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Lyapunov Stability of Systems of Linear Generalized Ordinary Differential Equations

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Abstract—Effective necessary and sufficient conditions are established for the stability in the Lyapunov sense of solutions of the linear system of generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t),$$

where $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $f : \mathbb{R}_+ \to \mathbb{R}^n$ ($\mathbb{R}_+ = [0, +\infty[$) are, respectively, matrix- and vector-functions with bounded total variation components on every closed interval from $\mathbb{R}_+$, having properties analogous to the case of systems of ordinary differential equations with constant coefficients. The obtained results are realized for linear systems of both impulsive equations and difference equations.
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1. STATEMENT OF THE PROBLEM AND FORMULATION OF THE RESULTS

Let $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $f : \mathbb{R}_+ \to \mathbb{R}^n$ ($\mathbb{R}_+ = [0, +\infty[$) be, respectively, matrix- and vector-functions with bounded total variation components on every closed interval from $\mathbb{R}_+$. Consider the system of linear generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t). \tag{1.1}$$

In this paper, the problem on the stability in the Lyapunov sense with respect to small perturbations is investigated for solutions of system (1.1). In particular, effective necessary and sufficient conditions are obtained for the stability and asymptotic stability of this system which generalize the previous one in [1,2]. They are the analogues of the well-known conditions for the stability of linear ordinary differential systems with constant coefficients (see, e.g., [3,4]).

To a considerable extent, the interest to the theory of generalized ordinary differential equations has been stimulated also by the fact that this theory enables one to investigate ordinary

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differential, impulsive, and difference equations from the unified viewpoint. In particular, in form (1.1) can be rewritten:

(a) the impulsive system

$$\frac{dx}{dt} = Q(t)x + q(t), \qquad \text{for } t \in \mathbb{R}_+, \tag{1.2}$$

$$x(t_k+) - x(t_k-) = G_k x(t_k-) + g_k, \qquad k = 1, 2, \dots, \tag{1.3}$$

where $Q : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and $q : \mathbb{R}_+ \to \mathbb{R}^n$ are, respectively, a matrix- and a vector-function with Lebesgue integrable components on every closed interval from $\mathbb{R}_+$; $G_k \in \mathbb{R}^{n\times n}$ $(k = 1, 2, \dots)$, $g_k \in \mathbb{R}^n$ $(k = 1, 2, \dots)$, $0 < t_1 < t_2 < \cdots$, $\lim_{k\to+\infty} t_k = +\infty$;

(b) the difference system

$$\Delta y(k-1) = G_1(k-1)y(k-1) + G_2(k)y(k) + G_3(k)y(k+1) + g_0(k), \qquad k = 1, 2, \dots, \tag{1.4}$$

where $G_j(k) \in \mathbb{R}^{n\times n}$ and $g_0(k) \in \mathbb{R}^n$ $(j = 1, 2, 3;\ k = 0, 1, \dots)$.

Quite a few questions of the theory of generalized ordinary differential equations (both linear and nonlinear) have been studied sufficiently well (see [1,2,5–15] and the references therein). In particular, some questions of stability have been investigated, e.g., in [1,9,10,14] (see also the references therein). Analogous questions are investigated, e.g., in [1,2,5,8,16–18] for impulsive and difference systems.

Throughout in the paper, the following notation and definitions will be used.

$\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = 0 \cup \mathbb{N}$, $\mathbb{R} =]-\infty, +\infty[$, $[a, b]$ $(a, b \in \mathbb{R})$ is a closed interval. I is an arbitrary closed or open interval from $\mathbb{R}$. $[t]$ is the integral part of $t \in \mathbb{R}$. $\mathbb{C}$ is the space of all complex numbers z; $|z|$ is the modulus of z.

$\mathbb{R}^{n\times m}(\mathbb{C}^{n\times m})$ is the space of all real (complex) $n \times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm

$$\|X\| = \sum_{i=1}^{n} \sum_{j=1}^{m} |x_{ij}|;$$

$|X| = (|x_{ij}|)_{i,j=1}^{n,m}$; $O_{n\times m}$ (or O) is the zero $n \times m$-matrix.

If $X \in \mathbb{C}^{n\times n}$, then X^{-1} is the matrix,inverse to X; $\det X$ is the determinant of X, $\ln X$ is the logarithm (the principal value) of X, and $r(X)$ is the spectral radius of X. $\operatorname{diag}(X_1, \dots, X_m)$, where $X_i \in \mathbb{C}^{n_i\times n_i}$ $(i = 1, \dots, m)$, $n_1 + \cdots n_m = n$, is a quasidiagonal $n \times n$-matrix; I_n is the identity $n \times n$-matrix; δ_{ij} is the Kronecker symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1, 2, \dots)$; $Z_n = (\delta_{i+1j})_{i,j=1}^{n}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$.

The inequalities between the real vectors (matrices) are understood componentwise.

If $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ is a matrix-function, then $\vee_a^b(X)$ is the sum of total variations on $[a, b]$ of its components x_{ij} $(i = 1, \dots, n;\ j = 1, \dots, m)$; $V(X)(t) = (v(x_{ij})(t))_{i,j=1}^{n,m}$, where $v(x_{ij})(0) = 0$, $v(x_{ij})(t) = \vee_0^t(x_{ij})$ for $t > 0$ $(i = 1, \dots, n;\ j = 1, \dots, m)$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits at the point $t \in \mathbb{R}_+$, $(X(0-) = X(0))$; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}([a, b]; \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : [a, b] \to \mathbb{R}^{n\times m}$ such that$\vee_a^b(X) < +\infty$.

$\mathrm{BV}_{\mathrm{loc}}(I; \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : I \to \mathbb{R}^{n\times m}$ such that $\vee_a^b(X) < +\infty$ for $a, b \in I$.

$L_{\mathrm{loc}}(I; \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : I \to \mathbb{R}^{n\times m}$ whose components are the functions measurable and Lebesgue integrable on every closed interval from $\mathbb{R}_+$.

$\tilde{C}_{\mathrm{loc}}(I; \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : I \to \mathbb{R}^{n\times m}$ whose components are the functions absolutely continuous on every closed interval from I.

$\tilde{C}_{\rm loc}(\mathbb{R}_+ \setminus \{t_k\}_{k=1}^{\infty}; \mathbb{R}^{n\times m})$, where $0 < t_1 < t_2 < \cdots$, is the set of all matrix-functions $X : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ whose restrictions to an arbitrary interval $I \subset \mathbb{R}_+ \setminus \{t_k\}_{k=1}^{\infty}$ belong to $\tilde{C}_{\rm loc}(I; \mathbb{R}^{n\times m})$.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if such is every its component.

s_j, $J : \mathrm{BV}_{\rm loc}(\mathbb{R}_+; \mathbb{R}) \to \mathrm{BV}_{\rm loc}(\mathbb{R}_+; \mathbb{R})$ $(j = 0, 1, 2)$ are the operators defined, respectively, by

$$\begin{aligned}
&s_1(x)(0) = s_2(x)(0) = 0;\\
&s_1(x)(t) = \sum_{o<\tau\le t} d_1x(\tau), \quad s_2(x)(t) = \sum_{o\le\tau<t} d_2x(\tau), \qquad \text{for } t > 0;\\
&s_0(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t), \qquad \text{for } t \in \mathbb{R}_+,
\end{aligned}$$

and

$$\begin{aligned}
&J(x)(0) = x(0),\\
&J(x)(t) = s_0(x)(t) - \sum_{0<\tau\le t} \ln|1 - d_1x(\tau)| + \sum_{0\le\tau<t} \ln|1 + d_2x(\tau)|, \qquad \text{for } t > 0.
\end{aligned}$$

If $g : \mathbb{R}_+ \to \mathbb{R}$ is a nondecreasing function, $x : \mathbb{R}_+ \to \mathbb{R}$ and $0 \le s < t$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,ds_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau)\,d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)\,d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)\,ds_0g(\tau)$ is the Lebesgue-Stieltjes integral over the open interval $]s,t[$ with respect to the measure corresponding to the function $s_0(g)$ (if $s = t$, then $\int_s^t x(\tau)\,dg(\tau) = 0$).

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_s^t x(\tau)\,ds_1(g)(\tau) - \int_s^t x(\tau)\,ds_2(g)(\tau), \qquad \text{for } 0 \le s \le t;$$

$L_{\rm loc}(\mathbb{R}_+, \mathbb{R}; g)$ is the set of all functions $x : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\left|\int_0^b x(t)\right| dg_j(t) < +\infty, \qquad \text{for } b \in \mathbb{R}_+, \quad j = 1, 2.$$

If $G = (g_{ik})_{i,k=1}^{l,n} \in \mathrm{BV}_{\rm loc}(\mathbb{R}_+; \mathbb{R}^{l\times n})$ and $X = (x_{kj})_{k,j=1}^{n,m} : \mathbb{R}_+ \to \mathbb{R}^{n\times m}$, then

$$S_j(G)(t) \equiv (s_j(g_{ik})(t))_{i,k=1}^{l,n}, \qquad j = 0, 1, 2,$$

and

$$\int_s^t dG(\tau)\cdot X(\tau) = \left(\sum_{k=1}^{n} \int_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\right)_{i,j=1}^{l,m}, \qquad \text{for } 0 \le s \le t.$$

$\mathcal{A} : \mathrm{BV}_{\rm loc}(\mathbb{R}_+; \mathbb{R}^{n\times n}) \times \mathrm{BV}_{\rm loc}(\mathbb{R}_+; \mathbb{R}^{n\times m}) \to \mathrm{BV}_{\rm loc}(\mathbb{R}_+; \mathbb{R}^{n\times m})$ is the operator defined by

$$\begin{aligned}
&\mathcal{A}(X,Y)(0) = Y(0),\\
&\mathcal{A}(X,Y)(t) = Y(t) + \sum_{0<\tau\le t} d_1X(\tau)\cdot(I_n - d_1X(\tau))^{-1}\,d_1Y(\tau)\\
&\qquad - \sum_{0\le\tau<t} d_2X(\tau)\cdot(I_n + d_2X(\tau))^{-1}\,d_2Y(\tau), \qquad \text{for } t > 0.
\end{aligned}$$

We say that the matrix-function $X \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ satisfies the Lappo-Danilevskiĭ condition if the matrices $S_0(X)(t)$, $S_1(X)(t)$, and $S_2(X)(t)$ are pairwise permutable for every $t \in \mathbb{R}_+$ and

$$\int_0^t S_0(X)(\tau)\,dS_0(X)(\tau) = \int_0^t dS_0(X)(\tau)\cdot S_0(X)(\tau), \qquad \text{for } t \in \mathbb{R}_+.$$

$E(J;D)$, where $J \subset \mathbb{N}_0$ and $D \subset \mathbb{R}^{n\times m}$, is the set of all matrix-functions $Y : J \to D$.

Δ is the first-order difference operator, i.e.,

$$\Delta y(k-1) = y(k) - y(k-1), \qquad k = 1,2,\dots, \quad \text{for } y \in E(\mathbb{N}_0;\mathbb{R}^n).$$

We use the following formulas:

$$\int_a^b f(t)\,d\left(\int_a^t g(s)\,dh(s)\right) = \int_a^b f(t)\,g(t)\,dh(t) \qquad \text{(substitution formula)};$$

$$\int_a^b f(t)\,dg(t) + \int_a^b f(t)\,dg(t) = f(b)g(b) - f(a)g(a) + \sum_{a<t\le b} d_1 f(t)\cdot d_1 g(t) - \sum_{a\le t<b} d_2 f(t)\cdot d_2 g(t) \qquad \text{(integration-by-parts formula)};$$

$$\int_a^b h(t)\,d(f(t)g(t)) = \int_a^b h(t)f(t)\,dg(t) + \int_a^b h(t)g(t)\,df(t) - \sum_{a<t\le b} h(t)d_1 f(t)\cdot d_1 g(t) + \sum_{a\le t<b} h(t)d_2 f(t)\cdot d_2 g(t)$$

(general integration-by-parts formula);

$$\int_a^b f(t)\,ds_1(g)t = \sum_{a<t\le b} f(t)d_1 g(t), \quad \int_a^b f(t)\,ds_2(g)t = \sum_{a\le t<b} f(t)d_2 g(t),$$

and

$$d_j\left(\int_a^t f(s)\,dg(s)\right) = f(t)d_j g(t), \qquad \text{for } t \in [a,b], \quad j = 1,2,$$

for $f,g,h \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$; $a,b \in \mathbb{R}_+$; $a < b$ (see [15, Theorems I.4.25, I.4.33, Lemma I.4.23]).

