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On the convergence rate of a difference solution of the Poisson equation with fully nonlocal constraints

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Abstract. We consider the Poisson equation in a rectangular domain. Instead of the classical specification of boundary data, we impose an integral constraints on the inner stripe adjacent to boundary having the width ξ. The corresponding finite-difference scheme is constructed on a mesh, which selection does not depend on the value ξ. It is proved the unique solvability of the scheme. An a priori estimate of the discretization error is obtained with the help of energy inequality method. It is proved that the scheme is convergent with the convergence rate of order $s-1$, when the exact solution belongs to the fractional Sobolev space of order s $(1 < s \leqslant 3)$.

Keywords: integral conditions, energy inequalities, difference scheme, convergence rate.

1 Introduction

Nonlocal boundary-value problems naturally arise in the mathematical modeling of many problems of ecology, physics, and engineering, when it is impossible to determine the boundary values of the unknown function (see, e.g., [1–5] and the references therein). At the same time, they are a very interesting generalization of classical boundary-value problems (see, e.g., [6]). The investigation of boundary-value problems with integral conditions goes back to Cannon [7]. The systematic investigation of a certain class of spatial nonlocal problems was carried out by Bitsadze and Samarskii [8]. Later, for elliptic equations, were posed and analyzed nonlocal boundary-value problems of various types (see, e.g., [9–14]).

In [15], we considered the nonlocal problem for the Poisson equation, when the Dirichlet–Neumann conditions are posed on a pair of adjacent sides of a rectangle, and integral constraints $\int_0^{l_k} u(x)\,\mathrm{d}x_k = 0$, $k = 1, 2$, were given instead of classical boundary conditions on the other pair. It is proved that corresponding difference scheme converges in the energy norm at the rate $O(|h|^{s-1})$, when the desired solution belongs to the Sobolev space W_2^s $(1 < s \leqslant 3)$. The proof bases on procedure of obtaining convergence estimate (compatible with smoothness of the exact solution) developed by Samarskii et al. [16] (see, also [17, 18]).

In this paper, we study the case, when the classical boundary conditions are completely replaced by nonlocal ones:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = -f(x), \quad x \in \Omega, \tag{1}$$

$$\int_0^{\xi_k} u(x)\,\mathrm{d}x_k = 0, \qquad \int_{l_k-\xi_k}^{k} u(x)\,\mathrm{d}x_k = 0, \quad 0 \leqslant x_{3-k} \leqslant l_{3-k},\ k = 1, 2, \tag{2}$$

where $\Omega = \{(x_1, x_2)\colon\ 0 < x_k < l_k,\ k = 1, 2\}$ be the rectangle; $l = \max\{l_1,\ l_2\}$. We assume that the solution u of the nonlocal boundary-value problem (1), (2) belongs to the fractional-order Sobolev space $W_2^s(\Omega)$, $s > 1$. For the corresponding difference scheme, estimate of convergence similar to [15], is obtained. Besides the fact that the operator of the difference scheme is not positive definite, basic difficulties comparing with [15] are as follows:

- It is not required that points with coordinates ξ_k or $l_k - \xi_k$ belong to the mesh, which complicates investigation;
- Full disregard of classical boundary conditions complicates obtaining a priori estimates.

2 Finite-difference scheme and main results

Consider the following grid domains on $\bar{\Omega}$: $\bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2$, $\omega = \omega_1 \times \omega_2$, where $\bar{\omega}_k = \{x_{k,i_k} = i_k h_k\colon\ i_k = 0, 1, \dots, n_k,\ h_k = l_k/n_k\}$, $\omega_k = \bar{\omega}_k \cap (0, l_k)$, $\omega_k^+ = \bar{\omega}_k \cap (0, l_k]$, $\hbar_k = h_k$ for $x_k \in \omega_k$, $\hbar_k = h_k/2$ for $x_k = 0, l_k$, $|h| = (h_1^2 + h_2^2)^{1/2}$.

For the values of net function in several points, we apply the notation $y_{ij} = y(ih_1, jh_2)$. When it does not lead to ambiguity, for simplicity, we use the notations $y_i = y(ih_1, x_2)$, $y_j = y(x_1, jh_2)$.

We define the finite-difference operators

$$v_{x_k} = \frac{v(x + h_k r_k) - v(x)}{h_k}, \quad v_{\bar{x}_k} = \frac{v(x) - v(x - h_k r_k)}{h_k}, \quad k = 1, 2,$$

where r_k is the unit vector on the x_k axis.

Let

$$\xi_k = (m_k + \theta_k)h_k, \quad 0 \leqslant \theta_k < 1,\ h_k \leqslant \xi_k \leqslant l_k/2,\ k = 1, 2,$$

where m_k is positive integer.

By H we denote the set of all discrete functions $v = v(x)$, defined on the grid $\bar{\omega}$ and satisfying conditions

$$\check{\mathcal{P}}'(v) = 0, \quad \hat{\mathcal{P}}'(v) = 0, \quad x_2 \in \bar{\omega}_2, \qquad \check{\mathcal{P}}''(v) = 0, \quad \hat{\mathcal{P}}''(v) = 0, \quad x_1 \in \omega_1, \tag{3}$$

where

$$\check{\mathcal{P}}'(v_j) := \sum_{i=0}^{m_1} h_1 v_{ij} - \frac{h_1}{2}(y_{0j} + v_{m_1j}) + \frac{\theta_1 h_1}{2}\big((2-\theta_1)v_{m_1j} + \theta_1 v_{m_1+1,j}\big),$$
$$\check{\mathcal{P}}''(v_i) := \sum_{j=0}^{m_2} h_2 v_{ij} - \frac{h_2}{2}(v_{i0} + v_{im_2}) + \frac{\theta_2 h_2}{2}\big((2-\theta_2)v_{im_2} + \theta_2 v_{i,m_2+1}\big),$$
$$\hat{\mathcal{P}}'(v_j) := \sum_{i=n_1-m_1}^{n_1} h_1 v_{ij} - \frac{h_1}{2}(v_{n_1-m_1,j} + v_{n_1j}) + \frac{\theta_1 h_1}{2}\big((2-\theta_1)v_{n_1-m_1,j} + \theta_1 v_{n_1-m_1-1,j}\big),$$
$$\hat{\mathcal{P}}''(v_i) := \sum_{j=n_2-m_2}^{n_2} h_2 v_{ij} - \frac{h_2}{2}(v_{i,n_2-m_2} + v_{in_2}) + \frac{\theta_2 h_2}{2}\big((2-\theta_2)v_{i,n_2-m_2} + \theta_2 v_{i,n_2-m_2-1}\big).$$

