საქართველოს მეცნიერებათა ეროვნული აკადემიის მოამბე, ტ. 12, №3, 2018
BULLETIN OF THE GEORGIAN NATIONAL ACADEMY OF SCIENCES, vol. 12, no.3, 2018

Mathematics

On the Cauchy Problem for Linear Systems of Impulsive Ordinary Differential Equations with Singularities

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(Presented by Academy Member Ivane Kiguradze)

ABSTRACT. In the present paper we consider the Cauchy problem for systems of linear systems of differential equations with singularities. The singularity is understood in the sense that the matrix and vector functions corresponding to the impulsive system, in general, are not integrable at the initial point. The sufficient conditions are for the unique solvability of the problem. *© 2018 Bull. Georg. Natl. Acad. Sci.*

Key words: impulsive differential equations, linear systems, singularities, Cauchy problem, unique solvability

1. Statement of the Problem and Basic Notation. Let $I \subset R$ be an interval non-degenerate in the point, $t_0 \in I$ and

$$I_{t_0} = I \setminus \{t_{t_0}\}.$$

Consider the linear system of impulsive differential equations with infinite and fixed points of impulses actions

$$\frac{dx}{dt} = P(t)x + q(t) \text{ for a .a. } t \in I_{t_0} \setminus \{\tau_l\}_{l=1}^{\infty}, \tag{1}$$

$$x(\tau_l+) - x(\tau_l-) = G_l x(\tau_l) + g_l \ \ (l = 1, 2, \dots), \tag{2}$$

where

$$P = (p_{ik})_{i,k=1}^{n} \in L_{loc}(I_{t_0}; R^{n\times n}),\ q = (q_i)_{i=1}^{n} \in L_{loc}(I_{t_0}, R^n),\ G_l = (g_{lik})_{i,k=1}^{n} \in R^{n\times n}\ (l = 1, 2, \dots),$$

$$g_l = (g_{lik})_{i,k=1}^{n} \in R^n\ (l = 1, 2, \dots);\ \tau_l \in I_{t_0},\ \tau_l > t_0\ (l = 1, 2, \dots),\ \tau_i \neq \tau_j \text{ if } i \neq j, \text{ and } \lim_{l\to\infty} \tau_l = t_0.$$

Let

$$H = diag(h_1, h_2, \dots, h_n) : I_{t_0} \to R^{n\times n}$$

be a diagonal matrix-function with continuous diagonal elements

$$h_k : I_{t_0} \to (0,+\infty) \ (k = 1,2,...,n).$$

We consider the problem of finding of solution $x : I_{t_0} \to R^n$ of the system (1), (2) satisfying the condition

$$\lim_{t \to t_0+} (H^{-1}(t)\, x(t)) = 0. \tag{3}$$

The analogous problem for the systems of ordinary differential equations (1) with singularities is investigated in [1-3].

The singularity of the system (1) is considered in the sense that the matrix P and vector q functions, in general, are not integrable at the point t_0. So that, in general, the solution of the problem (1), (2); (3) is not continuous at the point t_0 and, therefore, it cannot be the solution in the classical sense. But its restriction on every interval from I_{t_0} is a solution of the system (1).

In connection with this we give the example from [1]. Consider the problem

$$\frac{dx}{dt} = -\frac{\alpha}{t} x + \varepsilon |t|^{\varepsilon - 1 - \alpha},$$

$$\lim_{t \to 0} (t^{\alpha} x(t)) = 0.$$

The problem has the unique solution $x(t) = |t|^{\varepsilon - \alpha} \operatorname{sgn} t$ if $\alpha > 0$ and $\varepsilon \in (0,\alpha)$. This function is not solution of the considered equation on the set $I = R$, but its restrictions on $(-\infty;0)$ and $(0;+\infty)$ are the solutions of equation one.

We give sufficient conditions for the unique solvability of the problem (1), (2); (3).The analogous results belong to I. Kiguradze [2, 3] for the Cauchy problem for the systems of ordinary differential equations with singularities.

Some boundary value problems for linear impulsive systems with singularities are investigated in [4-6] (see, also the references therein).

In the paper the following notation and definitions will be used:

$R =]-\infty;+\infty[$, $R_+ = [0;+\infty[$; $[a,b]$ and $]a,b[$ are closed and open intervals, respectively.