By a solution of system (1.1) (of the system of generalized differential inequalities

$$dx(t) \le dA(t)\cdot x(t) + df(t))$$

we understand a vector-function $x \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^n)$ such that

$$x(t) - x(s) = \int_s^t dA(\tau)\cdot x(\tau) + f(t) - f(s)(\le), \qquad \text{for } 0 \le s \le t.$$

We assume that $A \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times n})$, $A(t) = (a_{ij}(t))_{i,j=1}^n$, $A(0) = O_{n\times n}$, $f \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^n)$, and

$$\det\left(I_n + (-1)^j d_j A(t)\right) \ne 0, \qquad \text{for } t \in \mathbb{R}_+, \quad j = 1,2. \tag{1.5}$$

Condition (1.5) guarantees the unique solvability of the Cauchy problem for system (1.1) (see [15, Theorem III.1.4]).

Let $X \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be a fundamental matrix of the homogeneous system

$$dx(t) = dA(t)\cdot x(t), \tag{1.1$_0$}$$

and let x be a solution of system (1.1). Then

$$x(t) = f(t) - f(t_0) + X(t)\left\{X^{-1}(t_0)x(t_0) - \int_{t_0}^{t} dX^{-1}(s)\cdot(f(s) - t(t_0))\right\}, \qquad \text{for } t_0, t \in \mathbb{R}_+$$

(variation-of-constants formula, see [15, Theorem III.2.13]).

If $\beta \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$ is such that

$$1 + (-1)^j d_j\beta(t) \neq 0, \qquad \text{for } t \in \mathbb{R}_+, \quad j = 1, 2,$$

then by $\gamma(\beta)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t) = \gamma(t)\, d\beta(t), \qquad \gamma(0) = 1.$$

It is known (see [11,12]) that $\gamma(\beta)(0) = 1$,

$$\gamma(\beta)(t) = \exp\left(s_0(\beta)(t) - s_0(\beta)(0)\right) \prod_{0<\tau\leq t} (1 - d_1\beta(\tau))^{-1} \prod_{0\leq\tau<t} (1 + d_2\beta(\tau)), \qquad \text{for } t > 0.$$

The stability in one or another sense of a solution of system (1.1) is defined in the same way as for systems of ordinary differential equations.

DEFINITION 1.1. *System (1.1) is called stable in one or another sense if every its solution is stable in the same sense.*

It is evident that system (1.1) is stable if and only if the zero solution of its corresponding homogeneous system (1.1_0) is stable in the same sense.

Therefore the stability is not the property of some solution of system (1.1); it is the common property of all solutions, and the vector-function f does not affect this property. Hence it is the property only of the matrix-function A. Thus, the following definition is natural.

DEFINITION 1.2. *A matrix-function A is called stable in one or another sense if system (1.1_0) is stable in the same sense.*

THEOREM 1.1. *Let the matrix-function $A \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$S_0(A)(t) = \sum_{l=1}^{m} s_0(\alpha_l)(t)\cdot B_l, \qquad \text{for } t \in \mathbb{R}_+, \tag{1.6}$$

and

$$I_n + (-1)^j d_jA(t) = \exp\left((-1)^j \sum_{l=1}^{m} d_j\alpha_l(t)\cdot B_l\right), \qquad \text{for } t \in \mathbb{R}_+, \quad j = 1, 2, \tag{1.7}$$

where $\alpha_l \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ $(l = 1,\dots,m)$, and $B_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m)$ are pairwise permutable constant matrices. Let, moreover, $(\lambda - \lambda_{li})^{n_{li}}$ $(i = 1,\dots,m_l;\ \sum_{i=1}^{m_l} n_{li} = n)$ be elementary divisors of the matrix B_l for every $l \in \{1,\dots,m\}$. Then:

(a) *the matrix-function A is stable if and only if*

$$\sup\left\{\prod_{l=1}^{m}\left(\sum_{i=1}^{m_l}(1+\alpha_l(t))^{n_{li}-1}\exp(\alpha_l(t)\,\mathrm{Re}\,\lambda_{li})\right) : t \in \mathbb{R}_+\right\} < +\infty; \tag{1.8}$$

(b) *the matrix-function A is asymptotically stable if and only if*

$$\lim_{t\to+\infty}\prod_{l=1}^{m}\left(\sum_{i=1}^{m_l}(1+\alpha_l(t))^{n_{li}-1}\exp(\alpha_l(t)\,\mathrm{Re}\,\lambda_{li})\right) = 0. \tag{1.9}$$

Corollary 1.1. *Let conditions (1.6) and (1.7) hold, where* $B_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m)$ *are pairwise permutable constant matrices, and* $\alpha_l \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ $(l = 1,\dots,m)$ *are such that*

$$\lim_{t\to+\infty} a_l(t) = +\infty, \qquad l = 1,\dots,m. \tag{1.10}$$

Then:

(a) *the matrix-function A is stable if and only if every eigenvalue of the matrices B_l ($l = 1,\dots,m$) has the nonpositive real part; in addition, every elementary divisor, corresponding to the eigenvalue with the zero real part, is simple;*
(b) *the matrix-function A is asymptotically stable if and only if every eigenvalue of the matrices B_l ($l = 1,\dots,m$) has the negative real part.*

If the matrix-function $A \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ has at most a finite number of discontinuity points in $[a,t]$ for every $t > 0$, then by $\nu_1(t)$ and $\nu_2(t)$ we denote, respectively, a number of points $\tau \in]0,t]$ for which $\|d_1A(\tau)\| \neq 0$ and a number of points $\tau \in [0,t[$, for which $\|d_2A(\tau)\| \neq 0$.

Corollary 1.2. *Let $A \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$S_0(A)(t) = \alpha(t)A_0, \qquad \text{for } t \in \mathbb{R}_+,$$

and

$$d_jA(t) = A_j, \qquad \text{if } \|d_jA(t)\| \neq 0, \quad t \in \mathbb{R}_+, \quad j = 1,2,$$

where $\alpha \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ is a continuous function satisfying

$$\lim_{t\to+\infty} \alpha(t) = +\infty,$$

and A_0, A_1, and $A_2 \in \mathbb{R}^{n\times n}$ are pairwise permutable constant matrices. Let, moreover, there exist numbers $\beta_1, \beta_2 \in \mathbb{R}_+$ such that

$$\limsup_{t\to+\infty} |\nu_j(t) - \beta_j\alpha(t)| < +\infty, \qquad j = 1,2. \tag{1.11}$$

Then:

(a) *the matrix-function A is stable if and only if every eigenvalue of the matrix $P = A_0 - \beta_1\ln(I_n - A_1) + \beta_2\ln(I_n + A_2)$ has the nonpositive real part; in addition, every elementary divisor, corresponding to the eigenvalue with the zero real part, is simple;*
(b) *the matrix-function A is asymptotically stable if and only if every eigenvalue of the matrix P has the negative real part.*

Corollary 1.3. *Let the matrix-function $A \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$S_0(A)(t) = C\,\mathrm{diag}\,(S_0(G_1)(t),\dots,S_0(G_m)(t))\,C^{-1}, \qquad \text{for } t \in \mathbb{R}_+,$$

and

$$I_n + (-1)^j d_jA(t) = C\,\mathrm{diag}\left(\exp\left((-1)^j d_jG_1(t)\right),\dots,\exp\left((-1)^j d_jG_m(t)\right)\right)C^{-1},$$
$$\text{for } t \in \mathbb{R}_+, \quad j = 1,2,$$

where $C \in \mathbb{C}^{n\times n}$ is a nonsingular complex matrix, $G_l(t) = \sum_{i=0}^{n_l-1}\alpha_{li}(t)Z_{n_l}^i$ ($l = 1,\dots,m$; $\sum_{l=1}^m n_l = n$), $\alpha_{li} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ ($l = 1,\dots,m$; $i = 1,\dots,n_l - 1$), and α_{l_0} is a complex-valued function such that $\mathrm{Re}\,\alpha_{l0}$ and $\mathrm{Im}\,\alpha_{l0} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$. Then:

(a) *the matrix-function A is stable if and only if*

$$\sup\left\{\exp\left(\mathrm{Re}\,\alpha_{l_0}(t)\right)\prod_{i=1}^{n_l-1}\left(1 + \alpha_{l_i}(t)\right)^{[(n_l-1)/i]} : t \in \mathbb{R}_+\right\} < +\infty, \qquad l = 1,\dots,m;$$

(b) *the matrix-function A is asymptotically stable if and only if*

$$\lim_{t\to+\infty}\exp\left(\mathrm{Re}\,\alpha_{l_0}(t)\right)\prod_{i=1}^{n_l-1}\left(1 + \alpha_{l_i}(t)\right)^{[(n_l-1)/i]} = 0, \qquad l = 1,\dots,m.$$

THEOREM 1.2. *Let $\alpha_{il} \in \mathbb{R}$ $(i, l = 1, \dots, n)$, and $\mu_i : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, \dots, n)$ be nondecreasing functions such that $s_0(\mu_i) \in \tilde{C}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}_+)$ $(i = 1, \dots, n)$ and*

$$\lim_{t\to+\infty} a_0(t) = +\infty, \quad \sigma_i = \liminf_{t\to+\infty} (\alpha_{ii} d_2 \mu_i(t)) > -1, \qquad i = 1, \dots, n, \tag{1.12}$$

where $a_0(t) \equiv \int_0^t \eta_0(s)\,ds + \sum_{0<s\le t} \ln|1 - \eta_1(s)| - \sum_{0\le s<t} \ln|1 + \eta_2(s)|$, $\eta_0(t) \equiv \min\{|\alpha_{ii}| \cdot (s_0(\mu_i)(t))' : i = 1, \dots, n\}$, $\eta_j(t) \equiv \max\{\alpha_{ii} d_j \mu_i(t) : i = 1, \dots, n\}$ $(j = 1, 2)$. Then the condition

$$\alpha_{ii} < 0, \qquad i = 1, \dots, n, \quad r(H) < 1, \tag{1.13}$$

where $H = ((1 - \delta_{il})(1 + |\sigma_i|)^{-1} |\alpha_{il}| |\alpha_{ii}|^{-1})_{i,l=1}^n$, is sufficient for the matrix-function $A(t) = (\alpha_{il}\mu_i(t))_{i,l=1}^n$ to be asymptotically stable; and if

$$\alpha_{il} \ge 0, \qquad i \ne l; \quad i, l = 1, \dots, n, \tag{1.14}$$

and

$$\sum_{l=1,\, l\ne i}^{n} \alpha_{il} d_1 \mu_i(t) < \min\{1 - \alpha_{ii} d_1 \mu_i(t), |1 + \alpha_{ii} d_1 \mu_i(t)|\}, \qquad \text{for } t \in \mathbb{R}_+, \quad i = 1, \dots, n, \tag{1.15}$$

then condition (1.13) is necessary as well.