We need the following averaging operators for functions defined on Ω:

$$T_1 u(x) = \frac{1}{h_1^2}\int_{x_1-h_1}^{x_1+h_1}\big(h_1 - |x_1 - t_1|\big)u(t_1, x_2)\,\mathrm{d}t_1,$$
$$T_2 u(x) = \frac{1}{h_2^2}\int_{x_2-h_2}^{x_2+h_2}\big(h_2 - |x_2 - t_2|\big)u(x_1, t_2)\,\mathrm{d}t_2.$$

We approximate the problem (1), (2) by the difference scheme

$$\Lambda y = -\varphi, \quad x \in \omega_1 \times \omega_2,\ y \in H, \tag{4}$$

where

$$\Lambda := \Lambda_1 + \Lambda_2, \qquad \Lambda_\alpha y := y_{\bar{x}_\alpha x_\alpha}, \qquad \varphi := T_1 T_2 f.$$

Theorem 1. *A solution of difference scheme* (4) *exists and is unique.*

Indeed, according to the Lemma 7, the homogeneous problem $\Lambda y = 0$ has only trivial solution $y = 0$. Therefore, the nonhomogeneous problem is uniquely solvable.

Theorem 2. *Let a solution $u(x)$ of the problem* (1), (2) *belong to the space $W_2^s(\Omega)$, $s > 1$. Then the convergence rate of the difference scheme* (4) *in the discrete weighted W_2^1-norm is determined by the estimate*

$$\|y - u\|_{W_2^1(\omega,\rho)} \leqslant c|h|^{s-1}\|u\|_{W_2^s(\Omega)}, \quad 1 < s \leqslant 3. \tag{5}$$

3 Auxiliary statements

Lemma 1. *For every discrete function $v \in H$, the following identities hold*:

$$\check{\mathcal{P}}''(v_i) = 0, \quad \hat{\mathcal{P}}''(v_i) = 0, \quad i = 0, n_1.$$

Proof. Indeed,

$$\begin{aligned}\check{\mathcal{P}}''(v_0) &= \check{\mathcal{P}}''(v_0) + \check{\mathcal{P}}''(v_{m_1})\\ &= \sum_{j=1}^{m_2-1} h_2(v_{0j} + v_{m_1 j}) + \frac{h_2}{2}(v_{00} + v_{0m_2} + v_{m_1 0} + v_{m_1 m_2})\\ &\quad + \frac{\theta_2 h_2}{2}\big((2-\theta_2)v_{0m_2} + \theta_2 v_{0,m_2+1} + (2-\theta_2)v_{m_1 m_2} + \theta_2 v_{m_1,m_2+1}\big).\end{aligned}$$

Hence using the relation

$$\begin{aligned}\sum_{j=1}^{m_2-1}(v_{0j} + v_{m_1 j}) = \sum_{i=1}^{m_1-1}(v_{i0} + v_{im_2}) + \sum_{i=1}^{m_1-1}\theta_2\big((2-\theta_2)v_{im_2} + \theta_2 v_{i,m_2+1}\big)\\ - \sum_{j=1}^{m_2-1}\theta_1\big((2-\theta_1)v_{m_1 j} + \theta_1 v_{m_1+1,j}\big),\end{aligned}$$

which follows from the equality

$$h_2 \sum_{j=1}^{m_2-1}\check{\mathcal{P}}'(v_j) = h_1 \sum_{i=1}^{m_1-1}\check{\mathcal{P}}''(v_i),$$

we obtain

$$\frac{1}{h_2}\check{\mathcal{P}}''(v_0) = \frac{1}{h_1}\big(\check{\mathcal{P}}'(v_0) + \check{\mathcal{P}}'(v_{m_2})\big) + \mathcal{Q} = \mathcal{Q}. \tag{6}$$

Here

$$\begin{aligned}\mathcal{Q} := {}& \theta_2(2-\theta_2)\left(\sum_{i=1}^{m_1-1} v_{im_2} + \frac{1}{2}(v_{0m_2} + v_{m_1 m_2})\right)\\ &+ \theta_2^2\left(\sum_{i=1}^{m_1-1} v_{i,m_2+1} + \frac{1}{2}(v_{0,m_2+1} + v_{m_1,m_2+1})\right)\\ &- \theta_1(2-\theta_1)\left(\sum_{j=1}^{m_2-1} v_{m_1 j} + \frac{1}{2}(v_{m_1 0} + v_{m_1 m_2})\right)\\ &- \theta_1^2\left(\sum_{j=1}^{m_2-1} v_{m_1+1,j} + \frac{1}{2}(v_{m_1+1,0} + v_{m_1+1,m_2})\right).\end{aligned}$$

By applying nonlocal conditions we see that

$$\begin{aligned}\mathcal{Q} = &-\theta_1\theta_2(2-\theta_2)\big((2-\theta_1)v_{m_1m_2}+\theta_1 v_{m_1+1,m_2}\big)\\ &-\theta_1\theta_2^2\big((2-\theta_1)v_{m_1,m_2+1}+\theta_1 v_{m_1+1,m_2+1}\big)\\ &+\theta_1\theta_2(2-\theta_1)\big((2-\theta_2)v_{m_1m_2}+\theta_2 v_{m_1,m_2+1}\big)\\ &+\theta_1^2\theta_2\big((2-\theta_2)v_{m_1+1,m_2}+\theta_2 v_{m_1+1,m_2+1}\big)=0.\end{aligned}$$