$R^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{i,j})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \max_{j=1...m} \sum_{i=1}^{n} |x_{ij}|,$$

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix.

if $X = (x_{ik})_{i,k=1}^{n,m}$, then

$$|X| = (|x_{ik}|)_{i,k=1}^{n,m} \text{ and } [X]_+ = \frac{|X| + X}{2}, \ [X]_- = \frac{|X| - X}{2}$$

$$R_+^{n\times m} = \{(x_{ij})_{i,j=1}^{n,m} : x_{ij} \geq 0 \ (i = 1,...,n; \ j = 1,...,m)\}.$$

$R^n = R^{n\times 1}$ is the space of all real column n_ vectors $x = (x_i)_{i=1}^n$.

If $X \in R^{n\times n}$, then X^{-1}, $det X$ and $r(X)$ are, the matrix inverse to X, the determinant of X and the spectral radius of X respectively; I_n is the identity $n \times n$ - matrix.

The inequalities between the matrices are understood component wise.

A matrix-function is said to be continuous, integrable, non decreasing, etc., if each of its components is such.

If $X: R \to R^{n\times m}$ is a matrix-function, then $\overset{b}{\underset{a}{V}}(X)$ is the sum of total variations on $[a,b]$ of its components x_{ik} $(i=1,...,n;\ k=1,...,m)$; if $a>b$, then we assume $\overset{b}{\underset{a}{V}}(X) = -\overset{a}{\underset{b}{V}}(X)$

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the martix-function $X:[a,b]\to R^{n\times m}$ at the point t; $\Delta(t) = X(t+) - X(t)$;

s_Δ and s_c are the operators defined, respectively, by

$$s_\Delta(X)(a_0) = s_c(X)(a_0) = 0;\ s_\Delta(X)(t) = s_\Delta(X)(s) + \sum_{s\le\tau<t} \Delta X(\tau)$$

$$s_c(X)(t) = s_c(X)(s) + X(t) - X(s) - \left(s_\Delta(X)(t) - s_\Delta(X)(s)\right) \text{ if } t_0 < s < t,$$

where $a_0 > t_0$ is some fixed point.

$\tilde{C}([a,b],D)$, where $D\subset R^{n\times m}$, is the set of all absolutely continuous matrix-functions $X:[a,b]\to D$.

$\tilde{C}_{loc}(I;R^{n\times m})$ is the set of all matrix-functions $X: I\to R^{n\times m}$ for which the restriction on $[a,b]$ belongs to $\tilde{C}_{loc}([a,b];R^{n\times m})$ for every closed interval $[a,b]$ from I.

$\tilde{C}_{loc}(I_{t_0}\setminus\{\tau_i\}_{i=1}^{\infty};D)$ is the set of all matrix-functions $X: I_{t_0}\to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from $I_{t_0}\setminus\{\tau_i\}_{i=1}^{\infty}$ belong to $\tilde{C}([a;b];D)$.

$L([a,b];D)$ is the set of all integrable matrix-functions $X:[a,b]\to D$.

$L_{loc}(I_{t_0};D)$ is the set of all matrix-functions $X: I_{t_0}\to D$ whose restrictions to an arbitrary closed interval $[a,b]$ from I_{t_0} belong to $L([a,b];D)$.

A vector-functions $x\in\tilde{C}_{loc}(I_{t_0}\setminus\{\tau_i\}_{i=1}^{\infty};R^n)$ is said to be a solution of the system (1), (2) if

$x'(t) = P(t)x(t) + q(t)$ for a. a. $t\in I_{t_0}\setminus\{\tau_i\}_{i=1}^{\infty}$ and there exist the onesided limits $x(\tau_l-)$ and $x(\tau_l+)$ $l=1,2,...$ such that the equalities (2) hold.

We will assume, without loss of generality that the solution x of the impulsive differential system (1), (2) is continuous from the left in the points of the impulses actions τ_l $(l=1,2,...)$, i.e. $x(\tau_l) = x(\tau_l-)$ $(l=1,2,...)$.

We assume that

$$\det(I_n + G_l) \ne 0 \ (l=1,2,...). \tag{4}$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems, i.e. for the case when $P\in L_{loc}(I;R^{n\times n})$ and $q\in L_{loc}(I;R^n)$ (See [7]).

Let $P_0\in L_{loc}(I_{t_0};R^{n\times n})$ and $G_{0l}\in R^{n\times n}$ $(l=1,2,...)$. Then a matrix-function $C_0: I_{t_0}\times I_{t_0}\to R^{n\times n}$ is said to be the Cauchy matrix of the homogeneous impulsive system

$$\frac{dx}{dt} = P_0(t)x, \tag{5}$$

$$x(\tau_l+) - x(\tau_l-) = G_{0l}x(\tau_l) \ (l=1,2,...), \tag{6}$$

if for every interval $J\subset I_{t_0}$ and $\tau\in J$, the restriction of the matrix-function $C_0(.,\tau): I_{t_0}\to R^{n\times n}$ on J is the fundamental matrix of the system (5),(6) satisfying the condition $C_0(\tau,\tau) = I_n$. Therefore, C_0 is the Cauchy matrix of the system (5),(6) if and only if the restriction of C_0 on $J\times J$, for every interval $J\subset I_{t_0}$, is the Cauchy matrix of the system in the sense of definition given in [7].