1.1. Impulsive Systems

By a solution of the impulsive system (1.2),(1.3) we understand a continuous from the left vector-function $x \in \tilde{C}_{\mathrm{loc}}(\mathbb{R}_+ \setminus T; \mathbb{R}^n)$ $(T = \{t_1, t_2, \dots\})$ satisfying both system (1.2) almost everywhere on $]t_k, t_{k+1}[$ and relation (1.3) at the point t_k for every $k \in \{1, 2, \dots\}$.

The stability in one or another sense of solutions of system (1.2),(1.3) as well as the stability of that system is defined as above.

Besides the homogeneous system, corresponding to the impulsive system (1.2),(1.3), is defined by the pair $(Q, \{G_k\}_{k=1}^{\infty})$. Therefore in this case we discuss the stability of this pair instead of the stability of the matrix-function A.

We assume

$$\det(I_n + G_k) \ne 0, \qquad k = 1, 2, \dots. \tag{1.16}$$

By $\nu(t)$ $(t > 0)$ we denote a number of the points t_k $(k = 1, 2, \dots)$ belonging to $[0, t[$.

THEOREM 1.3. *Let $Q \in L_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ and $G_k \in \mathbb{R}^{n\times n}$ $(k = 1, 2, \dots)$ be such that*

$$\int_0^t Q(\tau)\,d\tau = \sum_{l=1}^{m} \alpha_{0l}(t) B_l, \qquad \text{for } t \in \mathbb{R}_+, \tag{1.17}$$

and

$$G_k = \exp\left(\sum_{l=1}^{m} \alpha_{kl} B_l\right) - I_n, \qquad k = 1, 2, \dots. \tag{1.18}$$

Here $B_l \in \mathbb{R}^{n\times n}$ $(l = 1, \dots, m)$ are pairwise permutable constant matrices, $\alpha_{0l} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R})$ $(l = 1, \dots, m)$ are continuous functions, and $\alpha_{kl} \in \mathbb{R}$ $(l = 1, \dots, m;\ k = 1, 2, \dots)$ are numbers such that $\alpha_l(t) \ge 0$ for $t \in \mathbb{R}_+$ $(l = 1, \dots, m)$, where

$$\alpha_l(t) = \alpha_{0l}(t) + \sum_{0\le t_k<t} \alpha_{kl}, \qquad \text{for } t \in \mathbb{R}_+, \quad l = 1, \dots, m. \tag{1.19}$$

Let, moreover, $(\lambda - \lambda_{li})^{n_{li}}$ $(i = 1, \dots, m_l;\ \sum_{i=1}^{m_l} n_{li} = n)$ be the elementary divisors of the matrix B_l for every $l \in \{1, \dots, m\}$. Then the pair $(Q, \{G_k\}_{k=1}^{\infty})$ is stable (asymptotically stable) if and only if condition (1.8) (condition (1.9)) holds.

COROLLARY 1.4. *Let $Q \in L_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ and $G_k \in \mathbb{R}^{n\times n}$ $(k = 1,2,\dots)$ be such that conditions (1.17) and (1.18) hold, where $B_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m)$ are pairwise permutable constant matrices, $\alpha_{0l} \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ $(l = 1,\dots,m)$ are continuous functions, and $\alpha_{kl} \in \mathbb{R}$ $(l = 1,\dots,m;\ k = 1,2,\dots)$ are numbers such that the functions $\alpha_l(t)$ $(l = 1,\dots,m)$, defined by (1.19), are nonnegative and satisfy condition (1.10). Then:*

(a) *the pair $(Q,\{G_k\}_{k=1}^{\infty})$ is stable if and only if every eigenvalue of the matrices $B_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m)$ has the nonpositive real part; in addition, every elementary divisor, corresponding to the eigenvalue with the zero real part, is simple;*

(b) *the pair $(Q,\{G_k\}_{k=1}^{\infty})$ is asymptotically stable if and only if every eigenvalue of the matrices $B_l \in \mathbb{R}^{n\times n}$ $(l = 1,\dots,m)$ has the negative real part.*

COROLLARY 1.5. *Let*

$$Q(t) = \alpha(t)Q_0, \qquad \text{for } t \in \mathbb{R}_+, \quad G_k = G_0, \quad k = 1,2,\dots,$$

and there exist $\beta \in \mathbb{R}_+$, such that

$$\limsup_{t\to+\infty} |\nu(t) - \beta t| < +\infty,$$

where Q_0 and G_0 are permutable constant matrices, and $\alpha \in L_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$ is such that

$$\int_0^{+\infty} \alpha(t)\,dt = +\infty.$$

Then:

(a) *the pair $(Q,\{G_k\}_{k=1}^{\infty})$ is stable if and only if every eigenvalue of the matrix $P = Q_0 + \beta\ln(I_n + G_0)$ has the nonpositive real part; in addition, every elementary divisor, corresponding to the eigenvalue with the zero real part, is simple;*

(b) *the pair $(Q,\{G_k\}_{k=1}^{\infty})$ is asymptotically stable if and only if every eigenvalue of the matrix P has the negative real part.*

COROLLARY 1.6. *(See [18].) Let $Q(t) \equiv Q_0$, $G_k = G_0$ $(k = 1,2,\dots)$, and $t_{k+1} - t_k = \eta = \mathrm{const}$ $(k = 1,2,\dots)$, where Q_0 and G_0 are permutable constant matrices. Then the conclusion of Corollary 1.4 is true, where $P = Q_0 + \eta^{-1}\ln(I_n + G_0)$.*

THEOREM 1.4. *Let $\alpha_{il} \in \mathbb{R}$, $\nu_{ki} \in \mathbb{R}_+$ $(i,l = 1,\dots,n;\ k = 1,2,\dots)$, and $\nu_i \in L_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ $(i = 1,\dots,n)$ be functions such that the conditions*

$$\int_0^{+\infty} \eta(s)\,ds + \sum_{0\le t_k<\infty} \ln|1+\eta_k| = -\infty$$

and

$$\sigma_i = \liminf_{k\to+\infty} (\alpha_{ii}\nu_{ki}) > -1, \qquad i = 1,\dots,n,$$

hold, where $\eta(t) \equiv \min\{|\alpha_{ii}|\nu_i(t) : i = 1,\dots,n\}$, $\eta_k = \max\{\alpha_{ii}\nu_{ki} : i = 1,\dots,n\}$ $(k = 1,2,\dots)$. Then condition (1.13), where $H = ((1-\delta_{il})(1+|\sigma_i|)^{-1}|\alpha_{il}||\alpha_{ii}|^{-1})_{i,k=1}^{n}$, is sufficient for the pair $(Q,\{G_k\}_{k=1}^{\infty})$ to be asymptotically stable, where $Q(t) \equiv (\alpha_{il}\nu_i(t))_{i,l=1}^{n}$ and $Q_k = (\alpha_{il}\nu_{ki})_{i,l=1}^{n}$ $(k = 1,2,\dots)$; and if condition (1.14) holds, then condition (1.13) is necessary as well.

REMARK 1.1. From Theorems 1.3, 1.4, and Corollaries 1.4–1.6, if we assume $G_k = O_{n\times n}$, $\alpha_{kl} = 0$, $\nu_{ki} = 0$ $(l = 1,\dots,m;\ i = 1,\dots,n;\ k = 1,2,\dots)$ and $\beta = 0$, follow some results for the stability and asymptotic stability for the linear system of ordinary differential equations

$$\frac{dx}{dt} = Q(t)x + q(t), \qquad \text{for } t \in \mathbb{R}_+.$$

1.2. Difference Systems

Let $y_0 \in E(\mathbb{N}_0;\mathbb{R}^n)$ be a solution of the difference system (1.4) and let $G \in E(\mathbb{N}_0;\mathbb{R}^{n\times n})$ be an arbitrary matrix-function.

DEFINITION 1.3. *A solution $y_0 \in E(\mathbb{N}_0;\mathbb{R}^n)$ of system (1.4) is called G-stable if for every $\varepsilon > 0$ and $k_0 \in \mathbb{N}_0$ there exists $\delta = \delta(\varepsilon, k_0)$, such that for every solution y of system (1.4), satisfying*

$$\|(I_n + G(k_0))(y(k_0) - y_0(k_0))\| + \|y(k_0+1) - y_0(k_0+1)\| < \delta,$$

the estimate

$$\|(I_n + G(k))(y(k) - y_0(k))\| + \|y(k+1) - y_0(k+1)\| < \varepsilon, \qquad \text{for } k > k_0,$$

holds.

DEFINITION 1.4. *A solution $y_0 \in E(\mathbb{N}_0;\mathbb{R}^n)$ of system (1.4) is called G-asymptotically stable if it is G stable and for every $k_0 \in \mathbb{N}_0$ there exists $\delta = \delta(k_0) > 0$ such that for every solution y of system (1.4), satisfying*

$$\|(I_n + G(k_0))(y(k_0) - y_0(k_0))\| + \|y(k_0+1) - y_0(k_0+1)\| < \delta,$$

the condition

$$\lim_{k\to+\infty} (\|(I_n + G(k))(y(k) - y_0(k))\| + \|y(k+1) - y_0(k+1)\|) = 0$$

holds.

We say that y_0 is stable (asymptotically stable) if it is $O_{n\times n}$-stable ($O_{n\times n}$-asymptotically stable).

DEFINITION 1.5. *System (1.4) is called G-stable (G-asymptotically stable) if every its solution is G-stable (G-asymptotically stable).*

It is evident that system (1.4) is G-stable (G-asymptotically stable) if and only if its corresponding homogeneous system

$$\Delta y(k-1) = G_1(k-1)y(k-1) + G_2(k)y(k) + G_3(k)y(k+1), \qquad k = 1, 2, \dots, \tag{1.4$_0$}$$

is G-stable (G-asymptotically stable). On the other hand, system (1.4$_0$) is G-stable (G-asymptotically stable) if and only if its zero solution is G-stable (G-asymptotically stable).

Therefore the G-stability (G-asymptotic stability) of system (1.4) is the common property of all solutions and the vector-function g does not affect this property. Hence it is the property of the triple (G_1, G_2, G_3). Thus, the following definition is natural.

DEFINITION 1.6. *The triple (G_1, G_2, G_3) is said to be G-stable (G-asymptotically stable) if system (1.4$_0$) is G-stable (G-asymptotically stable).*

REMARK 1.2. It is evident that the triple (G_1, G_2, G_3) is G-stable if and only if every solution y of system (1.4$_0$) is G-bounded, i.e., there exists $M > 0$ such that

$$\|(I_n + G(k))y(k)\| + \|y(k+1)\| \le M, \qquad k = 0, 1, \dots.$$

Analogously, the triple (G_1, G_2, G_3) is G-asymptotically stable if and only if every solution y of system (1.4$_0$) is G-convergent to the zero, i.e.,

$$\lim_{k\to+\infty} (\|(I_n + G(k))y(k)\| + \|y(k+1)\|) = 0.$$

REMARK 1.3. If the matrix-function G is such that

$$\det(I_n + G(k)) \neq 0, \qquad k = 0, 1, \dots,$$

and

$$\|G(k)\| + \left\|(I_n + G(k))^{-1}\right\| < M, \qquad k = 0, 1, \dots,$$

for some $M > 0$, then the triple (G_1, G_2, G_3) is G-stable (G-asymptotically stable) if and only if it is stable (asymptotically stable).