From here and (6) follows the first identity of Lemma 1. The proof of the last part leads analogously. □

We define the weight functions

$$\rho^{(k)}=\rho^{(k)}(i_kh_k)=\begin{cases}(i_k-\frac{1}{2})\frac{h_k}{\xi_k}, & i_k=1,2,\dots,m_k,\\ 1, & i=m_k+2,\dots,n_k-m_k-1,\\ (n_k-i_k+\frac{1}{2})\frac{h_k}{\xi_k}, & i_k=n_k-m_k+1,\dots,n_k,\\ 1-\frac{\theta_k^2h_k}{2\xi_k}, & i_k=m_k+1,\ i_k=n_k-m_k,\end{cases}$$

$$\bar\rho^{(k)}=\bar\rho^{(k)}(i_kh_k)=\begin{cases}\frac{h_k}{4\xi_k}, & i_k=0,n_k,\\ \frac{i_kh_k}{\xi_k}, & i_k=1,2,\dots,m_k,\\ 1, & i_k=m_k+1,\dots,n_k-m_k-1,\\ \frac{(n_k-i_k)h_k}{\xi_k}, & i_k=n_k-m_k,\dots,n_k-1.\end{cases}$$

In H, we introduce the inner product and norm as

$$(y,v)=\sum_{\bar\omega}\hbar_1\hbar_2\big(\bar\rho^{(1)}+\bar\rho^{(2)}\big)yv,\qquad \|y\|=(y,y)^{1/2}.$$

Let, in addition,

$$(y,v)_\omega=\sum_\omega h_1h_2yv,$$

$$\|\nabla y\|^2=\sum_{\omega_1^+\times\bar\omega_2}h_1\hbar_2\rho^{(1)}\bar\rho^{(2)}(y_{\bar x_1})^2+\sum_{\bar\omega_1\times\omega_2^+}\hbar_1h_2\bar\rho^{(1)}\rho^{(2)}(y_{\bar x_2})^2,$$

$$\|y\|^2_{W_2^1(\rho,\omega)}=\|y\|^2+\|\nabla y\|^2,$$

$$G_1y_{ij}=\begin{cases}\frac{1}{\xi_1}\big(ih_1y_{ij}-h_1\sum_{k=0}^{i}y_{kj}+\frac{h_1}{2}(y_{ij}+y_{0j})\big), & 0\leqslant i\leqslant m_1,\\ y_{ij},\quad m_1+1\leqslant i\leqslant n_1-m_1-1,\\ \frac{1}{\xi_1}\big((n_1-i)h_1y_{ij}-h_1\sum_{k=i}^{n_1}y_{kj}+\frac{h}{2}(y_{ij}+y_{n_1j})\big), & n_1-m_1\leqslant i\leqslant n_1,\end{cases}$$

$$G_2y_{ij}=\begin{cases}\frac{1}{\xi_2}\big(jh_2y_{ij}-h_2\sum_{k=0}^{j}y_{ik}+\frac{h_2}{2}(y_{ij}+y_{i0})\big), & 0\leqslant j\leqslant m_2,\\ y_{ij},\quad m_2+1\leqslant i\leqslant n_2-m_2-1,\\ \frac{1}{\xi_2}\big((n_2-j)h_2y_{ij}-h_2\sum_{k=j}^{n_2}y_{ik}+\frac{h_2}{2}(y_{ij}+y_{in_2})\big), & n_2-m_2\leqslant j\leqslant n_2.\end{cases}$$

Lemma 2. *Let grid functions $v(x), y(x)$ be defined on $\bar{\omega}$, and $y(x)$ satisfy the conditions*

$$\check{\mathcal{P}}'(y) = 0,\ \hat{\mathcal{P}}'(y) = 0, \quad x_2 \in \bar{\omega}_2 \quad \textit{or} \quad \check{\mathcal{P}}''(y) = 0,\ \hat{\mathcal{P}}''(y) = 0, \quad x_1 \in \bar{\omega}_1.$$

Then

$$\sum_{\omega_k} h_k v_{\bar{x}_k x_k} G_k y = -\sum_{\omega_k^+} h_k \rho^{(k)} v_{\bar{x}_k} y_{\bar{x}_k}$$

for $k = 1$ or $k = 2$, respectively.

Proof. Using the summation by parts, we obtain

$$\sum_{i=1}^{m_1} h_1 (v_{\bar{x}_1 x_1})_i G_1 y_i = -\sum_{i=1}^{m_1} h_1 \rho^{(1)}(ih_1)(v_{\bar{x}_1})_i (y_{\bar{x}_1})_i + (v_{\bar{x}_1})_{m_1+1} G_1 y_{m_1}, \tag{7}$$

$$\begin{aligned} \sum_{i=m_1+1}^{n_1-m_1-1} h_1 (v_{\bar{x}_1 x_1})_i G_1 y_i = &- \sum_{i=m_1+1}^{n_1-m_1} h_1 (v_{\bar{x}_1})_i (y_{\bar{x}_1})_i + (v_{\bar{x}_1})_{n_1-m_1}\, y_{n_1-m_1} \\ &- (v_{\bar{x}_1})_{m_1+1}\, y_{m_1}, \end{aligned} \tag{8}$$

$$\begin{aligned} \sum_{i=n_1-m_1}^{n_1-1} h_1 (v_{\bar{x}_1 x_1})_i G_1 y_i = &- \sum_{i=n_1-m_1+1}^{n_1} h_1 \rho^{(1)}(ih_1)(v_{\bar{x}_1})_i (v_{\bar{x}_1})_i \\ &- (v_{\bar{x}_1})_{n_1-m_1} G_1 y_{n_1-m_1}. \end{aligned} \tag{9}$$