We assume

$$I_{t_0}^{+}(\delta) =]t_0, t_0+\delta] \cap I_{t_0}$$

for every $\delta > 0$.

2. Formulation of Main Results

Theorem 1. *Let there exist a matrix-function* $P_0 \in L_{loc}(I_{t_0}, R^{n\times n})$ *and constant matrices* $G_{0l} \in R^{n\times n}$ $(l = 1,2,...)$ *and* $B_0, B \in R_+^{n\times n}$ *such that the conditions (4),*

$$r(B) < 1 \tag{7}$$

and the estimates

$$|C_0(t,\tau)| \le H(t)B_0H^{-1}(\tau) \text{ for } \tau < t,\ \tau, t \in I_{t_0}^+(\delta),$$

$$\int_{t_0}^{t} |C_0(t,\tau)\cdot(P(\tau)-P_0(\tau))\cdot H(s)|ds + \sum_{t_0<\tau_l<t} |C_0(t,\tau_l)\cdot(I_n+G_{0l})^{-1}(G_l-G_{0l})| \le H(t)\cdot B \; for \; t \in I_{t_0}^+(\delta) \; hold for$$

some $\delta > 0$ *where* C_0 *is the Cauchy matrix of the system (5),(6).*

Let, moreover,

$$\lim_{t\to t_0+}\left\| \int_{t_0}^{t} H^{-1}(\tau)C_0(t,\tau)q(\tau)d\tau + \sum_{t_0<\tau_l<t} H^{-1}(\tau_l)C_0(t,\tau_l)(I_n+G_{0l})^{-1} g_l \right\| = 0 .$$

Then the problem (1), (2); (3) has the unique solution.

Theorem2. *Let there exist a constant matrix* $B = (b_{ik})_{i,k=1}^{n} \in R_+^{n\times n}$ *such that the condition (4) and*

$$[g_{lii}]_- < 1 \; (l = 1,2,...) \tag{8}$$

hold, and the estimates

$$c_i(t,\tau) \le b_0 \frac{h_i(t)}{h_i(\tau)} \quad for \; \tau < t,\ t,\tau \in I_{t_0}^+(\delta),$$

$$\left| \int_{t_0}^{t} c_i(t,\tau)h_i(\tau)[p_{ii}(\tau)]_+ d\tau + \sum_{t_0<\tau_l<t} c_i(t_l,\tau_l)h_i(\tau_l)[g_{lii}]_+ \right| \le b_{ii}h_i(t) \quad for \; t \in I_{t_0}^+(\delta) \; (i = 1,2,...n)$$

and

$$\left| \int_{t_0}^{t} c_i(t,\tau)h_k(\tau)|p_{ik}(\tau)|d\tau + \sum_{t_0<\tau_l<t} |c_i(t,\tau)|\cdot|1+g_{0ii}|^{-1} h_k(\tau)g_{ik} \right| \le b_{ik}h_i(t)$$

for $t \in I_{t_0}^+(\delta)$ $(i \ne k;\ i,k = 1,...,n)$

hold for some $b_0 > 0$ *and* $\delta > 0$. *Let, moreover,*

$$\lim_{t\to t_0+}\left(\int_{t_0}^{t} \frac{c_i(t,\tau)}{h_i(t)} q(\tau)d\tau + \sum_{t_0<\tau_l<t} \frac{c_i(t,\tau_l)}{h_i(t)}(1+g_{lii})g_l \right) = 0 \; (i = 1,...,n),$$

where c_i *is the Cauchy function of the impulsive differential equations*

$$\frac{dx}{dt} = p_{0ii}(t)x,$$

$$x(\tau_l+) - x(\tau_l) = g_{0lii}x(\tau_l) \; (l = 1,2,...)$$

$$p_{0ii}(t) \equiv -[p_{ii}(t)]_-, \; g_{0lii}(t) \equiv -[g_{lii}(t)]_- \; for \; i \in \{1,...,n\} \; and \; l \in \{1,2,...\}.$$

Then the problem (1),(2); (3) has the unique solution.