THEOREM 1.5. *Let the matrix-functions $G_1, G_2, G_3 \in E(\mathbb{N}_0; \mathbb{R}^{n\times n})$ be such that*

$$\det(I_n + G_1(k)) \neq 0, \qquad k = 1, 2, \dots, \tag{1.20}$$

and

$$G(k) = I_{2n} - \exp\left(-\sum_{l=1}^{m} \Delta\beta_l(k-1)\cdot B_l\right), \qquad k = 1, 2, \dots, \tag{1.21}$$

where $G(k) = (G_{ij}(k))_{i,j=1}^2$,

$$\begin{aligned} &G_{11}(k) \equiv (G_1(k) + G_2(k))\,(I_n + G_1(k))^{-1}, && G_{12}(k) \equiv G_3(k), \\ &G_{21}(k) \equiv -\,(I_n + G_1(k))^{-1}, && G_{22}(k)) \equiv I_n, \end{aligned} \tag{1.22}$$

$\beta_l \in E(\mathbb{N}_0; \mathbb{R}_+)$ $(l = 1, \dots, m)$, and $B_l \in \mathbb{R}^{2n\times 2n}$ $(l = 1, \dots, m)$ are pairwise permutable constant matrices. Let, moreover, $(\lambda - \lambda_{li})^{n_{li}}$ $(i = 1, \dots, m_l;\ \sum_{i=1}^{m_l} n_{li} = 2n)$ be elementary divisors of the matrix B_l for every $l \in \{1, \dots, m\}$. Then:

(a) *the triple (G_1, G_2, G_3) is stable if and only if*

$$\sup\left\{\prod_{l=1}^{m}\left(\sum_{i=1}^{m_l}(1+\beta_l(k))^{n_{li}-1}\exp(\beta_l(k)\operatorname{Re}\lambda_{li})\right) : k = 0, 1, \dots\right\} < +\infty;$$

(b) *the triple (G_1, G_2, G_3) is G_1-asymptotically stable if and only if*

$$\lim_{k\to+\infty}\prod_{l=1}^{m}\left(\sum_{i=1}^{m_l}(1+\beta_l(k))^{n_{li}-1}\exp(\beta_l(k)\operatorname{Re}\lambda_{li})\right) = 0.$$

COROLLARY 1.7. *Let the matrix-functions $G_1, G_2, G_3 \in E(\mathbb{N}_0; \mathbb{R}^{n\times n})$ be such that conditions (1.20),(1.21) and*

$$\lim_{k\to+\infty}\beta_l(k) = +\infty, \qquad l = 1, \dots, m,$$

hold, where $\beta_l \in E(\mathbb{N}_0; \mathbb{R}_+)$ $(l = 1, \dots, m)$, $B_l \in \mathbb{R}^{2n\times 2n}$ $(l = 1, \dots, m)$ are pairwise permutable constant matrices, and $G(k) = (G_{ij}(k))_{i,j=1}^2$ is defined by (1.22). Then:

(a) *the triple (G_1, G_2, G_3) is stable if and only if every eigenvalue of the matrices B_l $(l = 1, \dots, m)$ has the nonpositive real part; in addition, every elementary divisor, corresponding to the eigenvalue with the zero real part, is simple;*
(b) *the triple (G_1, G_2, G_3) is asymptotically stable if and only if every eigenvalue of the matrices B_l $(l = 1, \dots, m)$ has the negative real part.*

COROLLARY 1.8. *Let $G_j(k) \equiv G_{0j}$ $(j = 1, 2, 3)$ be constant matrix-functions such that*

$$\det(I_n + G_{01}) \neq 0, \qquad \det G_{03} \neq 0.$$

Let, moreover, $\lambda_1, \dots, \lambda_m$ be pairwise different eigenvalues of the $2n\times 2n$-matrix $G_0 = (G_{0ij})_{i,j=1}^2$, where $G_{011} = (G_{01} + G_{02})(I_n + G_{01})^{-1}$, $G_{012} = G_{03}$, $G_{021} = -(I_n + G_{01})^{-1}$, $G_{022} = I_n$. Then:

(a) *the triple (G_1, G_2, G_3) is stable if and only if $|1 - \lambda_i| \geq 1$ $(i = 1, \dots, m)$; in addition, if $|1 - \lambda_i| = 1$ for some $i \in \{1, \dots, m\}$, then every elementary divisor, corresponding to λ_i, is simple;*
(b) *the triple (G_1, G_2, G_3) is asymptotically stable if and only if $|1 - \lambda_i| > 1$ $(i = 1, \dots, m)$.*

THEOREM 1.6. *Let $G_j(k) \equiv G_{0j}$ $(j = 1, 2, 3)$ be the constant matrix-functions such that*

$$G_{01} = (I_n - M_1 A_1 + M_2 A_3)^{-1} S - I_n, \tag{1.23}$$

$$G_{02} = I_n + (M_1 A_1 + M_2 A_2 - 2I_n)(I_n + G_{01}), \tag{1.24}$$

and

$$G_{03} = (I_n - M_2 A_2) S, \tag{1.25}$$

where $A_j = (\alpha_{jil})_{i,l=1}^n$ $(j = 1, 2)$, $M_j = \operatorname{diag}(\mu_{j1}, \dots, \mu_{jn})$ $(j = 1, 2)$ and S are constant $n \times n$-matrices such that

$$\mu_{1i} > 0, \quad \mu_{2i} \geq 0, \qquad i = 1, \dots, n, \tag{1.26}$$

and

$$\det(I_n - M_1 A_1 + M_2 A_3) \neq 0, \qquad \det((I_n - M_2 A_2) S) \neq 0. \tag{1.27}$$

Then the condition

$$\alpha_{jii} < 0, \qquad j = 1, 2, \quad i = 1, \dots, n, \quad r(H) < 1, \tag{1.28}$$

where $H = (H_{mj})_{m,j=1}^2$, $H_{jj} = ((1 - \delta_{il})|\alpha_{jil}||\alpha_{jii}|^{-1})_{i,l=1}^n$ $(j = 1, 2)$, $H_{21} = (|\alpha_{3il}||\alpha_{2ii}|^{-1})_{i,l=1}^n$, $H_{12} = (|\alpha_{2il}\mu_{2i} - \delta_{il}||\alpha_{1ii}|^{-1}\mu_{1i}^{-1})_{i,l=1}^n$, is sufficient for the asymptotic stability of the triple (G_{01}, G_{02}, G_{03}); and if

$$\alpha_{jil} \geq 0, \quad \alpha_{2ii}\mu_{2i} \geq 1, \qquad j = 1, 2, 3, \quad i \neq l, \quad i, l = 1, \dots, n, \tag{1.29}$$

and

$$\begin{gathered} \alpha_{j+1ii}\mu_{2i} - \delta_{2j} + \sum_{l=1, l\neq i}^{n} (\alpha_{jil}\mu_{ji} + \alpha_{j+1il}\mu_{2i}) \\ < \min\{1 - \alpha_{jii}\mu_{ji}, |1 + \alpha_{jii}\mu_{ji}|\}, \qquad j = 1, 2, \quad i = 1, \dots, n, \end{gathered} \tag{1.30}$$

then condition (1.28) is necessary as well.

2. AUXILIARY PROPOSITIONS

LEMMA 2.1. *Let X be a fundamental matrix of system (1.1_0). Then*

$$dX^{-1}(t) = -X^{-1}(t)\, d\mathcal{A}(A, A)(t), \qquad \text{for } t \in \mathbb{R}_+.$$

PROOF. By Proposition III.2.15 from [15],

$$X^{-1}(t) - X^{-1}(s) = -X^{-1}(t)A(t) + X^{-1}(s)A(s) + \int_s^t dX^{-1}(\tau) \cdot A(\tau), \qquad \text{for } 0 \leq s \leq t. \tag{2.1}$$

Hence, using the integration-by-parts formula, the equalities

$$d_j X^{-1}(t) = -X^{-1}(t) d_j A(t) \cdot \left(I_n + (-1)^j d_j A(t)\right)^{-1}, \qquad \text{for } t \in \mathbb{R}_+, \quad j = 1, 2, \tag{2.2}$$

and the definition of the operator $\mathcal{A}$, we obtain

$$\begin{aligned} X^{-1}(t) - X^{-1}(s) = &- \int_s^t X^{-1}(\tau)\, dA(\tau) \\ &+ \sum_{s<\tau\leq t} d_1 X^{-1}(\tau) \cdot d_1 A(\tau) - \sum_{s\leq\tau<t} d_2 X^{-1}(\tau) \cdot d_2 A(\tau) \\ = &- \int_s^t X^{-1}(\tau)\, dA(\tau) - \sum_{s<\tau\leq t} X^{-1}(\tau) d_1 A(\tau) \cdot (I_n - d_1 A(\tau))^{-1} d_1 A(\tau) \\ &+ \sum_{s\leq\tau<t} X^{-1}(\tau) d_2 A(\tau) \cdot (I_n + d_2 A(\tau))^{-1} d_2 A(\tau) = -\int_s^t X^{-1}(\tau)\, d\mathcal{A}(\tau) \\ &\text{for } 0 \leq s < t. \end{aligned}$$

∎

LEMMA 2.2. *Let the matrix-function $B \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ satisfy the Lappo-Danilevskiĭ condition. Then*

$$\begin{aligned}\int_a^b d\exp(B(t))\cdot\exp(-B(t)) &= S_0(B)(b) - S_0(B)(a)\\ &+ \sum_{a<t\le b}(I_n - \exp(-d_1B(t))) + \sum_{a\le t<b}(\exp(d_2B(t)) - I_n), \qquad \text{for } 0\le a<b.\end{aligned} \tag{2.3}$$

PROOF. Since $S_0(B)(t)$, $S_1(B)(t)$, and $S_2(B)(t)$ $(t\in\mathbb{R}_+)$ are pairwise permutable matrices, we have in addition

$$S_0(B)(t)\cdot d_jB(t) = d_jB(t)\cdot S_0(B)(t), \qquad \text{for } t\in\mathbb{R}_+,\quad j=1,2,$$

and

$$S_j(B)(t)\cdot d_{3-j}B(t) = d_{3-j}B(t)\cdot S_j(B)(t), \qquad \text{for } t\in\mathbb{R}_+,\quad j=1,2.$$

Therefore, according to the general integration-by-parts formula, we find

$$\begin{aligned}\int_a^b d\exp(B(t))\cdot\exp(-B(t)) &= \int_a^b d\exp(S_0(B)(t))\cdot\exp\left(S_1(B)(t)+S_2(B)(t)\right)\cdot\exp(-B(t))\\ &\quad+\int_a^b \exp(S_0(B)(t))\,d\exp\left(S_1(B)(t)+S_2(B)(t)\right)\cdot\exp(-B(t))\\ &=\int_a^b d\exp(S_0(B)(t))\cdot\exp(-S_0(B)(t))\\ &\quad+\sum_{a<t\le b}\exp(S_0(B)(t))\,d_1\exp\left(S_1(B)(t)+S_2(B)(t)\right)\cdot\exp(-B(t))\\ &\quad+\sum_{a\le t<b}\exp(S_0(B)(t))\,d_2\exp\left(S_1(B)(t)+S_2(B)(t)\right)\cdot\exp(-B(t)).\end{aligned}$$

Hence,

$$\begin{aligned}\int_a^b d\exp(B(t))\cdot\exp(-B(t)) &= \int_a^b d\exp(S_0(B)(t))\cdot\exp(-S_0(B)(t))\\ &+\sum_{a<t\le b}(I_n-\exp(-d_1B(t)))+\sum_{a\le t<b}(\exp(d_2B(t))-I_n).\end{aligned} \tag{2.4}$$

Due to the Lappo-Danilevskiĭ condition, we easily get

$$\int_a^b dS_0^k(B)(t)\cdot S_0^m(B)(t) = \frac{k}{k+m}\left(S_0^{k+m}(B)(b) - S_0^{k+m}(B)(a)\right)$$

for every natural k and m.

