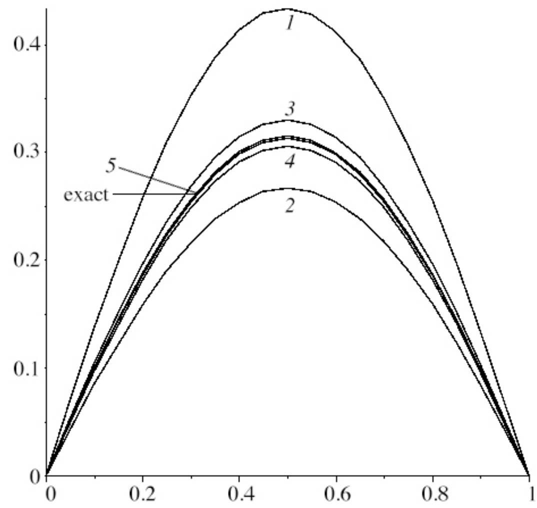

Fig. 2. Case of five iterations, $n = 20$.

We carried out nine iterations. In the case of division of the interval $[0, 1]$ into $n = 10, 20$ parts (with $h = 0.1, 0.05$, respectively) the integrals were computed using the generalized trapezoid rule. Here we applied the following definition of the kth iteration error:

$$\text{error } k = \max_{i=0,1,\ldots,n} \big\{ abs\big(u_k(x_i) - u_{k-1}(x_i)\big)\big\}, \quad x_i = ih, \quad k = 1, 2, \ldots, 9.$$

The error of numerical values are given in the table below.

n	error k						
	1	2	3	4	5	7	9
10	0.43203	0.16734	0.06405	0.02473	0.00953	0.00142	0.00021
20	0.43328	0.16715	0.06365	0.02446	0.00938	0.00138	0.00020

Initial approximation of approximate solution $u_0(x) = 0$. In the case of five iterations for $n = 10, 20$ the exact and approximate solutions are graphically illustrated (Figs. 1, 2).

In Figures 1 and 2 the numbers *1*, *2*, *3*, *4*, *5* corespond to the numbers of iterations, respectively. And the graph of the fifth iteration $u_5(x)$ actually coincides with the graph of the exact solution $u(x)$.

Numerical experiments clearly show the convergence of approximate solutions with the increase of the number of iterations and the influence of n number of the interval divisions on the rate of convergence.

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