Adding the equalities (7)–(9) and applying following from the nonlocal conditions identities

$$G_1 y_{m_1} = y_{m_1} + \frac{\theta_1^2 h_1^2}{2\xi_1} (y_{\bar{x}_1})_{m_1+1}, \qquad G_1 y_{n_1-m_1} = y_{n_1-m_1} - \frac{\theta_1^2 h_1^2}{2\xi_1} (y_{\bar{x}_1})_{n_1-m_1},$$

we verify the validity of the Lemma 2 in the case $k = 1$. The case $k = 2$ may be proved analogously. □

Lemma 3. *If a grid function $y(x)$, defined on $\bar{\omega}$, satisfies the conditions*

$$\check{\mathcal{P}}'(y) = 0, \quad \hat{\mathcal{P}}'(y) = 0, \quad x_2 \in \bar{\omega}_2 \quad \textit{or} \quad \check{\mathcal{P}}''(y) = 0, \quad \hat{\mathcal{P}}''(y) = 0, \quad x_1 \in \bar{\omega}_1,$$

then

$$\frac{7}{8} \sum_{\bar{\omega}_k} \hbar_k \bar{\rho}^{(k)} y^2 \leqslant \sum_{\omega_k} h_k y G_k y \leqslant \sum_{\bar{\omega}_k} \hbar_k \bar{\rho}^{(k)} y^2$$

for $k = 1$ or $k = 2$, respectively.

Proof. It may be showed that the identity

$$\begin{aligned} \sum_{i=1}^{n_1-1} h_1 y_{ij} G_1 y_{ij} = \frac{h_1^2}{\xi_1} \sum_{i=1}^{m_1} i y_{ij}^2 + h_1 \sum_{i=m_1+1}^{n_1-m_1-1} y_{ij}^2 + \frac{h_1^2}{\xi_1} \sum_{i=n_1-m_1}^{n_1-1} (n_1 - i) y_{ij}^2 \\ + \frac{h_1^2}{8\xi_1} \big(y_{0j}^2 + y_{n_1 j}^2 - \check{S}^2 - \hat{S}^2 \big) \end{aligned} \tag{10}$$

holds, where

$$\check{S} := 2\sum_{k=0}^{m_1} y_{kj} - y_{0j}, \qquad \hat{S} := 2\sum_{k=n_1-m_1}^{n_1} y_{kj} - y_{n_1 j}.$$

Let us note that, according to nonlocal conditions,

$$\check{S} = (1-\theta_1)^2 y_{m_1 j} - \theta_1^2 y_{m_1+1,j}.$$

In addition, the inequality

$$\check{S}^2 \leqslant (1-\theta_1)^2 y_{m_1 j}^2 + \theta_1^2 y_{m_1+1,j}^2 := A \tag{11}$$

follows from

$$A - \check{S}^2 \geqslant \bigl((1-\theta_1)^2 + \theta_1^2\bigr)A - \check{S}^2 = (1-\theta_1)^2\theta_1^2(y_{m_1 j} + y_{m_1+1,j})^2 \geqslant 0.$$

We can obtain analogously that

$$\hat{S}^2 \leqslant (1-\theta_1)^2 y_{n_1-m_1,j}^2 + \theta_1^2 y_{n_1-m_1-1,j}^2. \tag{12}$$

Adding inequalities (11), (12) and replacing in the right-hand side $(1-\theta_1)^2 \leqslant m_1$ and $\theta_1^2 \leqslant \xi_1/h_1$, we obtain

$$\begin{aligned}\frac{h_1^2}{8\xi_1}\bigl(\check{S}^2 + \hat{S}^2\bigr) &\leqslant \frac{m_1 h_1^2}{8\xi_1}\bigl(y_{m_1 j}^2 + y_{n_1-m_1,j}^2\bigr) + \frac{h_1}{8}\bigl(y_{m_1+1,j}^2 + y_{n_1-m_1-1,j}^2\bigr)\\ &\leqslant \frac{1}{8}\sum_{i=0}^{n_1} \hbar_1 \bar{\rho}_i^{(1)} y_{ij}^2.\end{aligned}$$

From this inequality and (10) follows the validity of Lemma 3 in the case $k=1$. We can consider the case $k=2$ analogously. □

Lemma 4. *If a grid function $y(x)$, defined on $\bar{\omega}$, satisfies the conditions*

$$\check{\mathcal{P}}'(y) = 0, \quad \hat{\mathcal{P}}'(y) = 0, \quad x_2 \in \bar{\omega}_2 \quad \textit{or} \quad \check{\mathcal{P}}''(y) = 0, \quad \hat{\mathcal{P}}''(y) = 0, \quad x_1 \in \bar{\omega}_1,$$

then

$$\sum_{\bar{\omega}_k} \hbar_k y^2 \leqslant l_k^2 \sum_{\omega_k^+} h_k \rho^{(k)} y_{\bar{x}_k}^2$$

for $k=1$ or $k=2$, respectively.

Proof. For arbitrary $y(x)$, defined on $\bar{\omega}$, the identity

$$\|y\|_{(1)}^2 := \sum_{i=0}^{n_1} \hbar_1 y_i^2 = J + \frac{l_1}{2}\bigl(y_{m_1}^2 + y_{n_1-m_1}^2\bigr) \tag{13}$$

is true, where

$$J = -\sum_{i=1}^{m_1} h_1\left(i-\frac{1}{2}\right)\left(y_i^2 - y_{i-1}^2\right) + \sum_{i=m_1+1}^{n_1-m_1} h_1\left(\frac{n_1}{2} - i + \frac{1}{2}\right)\left(y_i^2 - y_{i-1}^2\right)$$
$$+ \sum_{i=n_1-m_1+1}^{n_1} h_1\left(n_1 - i + \frac{1}{2}\right)\left(y_i^2 - y_{i-1}^2\right).$$

Let us estimate this sum.