By this and the definition of the exponential matrix, we obtain

$$\begin{aligned}&\int_a^b d\exp(S_0(B)(t))\cdot\exp(-S_0(B)(t))\\ &\quad=\exp(S_0(B)(b))-\exp(S_0(B)(a))\\ &\qquad+\sum_{m=1}^{\infty}\sum_{k=1}^{m}\frac{(-1)^{m-k+1}}{k!(m-k+1)!}\int_a^b dS_0^k(B)(t)\cdot S_0^{m-k+1}(B)(t)\\ &\quad=\exp(S_0(B)(b))-\exp(S_0(B)(a))\\ &\qquad+\sum_{m=1}^{\infty}\frac{S_0^{m+1}(B)(b)-S_0^{m+1}(B)(a)}{m+1}\cdot\sum_{k=0}^{m-1}\frac{(-1)^{m-k}}{k!(m-k)!}\\ &\quad=\exp(S_0(B)(b))-\exp(S_0(B)(a))-\sum_{m=1}^{\infty}\frac{S_0^{m+1}(B)(b)-S_0^{m+1}(B)(a)}{(m+1)!}.\end{aligned}$$

Thus

$$\int_a^b d\exp(S_0(B)(t)) \cdot \exp(-S_0(B)(t)) = S_0(B)(b) - S_0(B)(a). \tag{2.5}$$

By (2.4) and (2.5), condition (2.3) holds. ∎

LEMMA 2.3. *Let the matrix-function $A \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$S_0(A)(t) \equiv S_0(B)(t) \quad \text{and} \quad I_n + (-1)^j d_j A(t) \equiv \exp\left((-1)^j d_j B(t)\right), \qquad j = 1,2,$$

where the matrix-function $B \in BV_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ satisfies the Lappo-Danilevskiĭ condition. Then the matrix-function $\exp(B(t))$ is a solution of system (1.1_0).

PROOF. By (2.3),

$$\int_s^t d\exp(B(\tau)) \cdot \exp(-B(\tau)) = A(t) - A(s), \qquad \text{for } 0 \le t < s.$$

Consequently, using the substitution formula, we get

$$\begin{aligned}\int_s^t dA(\tau) \cdot \exp(B(\tau)) &= \int_s^t d\left(\int_s^\tau d\exp(B(\sigma)) \cdot \exp(-B(\sigma))\right) \cdot \exp(B(\tau)) \\ &= \exp(B(t)) - \exp(B(s)), \qquad \text{for } 0 \le t < s.\end{aligned}$$ ∎

LEMMA 2.4. *Let the matrix-function $A_0 \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^{n\times n})$ be such that*

$$\det\left(I_n + (-1)^j d_j A_0(t)\right) \neq 0, \qquad \text{for } t \ge t^*, \quad j = 1,2, \tag{2.6}$$

where $t^* \in \mathbb{R}_+$. Let, moreover:

(a) *the Cauchy matrix U_0 of the system*

$$dx(t) = dA_0(t) \cdot x(t) \tag{2.7}$$

satisfy the inequality

$$|U_0(t,t^*)| \le \Omega \exp\left(-\xi(t) + \xi(t^*)\right), \qquad \text{for } t \ge t^*, \tag{2.8}$$

where $\Omega \in \mathbb{R}_+^{n\times n}$, and $\xi \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$;

(b) *there exist $H \in \mathbb{R}_+^{n\times n}$ such that $r(H) < 1$ and*

$$\int_{t^*}^t \exp\left(\xi(t) - \xi(\tau)\right) |U_0(t,\tau)|\, dV\left(\mathcal{A}(A_0, A - A_0)\right)(\tau) < H, \qquad \text{for } t \ge t^*. \tag{2.9}$$

Then an arbitrary solution x of system (1.1_0) admits the estimate

$$|x(t)| \le (I_n - H)^{-1}\Omega |x(t^*)| \exp\left(-\xi(t) + \xi(t^*)\right), \qquad \text{for } t \ge t^*. \tag{2.10}$$

The proof of this lemma is given in [9].

LEMMA 2.5. *Let $t_0 \in [a,b]$, $\alpha, \beta \in \mathrm{BV}([a,b];\mathbb{R})$ and*

$$1 + (-1)^j d_j \alpha(t) \neq 0, \qquad \text{for } t \in [a,b]. \tag{2.11}$$

Let, moreover, $\xi \in BV([a,b];\mathbb{R})$ be a solution of the equation

$$d\xi(t) = \xi(t)\, d\alpha(t) + d\beta(t). \tag{2.12}$$

Then

$$\gamma^{-1}(t)\xi(t) - \gamma^{-1}(s)\xi(s) = \int_s^t \gamma^{-1}(\tau)\,d\beta(\tau) - \sum_{s<\tau\le t} d_1\gamma^{-1}(\tau)\cdot d_1\beta(\tau) + \sum_{s\le\tau<t} d_2\gamma^{-1}(\tau)\cdot d_2\beta(\tau), \qquad \text{for } a\le s<t\le b, \tag{2.13}$$

where $\gamma \in \mathrm{BV}([a,b];\mathbb{R})$ *is a solution of the Cauchy problem*

$$d\gamma(t) = \gamma(t)\,d\alpha(t), \qquad \gamma(t_0) = 1. \tag{2.14}$$

PROOF. By (2.11), problem (2.14) has the unique solution γ and $\gamma(t)\neq 0$ for $t\in[a,b]$.

Let $a\le s<t\le b$. By (2.1),(2.12) and the integration-by-parts formula, we have

$$\begin{aligned}
\gamma^{-1}(t)\xi(t) - \gamma^{-1}(s)\xi(s) &= \int_s^t \gamma^{-1}(\tau)\,d\xi(\tau) + \int_s^t \xi(\tau)\,d\gamma^{-1}(\tau)\\
&\quad - \sum_{s<\tau\le t} d_1\gamma^{-1}(\tau)\cdot d_1\beta(\tau) + \sum_{s\le\tau<t} d_2\gamma^{-1}(\tau)\cdot d_2\beta(\tau)\\
&= \int_s^t \gamma^{-1}(\tau)\xi(\tau)\,d\alpha(\tau) + \int_s^t \gamma^{-1}(\tau)\,d\beta(\tau) + \int_s^t \xi(\tau)\,d\gamma^{-1}(\tau)\\
&\quad - \sum_{s<\tau\le t} d_1\gamma^{-1}(\tau)\cdot(\xi(\tau)\,d_1\alpha(\tau) + d_1\beta(\tau))\\
&\quad + \sum_{s\le\tau<t} d_2\gamma^{-1}(\tau)\cdot(\xi(\tau)\,d_2\alpha(\tau) + d_2\beta(\tau)),
\end{aligned}$$

and

$$\gamma^{-1}(\tau) = \gamma^{-1}(s) - \int_s^\tau \gamma^{-1}(\sigma)\,d\alpha(\sigma) + \sum_{s<\sigma\le\tau} d_1\gamma^{-1}(\sigma)\cdot d_1\alpha(\sigma) - \sum_{s\le\sigma<\tau} d_2\gamma^{-1}(\sigma)\cdot d_2\alpha(\sigma), \qquad \text{for } s<\tau\le t.$$

Therefore, (2.13) holds, since by the latter equality

$$\int_s^t \xi(\tau)\,d\gamma^{-1}(\tau) = -\int_s^t \xi(\tau)\gamma^{-1}(\tau)\,d\alpha(\tau) + \sum_{s<\tau\le t} \xi(\tau)d_1\gamma^{-1}(\tau)\cdot d_1\alpha(\tau) - \sum_{s\le\tau<t} \xi(\tau)d_2\gamma^{-1}(\tau)\cdot d_2\alpha(\tau), \qquad \text{for } s<t. \qquad \blacksquare$$

LEMMA 2.6. *Let* $t_0\in[a,b]$, $C=(c_{ik})_{i,k=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$,

$$\det\left(I_n + (-1)^j d_jC(t)\right)\neq 0, \qquad \text{for } t\in[a,b]\setminus\{t_0\}, \quad j=1,2, \tag{2.15}$$

$$1+(-1)^j d_jc_{ii}(t) > 0, \qquad \text{for } (-1)^j(t-t_0)\ge 0, \quad j=1,2, \quad i=1,\dots,n, \tag{2.16}$$

and

$$1+(-1)^j\sum_{i=1}^n d_jc_{ik}(t) > 0, \qquad \text{for } (-1)^j(t-t_0)<0, \quad j=1,2, \quad k=1,\dots,n. \tag{2.17}$$

Let, moreover, the functions c_{il} $(i\neq l;\ i,l=1,\dots,n)$ *be nonincreasing on* $[a,t_0[$ *and nondecreasing on* $]t_0,b]$. *Then*

$$U(t,s)\ge 0, \qquad \text{for } a\le t\le s\le t_0 \text{ or } t_0\le s\le t<b, \tag{2.18}$$

where U ($U(s,s) \equiv I_n$) is the Cauchy matrix of the system

$$dx(t) = dC(t) \cdot x(t). \tag{2.19}$$

PROOF. First we note that in view of (2.16) and (2.17),

$$1 + (-1)^j d_j c_{ii}(t) > 0, \qquad \text{for } t \in [a,b], \quad j = 1,2, \quad i = 1,\dots,n, \tag{2.20}$$

since the functions c_{il} ($i \neq l$; $i,l = 1,\dots,n$) are nonincreasing on $[a,t_0[$ and nondecreasing $]t_0,b]$.

Let $s \in [a,b]$ ($s \neq t_0$) and $k \in \{1,\dots,n\}$ be fixed, and let $x_k(t,s) = (x_{ik}(t,s))_{i=1}^n$ be the k^{th} column of the matrix $U(t,s)$.

Assume

$$y(t) = (y_i(t))_{i=1}^n, \qquad \text{for } t \in [a,b],$$
$$y_i(t) = \gamma_s^{-1}(c_{ii})(t) \cdot x_{ik}(t,s), \qquad i = 1,\dots,n,$$

where $\gamma_s(c_{ii})(t) = \gamma^{-1}(c_{ii})(s) \cdot \gamma(c_{ii})(t)$. Here, in view of (2.20), $\gamma(c_{ii})(t)$ is positive for $t \in [a,b]$.

According to Lemma 2.5 and the integration-by-parts formula, we find

$$\begin{aligned}
y_i(t) - y_i(r) = &\sum_{l \neq i,\, l=1}^{n} \Bigg(\int_r^t \gamma_s^{-1}(c_{ii})(\tau) \cdot x_{lk}(\tau,s)\, dc_{il}(\tau) \\
&- \sum_{r<\tau\le t} d_1\gamma_s^{-1}(c_{ii})(\tau) \cdot x_{lk}(\tau,s)\, d_1 c_{il}(\tau) \\
&+ \sum_{r\le\tau<t} d_2\gamma_s^{-1}(c_{ii})(\tau) \cdot x_{lk}(\tau,s)\, d_2 c_{il}(\tau) \Bigg) \\
= &\sum_{l \neq i,\, l=1}^{n} \Bigg(\int_r^t \gamma_s^{-1}(c_{ii})(\tau) \cdot x_{lk}(\tau,s)\, ds_0(c_{il})(\tau) \\
&+ \sum_{r<\tau\le t} \gamma_s^{-1}(c_{ii})(\tau-) \cdot x_{lk}(\tau,s)\, d_1 c_{il}(\tau) \\
&+ \sum_{r\le\tau<t} \gamma_s^{-1}(c_{ii})(\tau+) \cdot x_{lk}(\tau,s)\, d_2 c_{il}(\tau) \Bigg) \\
= &\sum_{l \neq i,\, l=1}^{n} \Bigg(\int_r^t \gamma_s^{-1}(c_{ii})(\tau) \cdot \gamma_s(c_{ll})(\tau) y_l(\tau)\, ds_0(c_{il})(\tau) \\
&+ \sum_{r<\tau\le t} \gamma_s^{-1}(c_{ii})(\tau-) \cdot \gamma_s(c_{ll})(\tau)\, d_1 c_{il}(\tau) \\
&+ \sum_{r\le\tau<t} \gamma_s^{-1}(c_{ii})(\tau+) \cdot \gamma_s(c_{ll})(\tau)\, d_2 c_{il}(\tau) \Bigg), \qquad \text{for } a \le r \le t \le b, \quad i = 1,\dots,n.
\end{aligned}$$

Hence $y = (y_i)_{i=1}^n$ is a solution of the Cauchy problem

$$dy(t) = dC^*(t) \cdot y(t), \qquad y(s) = e_k, \tag{2.21}$$

where $e_k = (\delta_{ik})_{i=1}^n$, $C^*(t) = (c^*_{il}(t))_{i,l=1}^n$, $c^*_{ii}(t) \equiv 0$ and

$$\begin{aligned}
c^*_{il}(t) \equiv &\int_{t_0}^t \gamma_s^{-1}(c_{ii})(\tau) \cdot \gamma_s(c_{ii})(\tau)\, ds_0(c_{il})(\tau) \\
&+ \int_{t_0}^t \gamma_s^{-1}(c_{ii})(\tau-) \cdot \gamma_s(c_{ll})(\tau)\, ds_1(c_{il})(\tau) \\
&+ \int_{t_0}^t \gamma_s^{-1}(c_{ii})(\tau+) \cdot \gamma_s(c_{ll})(\tau)\, ds_2(c_{il})(\tau), \qquad i \neq l, \quad i,l = 1,\dots,n.
\end{aligned}$$

In view of the conditions of the lemma, the functions c^*_{il} $(i \neq l;\ i,l = 1,\dots,n)$ are nonincreasing on $[a, t_0[$ and nondecreasing on $]t_0, b]$.