Remark 1. The Cauchy functions $c_i(t,\tau)$ $(i = 1,...,n)$ mentioned in the theorem, for $t,\tau \in I_{t_0}^+(\delta)$, have the form:

$$C_i(t,\tau)=\begin{cases}\exp\int\limits_{\tau}^{t} p_{0ii}(s)ds\cdot\prod\limits_{\tau\le\tau_l<t}(1+g_{0lii}) & for\quad t>\tau\\ \exp\int\limits_{\tau}^{t} p_{0ii}(s)ds\cdot\prod\limits_{\tau\le\tau_l<t}(1+g_{0lii})^{-1} & for\quad t<\tau\ .\\ 1 & for\quad t=\tau\end{cases}$$

Corollary 1. *Let there exist a constant matrix* $B=(b_{ik})_{i,k=1}^{n}\in R_{+}^{n\times n}$ *such that the conditions (7) and (8) hold, and the estimates*

$$\int\limits_{\tau}^{t} p_{0ii}(\tau)d\tau+\sum_{\tau<\tau_l<t}\ln\left|1+g_{0lii}\right|\le-\lambda_1\ln\frac{t-t_0}{\tau-t_0}\ \ for\ \ \tau<t,\ \ \tau,t\in I_{t_0}^{+}(\delta),$$

$$\lim_{\tau\to t_0+}\left|\int\limits_{\tau}^{t}\left[p_{ii}(\tau)\right]_{+}dr+\sum_{\tau<\tau_l<t}\left[g_{lii}\right]_{+}\right|\le b_{ii}\ \ for\ \ t\in I_{t_0}^{+}(\delta)\ \ (i=1,\dots,n)$$

and

$$\lim_{\tau\to t_0+}\left|\int\limits_{\tau}^{t}\left|p_{ik}(\tau)\right|d\tau+\sum_{\tau\le\tau_l<t}\left|(1+g_{0lii})^{-1}\cdot g_{ik}\right|\right|\le b_{ik}\ \ for\ \ t\in I_{t_0}^{+}(\delta)\ \ (i\ne k;i,k=1,\dots,n)$$

hold for some $\mu_i\ge 0\ (i=1,\dots,n)$ *and* $\delta>0$. *Let moreover,*

$$\lim_{\tau\to t_0+}\int\limits_{t_0}^{t}\frac{|q_i(\tau)|}{|\tau-t_0|^{\mu_i}}d\tau+\sum_{t_0<\tau_l<t}\frac{\left|(1+g_{0lii})^{-1}g_{li}\right|}{|\tau_l-t_0|^{\mu_i}}=0\ \ (i=1,\dots,n)$$

Then the system (1), (2) has the unique solution satisfying the initial condition

$$\lim_{t\to t_0+}\frac{x_i(t)}{|t-t_0|^{\mu_i}}=0\ \ (i=1,\dots,n). \tag{9}$$

Remark 2. Let, in addition, of conditions of the **Corollary 1**, the condition

$$\limsup_{t\to t_0+}\xi_{ji}(t)<+\infty\ \ (j=1,2;\ \ i=1,\dots,n)$$

holds, where

$$\xi_{ji}(t)=\sum_{\tau\in I_{tj}}\sum_{k=1}^{n}|\tau-t_0|^{\mu_k}|g_{ik}|+|g_l|\ \ \text{for}\ \ t\in I_{t_0}^{+}(\delta)\ \ (j=1,2;\ \ i=1,\dots,n).$$

$I_{t_1}=]t,t_0+\delta]$ and $I_{t_2}=]\mathrm{t},\mathrm{t}_0+\delta[$, and δ is a small position number.

Then the solution of the problem (1), (2); (9) belongs to $\tilde{C}_{loc}(I,R^n)$.

მათემატიკა

კოშის ამოცანა სინგულარობებიან წრფივ იმპულსურ დიფერენციალურ განტოლებათა სისტემებისთვის

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***სოხუმის სახელმწიფო უნივერსიტეტი, მათემატიკისა და კომპიუტერულ მეცნიერებათა ფაკულტეტი, თბილისი, საქართველო*

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სტატიაში განხილულია კოშის ამოცანა სინგულარობებიან წრფივ იმპულსურ დიფერენციალურ განტოლებათა სისტემებისათვის. სინგულარობა გაიგება იმ აზრით, რომ ამ სისტემის შესაბამისი მატრიცული და ვექტორული ფუნქციები, საზოგადოდ, შეიძლება არ იყოს ინტეგრებადი საწყის წერტილში. მიღებულია მოცემული საწყისი ამოცანის ცალსახად ამოხსნადობის საკმარისი პირობები.

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Received July, 2018