If

$$J_1 := \sum_{i=1}^{m_1} h_1^3\left(i-\frac{1}{2}\right)^2 y_{\bar{x}_1,i}^2 + \sum_{i=m_1+1}^{n_1-m_1} h_1^3\left(\frac{n_1}{2} - i + \frac{1}{2}\right)^2 y_{\bar{x}_1,i}^2$$
$$+ \sum_{i=n_1-m_1+1}^{n_1} h_1^3\left(n - i + \frac{1}{2}\right)^2 y_{\bar{x}_1,i}^2,$$

then

$$|J| \leqslant \left(\sum_{i=1}^{n_1} h_1\left(y_i + y_{i-1}\right)^2\right)^{1/2} (J_1)^{1/2} \leqslant \frac{1}{8}\sum_{i=1}^{n_1} h_1(y_i + y_{i-1})^2 + 2J_1$$
$$\leqslant \frac{1}{2}\|y\|_{(1)}^2 + 2J_1.$$

Applying this inequality to (13), we have

$$\|y\|_{(1)}^2 \leqslant 4J_1 + l_1\left(y_{m_1}^2 + y_{n_1-m_1}^2\right). \tag{14}$$

From the nonlocal condition follows

$$\xi y_{m_1} = \sum_{i=1}^{m_1} h_1\left(i - \frac{1}{2}\right)(y_i - y_{i-1}) - \frac{\theta_1^2 h_1}{2}(y_{m_1+1} - y_{m_1}),$$

and, therefore,

$$y_{m_1}^2 \leqslant \frac{1}{2}\left(\sum_{i=1}^{m_1} h_1^2\left(i - \frac{1}{2}\right) y_{\bar{x}_1,i}^2 + \frac{\theta_1^2 h_1^2}{2} y_{\bar{x}_1,m_1+1}^2\right). \tag{15}$$

Based on the nonlocal condition, we have as well

$$y_{n_1-m_1}^2 \leqslant \frac{1}{2}\left(\sum_{i=n_1-m_1+1}^{n_1} h_1^2\left(n_1 - i + \frac{1}{2}\right) y_{\bar{x}_1,i}^2 + \frac{\theta_1^2 h_1^2}{2} y_{\bar{x}_1,n_1-m_1}^2\right). \tag{16}$$

From (14) with the help of (15), (16) we obtain

$$\|y\|_{(1)}^2 \leqslant \frac{9l_1}{2}\sum_{i=1}^{m_1} h_1^2\left(i - \frac{1}{2}\right) y_{\bar{x}_1,i}^2 + \frac{9l_1}{2}\sum_{i=n_1-m_1+1}^{n_1} h_1^2\left(n - i + \frac{1}{2}\right) y_{\bar{x}_1,i}^2$$
$$+ l_1^2 \sum_{i=m_1+1}^{n_1-m_1} h_1 y_{\bar{x}_1,i}^2 + \frac{l_1^2}{8\xi}\theta_1^2 h_1^2\left(y_{\bar{x}_1,m+1}^2 + y_{\bar{x}_1,n_1-m_1}^2\right).$$

If we increase the first and second sums by multiplication on the quantity $l_1/(2\xi_1) > 1$ and apply the inequality

$$1 + \frac{\theta_1^2 h_1}{8\xi_1} < \frac{9}{4}\left(1 - \frac{\theta_1^2 h_1}{2\xi_1}\right)$$

in the summands with the indices $i = m_1 + 1, n_1 - m_1$, we will be sure that the Lemma 4 is true. □

Lemma 5. *If a grid function $y(x)$, defined on $\bar{\omega}$, satisfies the conditions*

$$\check{\mathcal{P}}'(y) = 0, \quad \hat{\mathcal{P}}'(y) = 0, \quad x_2 \in \bar{\omega}_2 \quad \textit{or} \quad \check{\mathcal{P}}''(y) = 0, \quad \hat{\mathcal{P}}''(y) = 0, \quad x_1 \in \bar{\omega}_1,$$

then

$$\sum_{\omega_k} h_k (G_k y)^2 \leqslant 10 \sum_{\bar{\omega}_k} \hbar_k \bar{\rho}_k y^2, \quad k = 1, 2,$$

for $k = 1$ or $k = 2$, respectively.

Proof. Let

$$\check{S}_i := \sum_{k=0}^{i} h_1 y_k - \frac{h_1}{2}(y_i + y_0), \qquad \hat{S}_i := \sum_{k=i}^{n_1} h_1 y_k - \frac{h_1}{2}(y_i + y_{n_1}).$$

According to the definition of the operator G_1,

$$\sum_{i=1}^{n_1-1} h_1 (G_1 y_i)^2 = \sum_{i=1}^{m_1} \frac{h_1}{\xi^2}(i h_1 y_i - \check{S}_i)^2 + \sum_{i=m_1+1}^{n_1-m_1-1} h_1 (y_i)^2 + \sum_{i=n_1-m_1}^{n_1-1} \frac{h_1}{\xi_1^2}\big((n_1 - i) y_i - \hat{S}_i\big)^2.$$

We have from here

$$\sum_{i=1}^{n_1-1} h_1 (G_1 y_i)^2 \leqslant \sum_{i=1}^{m_1} \frac{2h_1}{\xi_1^2}(i h_1)^2 (y_i)^2 + \sum_{i=m_1+1}^{n_1-m_1-1} h_1 (y_i)^2 + \sum_{i=n_1-m_1}^{n_1-1} \frac{2h_1}{\xi_1^2}(n_1 - i)^2 h_1^2 y_i^2 + \sum_{i=1}^{m_1} \frac{2h_1}{\xi_1^2}(\check{S}_i)^2 + \sum_{i=n_1-m_1}^{n_1-1} \frac{2h_1}{\xi_1^2}(\hat{S}_i)^2. \tag{17}$$

It is not difficult to verify that

$$\sum_{i=1}^{m_1} h_1 (\check{S}_i)^2 = \left(m_1 + \frac{1}{2}\right) h_1 (\check{S}_{m_1})^2 - \sum_{i=1}^{m_1} \frac{h_1^2}{2}\left(i - \frac{1}{2}\right)(y_i + y_{i-1})(\check{S}_i + \check{S}_{i-1}).$$