Let

$$\Lambda_s(t) = \operatorname{diag}(\gamma_s(c_{11})(t),\dots,\gamma_s(c_{nn})(t))$$

and

$$Q(t) = \operatorname{diag}(c_{11}(t),\dots,c_{nn}(t)), \qquad \text{for } t \in [a,b].$$

Using (2.2), we have

$$\begin{aligned} I_n + (-1)^j d_j C^*(t) &= I_n + (-1)^j \left(\Lambda_s^{-1}(t) + (-1)^j d_j \Lambda_s^{-1}(t)\right)(d_j C(t) - d_j Q(t))\Lambda_s(t) \\ &= \left(\Lambda_s^{-1}(t) + (-1)^j d_j \Lambda_s^{-1}(t)\right)\left[\left(I_n + (-1)^j d_j Q(t)\right)\Lambda_s(t)\right. \\ &\quad \left. +(-1)^j \left(d_j C(t) - d_j Q(t)\right)\Lambda_s(t)\right], \qquad \text{for } t \in [a,b], \quad j = 1,2, \end{aligned}$$

and

$$I_n + (-1)^j d_j C^*(t) = \left(\Lambda_s^{-1}(t) + (-1)^j d_j \Lambda_s^{-1}(t)\right)\left(I_n + (-1)^j d_j C(t)\right)\Lambda_s(t), \quad \text{for } t \in [a,b], \quad j = 1,2. \tag{2.22}$$

Hence, due to (2.15), we obtain

$$\det\left(I_n + (-1)^j d_j C^*(t)\right) \neq 0, \qquad \text{for } t \in [a,b] \setminus \{t_0\}, \quad j = 1,2.$$

Therefore, according to Theorem 1.2 from [8],

$$\lim_{m \to +\infty} z_m(t) = y(t), \qquad \text{uniformly on } [a,b], \tag{2.23}$$

where

$$\begin{aligned} & z_m(s) = e_k, \qquad m = 0,1,\dots, \\ & z_0(t) = \left(I_n + (-1)^j d_j C^*(t)\right)^{-1} e_k, \qquad \text{for } (-1)^j (t-s) < 0, \quad j = 1,2, \\ & z_m(t) = \left(I_n + (-1)^j d_j C^*(t)\right)^{-1}\left[e_k + \int_s^t dC^*(\tau)\cdot z_{m-1}(\tau)\right. \\ & \qquad \left. +(-1)^j d_j C^*(t)\cdot z_{m-1}(t)\right], \qquad \text{for } (-1)^j (t-s) < 0, \quad j = 1,2, \quad m = 1,2,\dots. \end{aligned} \tag{2.24}$$

Taking into account the equalities

$$d_j \Lambda_s(t) = d_j Q(t) \cdot \Lambda_s(t), \qquad \text{for } t \in [a,b], \quad j = 1,2,$$

from (2.22) we have

$$\begin{aligned} I_n + (-1)^j d_j C^*(t) &= \left(\Lambda_s^{-1}(t) + (-1)^j d_j \Lambda_s^{-1}(t)\right)\left(I_n - Q_j(t)\right) \\ &\quad \times \left(\Lambda_s(t) + (-1)^j d_j \Lambda_s(t)\right), \qquad \text{for } t \in [a,b], \quad j = 1,2, \end{aligned} \tag{2.25}$$

where $Q_j(t) \equiv (-1)^j (d_j Q(t) - d_j C(t))(I_n + (-1)^j d_j Q(t))^{-1}$. On the other hand, by (2.17) and (2.20),

$$Q_j(t) \geq 0, \qquad \text{for } (-1)^j (t - t_0) \leq 0, \quad j = 1,2,$$

and

$$\|Q_j(t)\| < 1, \qquad \text{for } (-1)^j (t - t_0) < 0, \quad j = 1,2.$$

Therefore, due to (2.25),

$$\left(I_n + (-1)^j d_j C^*(t)\right)^{-1} \geq O_{n\times n}, \qquad \text{for } (-1)^j (t - t_0) < 0, \quad j = 1,2, \tag{2.26}$$

since by (2.20),

$$\Lambda_s(t) \geq O_{n\times n}, \qquad \text{for } t \in [a,b]. \tag{2.27}$$

From (2.24) and (2.26) we get

$$z_m(t) \geq \left(I_n + (-1)^j d_j C^*(t)\right)^{-1} e_k, \qquad \text{for } (-1)^j(t-s) < 0, \quad j = 1,2; \quad m = 0,1,\dots.$$

Using now (2.23) and (2.24), we obtain

$$y(s) \geq e_k, \quad y(t) \geq \left(I_n + (-1)^j d_j C^*(t)\right)^{-1} e_k, \qquad \text{for } (-1)^j(t-s) < 0, \quad j = 1,2. \tag{2.28}$$

On the other hand, by equalities

$$y(t) = \Lambda_s^{-1}(t) x_k(t,s), \qquad \text{for } t \in [a,b],$$

inequality (2.28) implies

$$x_k(t,s) \geq \Lambda_s(t)\left(I_n + (-1)^j d_j C^*(t)\right)^{-1} e_k,$$
$$\text{for } (-1)^j(t-s) < 0, \quad (-1)^j(t-t_0) < 0, \quad j = 1,2.$$

Since the latter inequalities are fulfilled for every $k \in \{1,\dots,n\}$, we have

$$U(t,s) \geq \Lambda_s(t)\left(I_n + (-1)^j d_j C^*(t)\right)^{-1}, \qquad \text{for } (-1)^j(t-s) < 0, \quad j = 1,2. \tag{2.29}$$

By (2.26) and (2.27), condition (2.29) implies (2.18). ∎

REMARK 2.1. In fact, we proved estimate (2.29) which is stronger than (2.18). Note also that the condition

$$\|d_j C(t)\| < 1, \qquad \text{for } t \in [a,b], \quad j = 1,2,$$

guarantees conditions (2.15)–(2.17).

LEMMA 2.7. *Let $t_0 \in [a,b]$, $c_0 \in \mathbb{R}^n$, $q \in \mathrm{BV}([a,b];\mathbb{R}^n)$, and a matrix-function $C = (c_{ik})_{i,k=1}^n \in \mathrm{BV}([a,b];\mathbb{R}^{n\times n})$, where c_{ik} ($i \neq k$; $i,k = 1,\dots,n$) are nondecreasing functions on $[a,b]$, be such that*

$$\det\left(I_n + d_j C(t)\right) \neq 0, \qquad \text{for } t \in [a,b]\setminus\{t_0\}, \qquad j = 1,2, \tag{2.30}$$
$$1 + d_j c_{ii}(t) > 0, \qquad \text{for } (-1)^j(t-t_0) \geq 0, \quad j = 1,2, \tag{2.31}$$

and

$$\sum_{i=1}^n d_j c_{ik}(t) < 1, \qquad \text{for } (-1)^j(t-t_0) < 0, \quad j = 1,2, \quad k = 1,\dots,n. \tag{2.32}$$

Let, moreover, a vector-function $x : [a,b] \to \mathbb{R}^n$, $x \in \mathrm{BV}_{\mathrm{loc}}([a,t_0[,\ \mathbb{R}^n) \cap \mathrm{BV}_{\mathrm{loc}}(]t_0,b],\mathbb{R}^n)$, be a solution of the system of linear differential inequalities

$$dx(t)\cdot \operatorname{sgn}(t-t_0) \leq dC(t)\cdot x(t) + dq(t) \tag{2.33}$$

on the intervals $[a,t_0[$ and $]t_0,b]$, satisfying the condition

$$x(t_0) + (-1)^j d_j x(t_0) \leq c_0 + d_j C(t_0)\cdot c_0 + d_j q(t_0), \qquad j = 1,2. \tag{2.34}$$

Then the estimate

$$x(t) \leq y(t), \qquad \text{for } t \in [a,b]\setminus\{t_0\} \tag{2.35}$$

holds, where $y \in \mathrm{BV}([a,b];\mathbb{R}^n)$ *is a solution of the system*

$$dy(t) = (dC(t) \cdot y(t) + dq(t)) \operatorname{sgn}(t - t_0) \tag{2.36}$$

on the intervals $[a, t_0[$ *and* $]t_0, b]$, *satisfying the conditions*

$$(-1)^j d_j y(t_0) = d_j C(t_0) \cdot y(t_0) + d_j q(t_0), \qquad j = 1, 2, \tag{2.37}$$

and

$$y(t_0) = c_0. \tag{2.38}$$

PROOF. Assume $t_0 < b$ and consider the closed interval $[t_0, b]$. Then problem (2.36)–(2.38) has the form

$$dy(t) = dC(t) \cdot y(t) + dq(t), \qquad y(t_0) = c_0.$$

Let Z ($Z(t_0) = I_n$) be a fundamental matrix of the system

$$dz(t) = dC(t) \cdot z(t), \qquad \text{for } t \in [a, b]. \tag{2.39}$$

Then by the variation of constants formula,

$$y(t) = q(t) - q(s) + Z(t)\left\{Z^{-1}(s)y(s) - \int_s^t dZ^{-1}(\tau) \cdot (q(\tau) - q(s))\right\}, \qquad \text{for } s, t \in [t_0, b]. \tag{2.40}$$

Put

$$g(t) = -x(t) + x(t_0) + \int_{t_0}^t dC(\tau) \cdot x(\tau) + q(t) - q(t_0), \qquad \text{for } t \in [t_0, b].$$

Evidently,

$$dx(t) = dC(t) \cdot x(t) + d(q(t) - g(t)), \qquad \text{for } t \in [t_0, b].$$

Let ε be an arbitrary positive number. Then

$$x(t) = q(t) - q(t_0+\varepsilon) - g(t) + g(t_0+\varepsilon) + Z(t)\left\{Z^{-1}(t_0+\varepsilon)x(t_0+\varepsilon) \right.$$
$$\left. - \int_{t_0+\varepsilon}^t dZ^{-1}(\tau) \cdot (q(\tau) - q(t_0+\varepsilon) - g(\tau) + g(t_0+\varepsilon))\right\}, \qquad \text{for } t \in [t_0+\varepsilon, b].$$

Hence, by (2.40), we get

$$x(t) = y(t) + Z(t)Z^{-1}(t_0+\varepsilon)\,(x(t_0+\varepsilon) - y(t_0+\varepsilon)) + g_\varepsilon(t), \qquad \text{for } t \in [t_0+\varepsilon, b], \tag{2.41}$$

where

$$g_\varepsilon(t) = -g(t) + g(t_0+\varepsilon) + Z(t)\int_{t_0+\varepsilon}^t dZ^{-1}(\tau) \cdot (g(\tau) - g(t_0+\varepsilon)).$$

Using the integration-by-parts formula, we have

$$\begin{aligned} g_\varepsilon(t) = &-\int_{t_0+\varepsilon}^t U(t,\tau)\, ds_0(g)(\tau) - \sum_{t_0+\varepsilon<\tau\le t} U(t,\tau-)\, d_1 g(\tau) \\ &- \sum_{t_0+\varepsilon\le\tau<t} U(t,\tau+)\, d_2 g(\tau), \qquad \text{for } t \in [t_0+\varepsilon, b], \end{aligned} \tag{2.42}$$

where $U(t,\tau) = Z(t)Z^{-1}(\tau)$ is the Cauchy matrix of system (2.39).