We have from here

$$2\sum_{i=1}^{m_1} h_1 (\check{S}_i)^2 \leqslant (4m_1 + 1) h_1 (\check{S}_{m_1})^2 + 2\sum_{i=1}^{m_1} h_1 \left(h_1 \left(i - \frac{1}{2}\right)(y_i + y_{i-1})\right)^2. \tag{18}$$

Noting that

$$(\check{S}_{m_1})^2 \leqslant \frac{h_1^2}{2}\big(y_{m_1}^2 + y_{m_1+1}^2\big),$$

we obtain from (18)

$$2\sum_{i=1}^{m_1} h_1 \check{S}_i^2 \leqslant 8\sum_{i=1}^{m_1} h_1^2 i\xi_1 y_i^2 + 3h_1\xi_1^2 y_{m_1+1}^2 + h_1^3 y_0^2. \tag{19}$$

We can obtain analogously that

$$2\sum_{i=n_1-m_1}^{n_1-1} h_1(\hat{S}_i)^2 \leqslant 8\sum_{i=n_1-m_1}^{n_1-1} h_1^2(n_1-i)\xi_1 y_i^2 + 3h_1\xi_1^2 y_{n_1-m_1-1}^2 + h_1^3 y_{n_1}^2. \tag{20}$$

From the inequalities (17), (19), (20) follows validity of the Lemma 5 in the case $k = 1$. We can consider the case $k = 2$ analogously. □

Lemma 6. *For every $y \in H$, the following inequalities hold*:

$$\frac{7}{8}\|\nabla y\|^2 \leqslant (-\Lambda y, G_1G_2y)_\omega \leqslant \|\nabla y\|^2, \tag{21}$$

$$\|y\|^2_{W_2^1(\rho,\omega)} \leqslant c(-\Lambda y, G_1G_2y)_\omega, \quad c = \frac{8}{7} + l^2.$$

Proof. Based on the Lemma 2,

$$(-\Lambda y, G_1G_2y)_\omega = \sum_{\omega_1^+\times\omega_2} h_1h_2\rho^{(1)} y_{\bar{x}_1} G_2 y_{\bar{x}_1} + \sum_{\omega_1\times\omega_2^+} h_1h_2\rho^{(2)} y_{\bar{x}_2} G_1 y_{\bar{x}_2}.$$

Taking into account in addition following from the Lemma 3 inequalities

$$\begin{aligned}
&\frac{7}{8}\sum_{\bar{\omega}_1\times\omega_2^+} \hbar_1 h_2 \bar{\rho}^{(1)}\rho^{(2)}(y_{\bar{x}_2})^2 \\
&\quad\leqslant \sum_{\omega_1\times\omega_2^+} h_1h_2\rho^{(2)} y_{\bar{x}_2} G_1 y_{\bar{x}_2} \leqslant \sum_{\bar{\omega}_1\times\omega_2^+} \hbar_1 h_2 \bar{\rho}^{(1)}\rho^{(2)}(y_{\bar{x}_2})^2, \\
&\frac{7}{8}\sum_{\omega_1^+\times\bar{\omega}_2} h_1 \hbar_2 \rho^{(1)}\bar{\rho}^{(2)}(y_{\bar{x}_1})^2 \\
&\quad\leqslant \sum_{\omega_1^+\times\omega_2} h_1h_2\rho^{(1)} y_{\bar{x}_1} G_2 y_{\bar{x}_1} \leqslant \sum_{\omega_1^+\times\bar{\omega}_2} h_1 \hbar_2 \rho^{(1)}\bar{\rho}^{(2)}(y_{\bar{x}_1})^2,
\end{aligned}$$

we ensure the validity for the first inequality of lemma.

According to Lemma 4,

$$\|y\|^2 \leqslant c_1\|\nabla y\|^2, \quad c_1 = \max\big(l_1^2; l_2^2\big).$$

From here and (21) it follows Lemma 6. □

To determine the convergence rate of the finite-difference scheme (4), we apply the following lemma.

Lemma 7. *Assume that the linear functional $\eta(u)$ is bounded in $W_2^s(E)$, where $s = \bar{s}+\epsilon$, $\bar{s}$ is an integer, $0 < \epsilon \leqslant 1$, and $\eta(P) = 0$ for every polynomial P of degree $\leqslant \bar{s}$ in two variables. Then there exists a constant c, independent of u, such that $|\eta(u)| \leqslant c\|u\|_{W_2^s(E)}$.*

This lemma is a particular case of Dupont–Scott approximation theorem [19] and it represents a generalization of the Bramble–Hilbert lemma [20] (see also [16, p. 29]).

4 The problem for the error

Let us define on the particular subintervals the components of approximation errors for the integral conditions (2):

$$\zeta'_{ij} := \int\limits_{(j-1)h_2}^{jh_2} u(ih_1, t_2)\,\mathrm{d}t_2 - \frac{h_2}{2}(u_{i,j-1} + u_{ij}), \quad j \neq m_2+1,\ n_2-m_2,$$

$$\zeta'_{i,m_2+1} := \int\limits_{m_2h_2}^{(m_2+\theta_2)h_2} u(ih_1, t_2)\,\mathrm{d}t_2 - \frac{\theta_2h_2}{2}\big((2-\theta_2)u_{i,m_2} + \theta_2u_{i,m_2+1}\big),$$

$$\zeta'_{i,n_2-m_2} := \int\limits_{(n_2-m_2-\theta_2)h_2}^{(n_2-m_2)h_2} u(ih_1, t_2)\,\mathrm{d}t_2 - \frac{\theta_2h_2}{2}\big((2-\theta_2)u_{i,n_2-m_2} + \theta_2u_{i,n_2-m_2-1}\big),$$

$$\zeta''_{ij} := \int\limits_{(i-1)h_1}^{ih_1} u(t_1, jh_2)\,\mathrm{d}t_1 - \frac{h_1}{2}(u_{i-1,j} + u_{ij}), \quad i \neq m_1+1,\ n_1-m_1,$$

$$\zeta''_{m_1+1,j} := \int\limits_{m_1h_1}^{(m_1+\theta_1)h_1} u(t_1, jh_2)\,\mathrm{d}t_1 - \frac{\theta_1h_1}{2}\big((2-\theta_1)u_{m_1,j} + \theta_1u_{m_1+1,j}\big),$$

$$\zeta''_{n_1-m_1,j} := \int\limits_{(n_1-m_1-\theta_1)h_1}^{(n_1-m_1)h_1} u(t_1, jh_2)\,\mathrm{d}t_1 - \frac{\theta_1h_1}{2}\big((2-\theta_1)u_{n_1-m_1,j} + \theta_1u_{n_1-m_1-1,j}\big).$$

Lemma 8. *Let u be the solution of the problem* (1), (2) *and y be the solution of the finite-difference scheme* (4). *Then discretization error $z = y - u$ satisfies the following problem*:

$$\Lambda z = \eta^{(1)}_{\bar{x}_1x_1} + \eta^{(2)}_{\bar{x}_2x_2}, \quad x \in \omega, \tag{22}$$

$$\begin{aligned} \check{\mathcal{P}}'(z) &= \check{\chi}^{(2)}(x_2), \quad \hat{\mathcal{P}}'(z) = \hat{\chi}^{(2)}(x_2), \quad x_2 \in \bar{\omega}_2, \\ \check{\mathcal{P}}''(z) &= \check{\chi}^{(1)}(x_1), \quad \hat{\mathcal{P}}''(z) = \hat{\chi}^{(1)}(x_1), \quad x_1 \in \omega_1, \end{aligned} \tag{23}$$

where

$$\eta^{(1)} := T_2 u - u, \qquad \eta^{(2)} := T_1 u - u,$$
$$\check{\chi}_i^{(1)} := \sum_{j=1}^{m_2+1} \zeta_{ij}, \qquad \hat{\chi}_i^{(1)} := \sum_{j=n_2-m_2}^{n_2} \zeta_{ij},$$
$$\check{\chi}_j^{(2)} := \sum_{i=1}^{m_1+1} \zeta_{ij}, \qquad \hat{\chi}_j^{(2)} := \sum_{i=n_1-m_1}^{n_1} \zeta_{ij}.$$

Indeed, (22) can be obtained from substituting $y = z + u$ into (4) and taking into account $T_k(\partial^2 u/\partial x^2) = u_{\bar{x}_k x_k}$.

Further, in view of the conditions (2), (3), we have

$$\check{\mathcal{P}}'(z) = \check{\mathcal{P}}'(y) - \check{\mathcal{P}}'(u) = \int_0^{\xi_1} u(t_1, x_2)\,\mathrm{d}t_1 - \check{\mathcal{P}}'(u) = \check{\chi}^{(2)}(x_2).$$

We can verify other equalities of (23) analogously.

As we see, the nonlocal conditions for the error problem, unlike the difference scheme, are not homogeneous. Therefore, in order to use the results obtained in the Section 3, we pass to the new unknown function.

First of all, let us define the functions

$$\check{\beta}^{(k)}(x_k) = \frac{2l_k - \xi_k - 2x_k}{2\xi_k(l_k - \xi_k)}, \quad \hat{\beta}^{(k)}(x_k) = \frac{2x_k - \xi_k}{2\xi_k(l_k - \xi_k)}, \quad k = 1, 2.$$

For them, the following hold:

$$\check{\mathcal{P}}'(\check{\beta}^{(1)}) = 1, \qquad \check{\mathcal{P}}''(\check{\beta}^{(2)}) = 1, \qquad \hat{\mathcal{P}}'(\check{\beta}^{(1)}) = 0, \qquad \hat{\mathcal{P}}''(\check{\beta}^{(2)}) = 0,$$
$$\check{\mathcal{P}}'(\hat{\beta}^{(1)}) = 0, \qquad \check{\mathcal{P}}''(\hat{\beta}^{(2)}) = 0, \qquad \hat{\mathcal{P}}'(\hat{\beta}^{(1)}) = 1, \qquad \hat{\mathcal{P}}''(\hat{\beta}^{(2)}) = 1.$$

Let

$$\begin{aligned} w(x) = z(x) &- \check{\beta}^{(1)}(x_1)\check{\chi}^{(2)}(x_2) - \hat{\beta}^{(2)}(x_2)\hat{\chi}^{(1)}(x_1) - \check{\beta}^{(2)}(x_2)\check{\chi}^{(1)}(x_1) \\ &+ \check{\beta}^{(1)}(x_1)\check{\beta}^{(2)}(x_2)\check{\mathcal{P}}'(\check{\chi}^{(1)}) + \check{\beta}^{(1)}(x_1)\hat{\beta}^{(2)}(x_2)\check{\mathcal{P}}'(\hat{\chi}^{(1)}) - \hat{\beta}^{(1)}(x_1)\hat{\chi}^{(2)}(x_2) \\ &+ \hat{\beta}^{(1)}(x_1)\check{\beta}^{(2)}(x_2)\hat{\mathcal{P}}'(\check{\chi}^{(1)}) + \hat{\beta}^{(1)}(x_1)\hat{\beta}^{(2)}(x_2)\hat{\mathcal{P}}'(\hat{\chi}^{(1)}). \end{aligned} \tag{24}$$

We can verify straightforward that $\check{P}'(w) = 0$ and $\hat{P}'(w) = 0$.