On the other hand, conditions (2.30)–(2.32) guarantee conditions (2.15)–(2.17). Hence, according to Lemma 2.6, estimate (2.18) holds, and by (2.42),

$$g_\varepsilon(t) \le 0, \qquad \text{for } t \in [t_0 + \varepsilon, b],$$

since by (2.33) the function g is nondecreasing on $]t_0, b]$. From this and (2.41),

$$x(t) \le y(t) + U(t, t_0 + \varepsilon)(x(t_0 + \varepsilon) - y(t_0 + \varepsilon)), \qquad \text{for } t \in [t_0 + \varepsilon, b].$$

Passing to the limit as $\varepsilon \to 0$ in the latter inequality and taking into account (2.18) and (2.34), we get

$$x(t) \le y(t), \qquad \text{for } t \in]t_0, b],$$

since by (2.37) and (2.38)

$$y(t_0+) = c_0 + d_2C(t_0) \cdot c_0 + d_2q(t_0).$$

Analogously we can show the validity of inequality (2.35) for $t \in [a, t_0[$. ∎

REMARK 2.2. It is evident that if in Lemma 2.7 we assume

$$x(t_0) \le c_0,$$

then inequality (2.35) is fulfilled on the whole $[a, b]$. Moreover, note that in this case inequalities (2.34) follow from the inequalities

$$(-1)^j d_j x(t_0) \le d_j C(t) \cdot c_0 + d_j q(t), \qquad j = 1, 2.$$

In particular, Lemma 2.7 yields the following proposition.

PROPOSITION 2.1. *Let $t_0 \in [a, b]$, $c_0 \in \mathbb{R}^n$, $q \in \mathrm{BV}([a, b]; \mathbb{R}^n)$, and $C = (c_{ik})_{i,k=1}^n : [a, b] \to \mathbb{R}^{n\times n}$ be a nondecreasing matrix-function satisfying conditions (2.30) and (2.32). Let, moreover, $x : [a, b] \to \mathbb{R}^n$, $x \in \mathrm{BV}([a, t_0[; \mathbb{R}^n) \cap \mathrm{BV}(]t_0, b]; \mathbb{R}^n)$, be a solution of the system of linear integral inequalities*

$$x(t) \le c_0 + \left(\int_{t_0}^t dC(\tau) \cdot x(\tau) + q(t) - q(t_0)\right) \cdot \operatorname{sgn}(t - t_o), \qquad \text{for } t \in [a, b], \tag{2.43}$$

satisfying (2.34). Then the conclusion of Lemma 2.7 is true.

PROOF. Let us introduce the vector-function

$$\tilde{x}(t) = c_0 + \left(\int_{t_0}^t dC(\tau) \cdot x(\tau) + q(t) - q(t_0)\right) \cdot \operatorname{sgn}(t - t_0), \qquad \text{for } t \in [a, b].$$

It is clear that $\tilde{x} \in \mathrm{BV}([a, t_0[; \mathbb{R}^n) \cap \mathrm{BV}(]t_0, b]; \mathbb{R}^n)$. Moreover, by (2.43) $\tilde{x}$ satisfies (2.34) and

$$x(t) \le \tilde{x}(t), \qquad \text{for } t \in [a, b]. \tag{2.44}$$

Since C is a nondecreasing matrix-function, from the latter inequality we find that x satisfies (2.33) on the intervals $[a, t_0[$ and $]t_0, b]$. Therefore, according to Lemma 2.7 and (2.44), the proposition is proved. ∎

REMARK 2.3. Let the function $\beta \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R})$ be such that

$$1 + (-1)^j d_j\beta(t) > 0, \qquad \text{for } t \in \mathbb{R}_+, \quad j = 1, 2.$$

Then if one of the functions β, $J(\beta)$, and $\mathcal{A}(\beta, \beta)$ is nondecreasing (nonincreasing), then all the others will be same.

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 1.1. It is evident that the matrix-function

$$B(t) \equiv \sum_{l=1}^{m} \alpha_l(t) B(t)$$

satisfies the Lappo-Danilevskiĭ condition. Therefore, by Lemma 2.3, the matrix-function

$$X(t) = \prod_{l=1}^{m} \exp(\alpha_l B_l), \qquad \text{for } t \in \mathbb{R}_+, \tag{3.1}$$

is a fundamental matrix of system (1.1_0).

According to the Jordan theorem,

$$B_l = C_l \operatorname{diag}\left(J_{n_{l1}}(\lambda_{l1}), \dots, J_{n_{lm_l}}(\lambda_{lm_l})\right) C_l^{-1}, \qquad l = 1, \dots, m,$$

where $J_{n_{li}}(\lambda_{li}) = \lambda_{li} I_{n_{li}} + Z_{n_{li}}$ is the Jordan box corresponding to the elementary divisor $(\lambda - \lambda_{li})^{n_{li}}$ for every $l \in \{1, \dots, m\}$ and $i \in \{1, \dots m_l\}$, and $C_l \in \mathbb{C}^{n\times n}$ $(l = 1, \dots, m)$ are nonsingular complex matrices. Hence,

$$\begin{gathered} \exp(\alpha_l(t) B_l) = C_l \operatorname{diag}\left(\exp(\alpha_l(t) J_{n_{l1}}(\lambda_{l1})), \dots, \exp\left(\alpha_l(t) J_{n_{lm_l}}(\lambda_{lm_l})\right)\right) C_l^{-1}, \\ \text{for } t \in \mathbb{R}_+, \quad l = 1, \dots, m, \end{gathered} \tag{3.2}$$

where

$$\exp(\alpha_l(t) J_{n_{li}}(\lambda_{li})) = \exp(\lambda_{li}\alpha_l(t)) \sum_{j=0}^{n_{li}-1} \frac{\alpha_l^j(t)}{j!} Z_{n_{li}}^j, \qquad \text{for } t \in \mathbb{R}_+, \quad l = 1, \dots, m. \tag{3.3}$$

In view of (3.2) and (3.3), it is evident that

$$\exp(\alpha_l(t) B_l) = \left(\sum_{i=1}^{m_l} p_{lijk}(\alpha_l(t)) \exp(\lambda_{li}\alpha_l(t))\right)_{i,k=1}^{n}, \qquad \text{for } t \in \mathbb{R}_+, \quad l = 1, \dots, m, \tag{3.4}$$

where $p_{lijk}(s)$ is a polynomial with respect to the variable s, whose degree is at most $n_{li} - 1$ $(i, k = 1, \dots, n;\ l = 1, \dots, m)$.

Substituting (3.4) in (3.1), we find

$$\begin{aligned} &\beta_1 \prod_{l=1}^{m} \left(\sum_{l=1}^{m_l} (1 + \alpha_l(t))^{n_{li}-1} \exp(\alpha_l(t) \operatorname{Re} \lambda_{li})\right) \\ &\qquad \le \|X(t)\| \le \beta_2 \prod_{l=1}^{m} \left(\sum_{l=1}^{m_l} (1 + \alpha_l(t))^{n_{li}-1} \exp(\alpha_l(t) \operatorname{Re} \lambda_{li})\right), \qquad \text{for } t \in \mathbb{R}_+, \end{aligned}$$

where β_1 and β_2 are some positive numbers.

The latter estimates imply the validity of the theorem. ∎

Proof of Corollary 1.1. The corollary immediately follows from Theorem 1.1 since conditions (1.8) and (1.9) are equivalent to the conditions imposed on the real parts of the eigenvalues λ_{li} $(l = 1, \dots, m;\ i = 1, \dots, m_l)$ of the matrices B_l $(l = 1, \dots, m)$. ∎

Proof of Corollary 1.2. Let

$$\alpha_1(t) \equiv \alpha(t), \qquad \alpha_2(t) \equiv \beta_1\alpha(t) - \nu_1(t), \qquad \alpha_3(t) \equiv \nu_2(t) - \beta_2\alpha(t),$$

and

$$B_1 = A_0 - \beta_1 \ln(I_n - A_1) + \beta_2 \ln(I_n + A_2), \qquad B_2 = \ln(I_n - A_1), \qquad B_3 = \ln(I_n + A_2).$$

Then we have

$$S_0(A)(t) = \sum_{l=1}^{3} s_0(\alpha_l)(t) \cdot B_l, \qquad \text{for } t \in \mathbb{R}_+, \quad j = 1, 2,$$

and

$$\begin{aligned} \exp\left((-1)^j \sum_{l=1}^{3} d_j \alpha_l(t) \cdot B_l\right) &= \exp\left(\ln(I_n + (-1)^j A_j)\right) \\ &= I_n + (-1)^j A_j = I_n + (-1)^j d_j A(t), \qquad \text{if } \|d_j A(t)\| \neq 0, \\ &\qquad\qquad \text{for } t \in \mathbb{R}_+, \quad j = 1, 2, \end{aligned}$$

since the function α is continuous, and $d_j \nu_i(t) \equiv \delta_{ij}$ $(i, j = 1, 2)$.

Hence the conditions of Theorem 1.1 are fulfilled. The corollary follows from (1.8) and (1.9) since due to (1.11) the functions α_2 and α_3 are bounded on $\mathbb{R}_+$. ∎

PROOF OF COROLLARY 1.3. The corollary follows from Theorem 1.1 if we choose the functions α_l $(l = 1, \dots, m)$ and the matrices B_l $(l = 1, \dots, m)$ in a suitable way. But the proof of Corollary 1.3 is easier if we use same way as in proof of Theorem 1.1.

By Lemma 2.3 the matrix-function

$$X(t) \equiv C \operatorname{diag}\left(\exp(G_1(t)), \dots, \exp(G_m(t))\right) C^{-1}$$

is a fundamental matrix of system (1.1_0). Moreover, obviously

$$\exp(G_l(t)) = \prod_{i=0}^{n_l-1} \exp\left(\alpha_{li}(t) Z_{n_l}^i\right) = \exp(\alpha_{l0}(t)) \prod_{i=1}^{n_l-1} \sum_{j=1}^{[(n_l-1)/i]} \frac{\alpha_l^j(t)}{j!} Z_{n_l}^{ij}, \qquad \text{for } t \in \mathbb{R}_+, \quad l = 1, \dots, m.$$

Hence, as in Theorem 1.1, the statement of the corollary follows. ∎

PROOF OF THEOREM 1.2. Let us prove the first part. Let $a_{il}(t) \equiv \alpha_{il}\mu_i(t)$ $(i, l = 1, \dots, n)$, and $U_0(t, \tau)$ be the Cauchy matrix of system (2.7), where $A_0(t) \equiv \operatorname{diag}(a_{11}(t), \dots, a_{nn}(t))$. Then

$$U_0(t, \tau) = \operatorname{diag}\left(\gamma(a_{11})(t) \cdot \gamma^{-1}(a_{11})(\tau), \dots, \gamma(a_{nn})(t) \cdot \gamma^{-1}(a_{nn})(\tau)\right), \qquad \text{for } t \in \mathbb{R}_+,$$

where $\gamma(a_{ii})(t)$ $(i = 1, \dots, n)$ are defined as above.