For the verification of the conditions $\check{P}''(w) = 0$ and $\hat{P}''(w) = 0$, we apply the consequences of (23), respectively,

$$\check{P}''(\check{\chi}^{(2)}) = \check{P}'(\check{\chi}^{(1)}), \qquad \check{P}''(\hat{\chi}^{(2)}) = \hat{P}'(\check{\chi}^{(1)})$$

and

$$\hat{P}''(\check{\chi}^{(2)}) = \check{P}'(\hat{\chi}^{(1)}), \qquad \hat{P}''(\hat{\chi}^{(2)}) = \hat{P}'(\hat{\chi}^{(1)}).$$

It may be proved that the function $w(x)$ represents a solution of the following problem:

$$\Lambda w = \psi, \quad x \in \omega,\ w \in H, \tag{25}$$

where

$$\psi := \sum_{k=1}^{2} \bigl(\Lambda_k \eta^{(k)} - \check{\beta}^{(3-k)} \Lambda_k \check{\chi}^{(k)} - \hat{\beta}^{(3-k)} \Lambda_k \hat{\chi}^{(k)}\bigr).$$

5 Proof of Theorem 2

It follows from (24)

$$\|\nabla z\| \leqslant \|\nabla w\| + c(J_1 + J_2 + J_3), \tag{26}$$

where

$$\begin{aligned}
J_1 :&= \bigl\|\check{\chi}^{(1)}\bigr\|_{\bar{\omega}_1} + \bigl\|\check{\chi}^{(2)}\bigr\|_{\bar{\omega}_2} + \bigl\|\hat{\chi}^{(1)}\bigr\|_{\bar{\omega}_1} + \bigl\|\hat{\chi}^{(2)}\bigr\|_{\bar{\omega}_2},\\
J_2 :&= \bigl\|\check{\chi}^{(1)}_{\bar{x}_1}\bigr\|_{\omega_1^+} + \bigl\|\check{\chi}^{(2)}_{\bar{x}_2}\bigr\|_{\omega_2^+} + \bigl\|\hat{\chi}^{(1)}_{\bar{x}_1}\bigr\|_{\omega_1^+} + \bigl\|\hat{\chi}^{(2)}_{\bar{x}_2}\bigr\|_{\omega_2^+},\\
J_3 :&= \bigl|\check{\mathcal{P}}'\bigl(\check{\chi}^{(1)}\bigr)\bigr| + \bigl|\check{\mathcal{P}}'\bigl(\hat{\chi}^{(1)}\bigr)\bigr| + \bigl|\hat{\mathcal{P}}'\bigl(\check{\chi}^{(1)}\bigr)\bigr| + \bigl|\hat{\mathcal{P}}'\bigl(\hat{\chi}^{(1)}\bigr)\bigr|.
\end{aligned}$$

According to (25), we have $(\Lambda w, G_1 G_2 w)_\omega = (\psi, G_1 G_2 w)_\omega$. If we apply the first inequality of Lemma 6 in the left-hand side of this identity, and in the right-hand side the Lemmas 2 and 5, we obtain

$$\|\nabla w\| \leqslant c\bigl(\bigl\|\eta^{(1)}_{\bar{x}_1}\bigr\|_{\omega_1^+ \times \omega_2} + \bigl\|\eta^{(2)}_{\bar{x}_2}\bigr\|_{\omega_1 \times \omega_2^+} + J_2\bigr). \tag{27}$$

The second inequality of the Lemma 6 together with (2), (27) gives an a priori estimate for the problem (22)

$$\|z\|_{W_2^1(\omega,\rho)} \leqslant c\bigl(\bigl\|\eta^{(1)}_{\bar{x}_1}\bigr\|_{\omega_1^+ \times \omega_2} + \bigl\|\eta^{(2)}_{\bar{x}_2}\bigr\|_{\omega_1 \times \omega_2^+} + J_1 + J_2 + J_3\bigr). \tag{28}$$

For the estimation of J_1, notice that the summands ζ', ζ'', as linear functionals with respect to $u(x)$, vanish on the polynomials of first order and are bounded on W_2^s, $s > 1$. Consequently, using Lemma 7, we have $J_1 \leqslant c|h|^s \|u\|_{W_2^s(\Omega)}$, $1 < s \leqslant 2$, from which $J_1 \leqslant c|h|^{s-1} \|u\|_{W_2^s(\Omega)}$, $1 < s \leqslant 3$.

For the estimation of J_2, notice that the summands $\zeta'_{\bar{x}_1}$, $\zeta''_{\bar{x}_2}$, as linear functionals with respect to $u(x)$, vanish on the polynomials of second order and are bounded on W_2^s, $s > 1$. Consequently, using Lemma 7, we receive $J_2 \leqslant c|h|^{s-1} \|u\|_{W_2^s(\Omega)}$, $1 < s \leqslant 3$.

For the estimation of J_3, we represent its summands in the expanded form, for example,

$$\check{\mathcal{P}}'\bigl(\check{\chi}^{(1)}\bigr) = \frac{h_1}{2} \sum_{i=1}^{m_1} \sum_{j=1}^{m_2+1} (\zeta'_{ij} + \zeta'_{i-1,j}) + \frac{\theta_1 h_1}{2} \sum_{j=1}^{m_2+1} \bigl((2-\theta_1)\chi^{(1)}_{m_1 j} + \theta_1 \chi^{(1)}_{m_1+1,j}\bigr).$$

This may be estimated analogously to J_1.

The norms of the functionals $\eta^{(k)}_{\bar{x}_k}$, $k = 1, 2$, are less than $c|h|^{s-1}\|u\|_{W_2^s(\Omega)}$, $1 < s \leqslant 3$. The obtaining of these estimates are considered in detail, for example, in [16, pp.148–149].

As a result from (28) it follows the validity of Theorem 2.

6 Conclusion

A nonlocal problem posed for Poisson equation is considered—classical boundary conditions are fully replaced with integral conditions on the inner stripe adjacent to boundary having the width ξ. The corresponding difference scheme is constructed for which convergence with rate $s-1$ is proved when the exact solution belongs to Sobolev space W_2^s, $1 < s \leqslant 3$, with fractional exponent.

The obtained results may be expanded: for a case when the width of the stripe defined by integral conditions is different at all sides of the rectangle; for a system of statical theory of elasticity with constant coefficients, also for three dimensional case.

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