According to Lemma 2.1,

$$\gamma^{-1}(a_{ii})(t) - \gamma^{-1}(a_{ii})(\tau) = -\int_\tau^t \gamma^{-1}(a_{ii})(s)\, d\mathcal{A}(a_{ii}, a_{ii})(s), \qquad \text{for } 0 \le \tau \le t, \quad i = 1, \dots, n. \tag{3.5}$$

Due to (1.12), there exists $t^* \in \mathbb{R}_+$ such that

$$d_2 a_{ii}(t) > -1, \qquad \text{for } t \ge t^*, \quad i = 1, \dots, n.$$

Therefore,

$$1 + (-1)^j d_j a_{ii}(t) > 0, \qquad \text{for } t \ge t^*, \quad j = 1, 2; \quad i = 1, \dots, n, \tag{3.6}$$

since by (1.13) the functions a_{ii} $(i = 1, \dots, n)$ are nonincreasing. By virtue of Remark 2.3, the functions $J(a_{ii})$ $(i = 1, \dots, n)$ are nonnegative, nonincreasing, and

$$-J(a_{ii})(t) + J(a_{ii})(\tau) \geq a_0(t) - a_0(\tau), \qquad \text{for } t \geq \tau \geq t^*, \quad i = 1, \dots, n. \tag{3.7}$$

In view of (1.13), there exists $\varepsilon \in]0, 1[$, such that

$$r(H_\varepsilon) < 1,$$

where $H_\varepsilon = ((1-\varepsilon)^{-1} h_{ik})_{i,k=1}^n$, $h_{ik} = (1 - \delta_{ik})(1 + |\sigma_i|)^{-1} |\alpha_{ik}| |\alpha_{ii}|^{-1}$ $(i, k = 1, \dots, n)$.

Assume $\xi(t) \equiv \varepsilon a_0(t)$. Then by (1.13) conditions (2.6) and (2.8) are fulfilled for $\Omega = I_n$. Moreover,

$$\begin{gathered} |s_0(a_{ik})(t) - s_0(a_{ik})(\tau)| \leq -h_{ik} \left(s_0(a_{ii})(t) - s_0(a_{ii})(\tau) \right), \\ \text{for } t \geq \tau \geq t^*, \quad i \neq k, \quad i, k = 1, \dots, n, \end{gathered} \tag{3.8}$$

$$\begin{gathered} |d_j a_{ik}(t)| \leq -h_{ik} d_j a_{ii}(t) \cdot (1 + d_j a_{ii}(t))^{j-1}, \\ \text{for } t \geq t^*, \quad j = 1, 2, \quad i \neq k, \quad i, k = 1, \dots, n. \end{gathered} \tag{3.9}$$

Let $b_{ik}(t) \equiv \mathcal{A}(a_{ii}, a_{ik})(t)$ $(i, k = 1, \dots, n)$. Using (3.5)–(3.8), we get

$$\begin{gathered} \exp(J(a_{ii})(t)) = \gamma(a_{ii})(t), \qquad \text{for } t \geq t^*, \quad i = 1, \dots, n, \\ \int_{t^*}^t \exp(\xi(t) - \xi(\tau) + J(a_{ii})(t) - J(a_{ii})(\tau)) \, dv(b_{ik})(\tau) \\ \leq \int_{t^*}^t \exp((1-\varepsilon)(J(a_{ii})(t) - J(a_{ii})(\tau))) \, dv(b_{ik})(\tau), \\ \text{for } t \geq t^*, \quad i \neq k, \quad i, k = 1, \dots, n, \end{gathered} \tag{3.10}$$

$$\begin{gathered} |s_0(b_{ik})(t) - s_0(b_{ik})(\tau)| \leq (1-\varepsilon)^{-1} h_{ik} \left[(\varepsilon - 1) s_0(a_{ii})(t) - (\varepsilon - 1) s_0(a_{ii})(\tau) \right], \\ \text{for } t \geq \tau \geq t^*, \quad i \neq k, \quad i, k = 1, \dots, n. \end{gathered} \tag{3.11}$$

Then

$$\begin{aligned} &(1-\varepsilon)^{-1}(-1)^j \left[1 - \left(1 + (-1)^j d_j a_{ii}(t)\right)^{\varepsilon - 1} \right] \\ &\qquad \leq d_j a_{ii}(t) \cdot \left(1 + (-1)^j d_j a_{ii}(t)\right)^{j-2}, \qquad \text{for } t \geq t^*, \quad j = 1, 2, \quad i = 1, \dots, n. \end{aligned}$$

From this and (3.9) we conclude

$$\begin{gathered} |d_j b_{ik}(t)| \leq (1-\varepsilon)^{-1}(-1)^j h_{ik} \left[\left(1 + (-1)^j d_j a_{ii}(t)\right)^{\varepsilon - 1} - 1 \right] \\ \text{for } t \geq t^*, \quad j = 1, 2, \quad i \neq k, \quad i, k = 1, \dots, n. \end{gathered} \tag{3.12}$$

By (2.3), (3.10)–(3.12), and the definition of $J(a_{ii})$ $(i = 1, \dots, n)$, we find

$$\begin{aligned} &\int_{t^*}^t \exp((1-\varepsilon)(J(a_{ii})(t) - J(a_{ii})(\tau))) \, dv(b_{ik})(\tau) \\ &\leq (1-\varepsilon)^{-1} h_{ik} \int_{t^*}^t \exp((1-\varepsilon)(J(a_{ii})(t) - J(a_{ii})(\tau))) \\ &\quad \times d\left(\int_{t^*}^\tau \exp((1-\varepsilon) J(a_{ii})(s)) \, d\exp((\varepsilon - 1) J(a_{ii})(s)) \right) \\ &= (1-\varepsilon)^{-1} h_{ik} \exp((1-\varepsilon) J(a_{ii})(t)) \left[\exp((\varepsilon - 1) J(a_{ii})(t)) \right. \\ &\quad \left. - \exp((\varepsilon - 1) J(a_{ii})(t^*)) \right] \leq (1-\varepsilon)^{-1} h_{ik}, \qquad \text{for } t \geq t^*, \quad i \neq k, \quad i, k = 1, \dots, n. \end{aligned}$$

Consequently, estimate (2.9) is fulfilled. Therefore, by Lemma 2.4 every solution x of system (1.1_0) admits estimate (2.10). Thus A is asymptotically stable since by the first condition in (1.12) $\xi(t) \to \infty$ as $t \to \infty$.

Let us prove the second part. Assume the contrary. Let conditions (1.14) and (1.15) be fulfilled, A be asymptotically stable, but condition (1.13) be violated. Then either

$$a_{i_0 i_0} \geq 0 \tag{3.13}$$

for some $i_0 \in \{1, \dots, n\}$, or

$$\alpha_{ii} < 0, \qquad i = 1, \dots, n, \tag{3.14}$$

but

$$r(H) \geq 1. \tag{3.15}$$

If condition (3.13) holds, then in view of (1.14) the vector-function $x(t) \equiv (\delta_{ii_0})_{i=1}^n$ is a solution of the system of generalized differential inequalities

$$dx(t) \leq dA(t) \cdot x(t), \qquad \text{for } t \in \mathbb{R}_+. \tag{3.16}$$

Moreover, with regard to (1.12), (1.15), and the Hadamard's condition on the nonsingularity of matrices (see [19, p. 382]) it is not difficult to verify that the conditions of Lemma 2.7 are fulfilled for sufficiently large $t_0 > 0$. By this lemma,

$$x(t) \leq U(t, t_0) x(t_0), \qquad \text{for } t > t_0,$$

where $U(t, \tau)$ is the Cauchy matrix of system (1.1_0). Hence, due to the asymptotic stability of A, we have

$$\|x(t)\| \leq \|U(t, t_0) x(t_0)\| \to 0, \qquad \text{as } t \to +\infty. \tag{3.17}$$

But this is impossible since $\|x(t)\| \equiv 1$. Therefore (3.14) holds.

From (3.14) we find

$$\sigma_i \leq 0, \qquad i = 1, \dots, n \quad \text{and} \quad \sigma \leq 0, \tag{3.18}$$

where $\sigma = \max\{\sigma_i : i = 1, \dots, n\}$.

Assume now that (3.15) is fulfilled. Then there exist a complex vector $(c_i)_{i=1}^n$ and a complex number λ such that

$$\sum_{k=1}^n |c_k| = 1, \qquad |\lambda| = r(H) \geq 1,$$

and

$$\sum_{k=1}^n (1 - \delta_{ik})(1 - \sigma_i)^{-1} |\alpha_{ik}| |\alpha_{ii}|^{-1} c_k = \lambda c_i, \qquad i = 1, \dots, n.$$

Therefore,

$$|\alpha_{ii}||c_i| \leq \sum_{k=1, k \neq i}^n (1 - \sigma_i)^{-1} |\alpha_{ik}||c_k| \leq (1 - \sigma)^{-1} \sum_{k=1, k \neq i}^n |\alpha_{ik}||c_k|, \qquad i = 1, \dots, n.$$

The last inequalities, (1.14) and (3.18), imply

$$\begin{aligned}
0 &\leq (1 - \sigma)^{-1} \sum_{k=1, k \neq i}^n \alpha_{ik} |c_k| + \alpha_{ii} |c_i| \\
&= (1 - \sigma)^{-1} \sum_{k=1, k \neq i}^n \alpha_{ik} |c_k| + (1 - \sigma)^{-1} \alpha_{ii} |c_i| - \sigma (1 - \sigma)^{-1} \alpha_{ii} |c_i| \\
&\leq (1 - \sigma)^{-1} \sum_{k=1}^n \alpha_{ik} \, |c_k|, \qquad i = 1, \dots, n,
\end{aligned}$$

and

$$0 \le \sum_{k=1}^{n} \alpha_{ik}|c_k|, \qquad i = 1, \dots, n.$$

Consequently, the vector-function $x(t) \equiv (|c_k|)_{k=1}^n$ is a solution of the system of differential inequalities (3.16). As above we can show that (3.17) holds. But this is impossible since $\|x(t)\| \equiv 1$. The obtained contradiction proves the theorem. ∎

To prove the results concerning the impulsive system (1.2),(1.3), we use the following concept.

It is easy to show that the vector-function $x \in \tilde{C}_{\mathrm{loc}}(\mathbf{R}_+ \setminus T; \mathbf{R}^n)$ $(T = \{t_1, t_2, \dots\})$ is a solution of the impulsive system (1.2),(1.3) if and only if it is a solution of system (1.1), where

$$\begin{aligned} &A(0) = O_{n\times n}, && f(0) = O_n; \\ &A(t) = \int_0^t Q(\tau)\,d\tau + \sum_{0\le t_j<t} G_j, && f(t) = \int_0^t q(\tau)\,d\tau + \sum_{0\le t_j<t} g_j, \qquad \text{for } t > 0. \end{aligned}$$

Therefore system (1.2),(1.3) is a particular case of system (1.1). In addition, condition (1.5) is equivalent to condition (1.16). Thus Theorems 1.3 and 1.4 and Corollaries 1.4, 1.5 are particular cases of Theorems 1.1, 1.2 and Corollaries 1.2, 1.3, respectively. Corollary 1.4 follows from Corollary 1.3.

Consider now the difference system (1.4).

Proof or Theorem 1.5. We construct a system of the form (1.1) corresponding to system (1.4) in order to apply Theorem 1.1.

Let $y \in E(\mathbf{N}_0; \mathbf{R}^n)$ be a solution of the difference system (1.4). Then the vector-function $z = (z_i)_{i=1}^2 \in E(\mathbf{N}_0; \mathbf{R}^{2n})$, where

$$z_1(k) = (I_n + G_1(k))y(k) \quad \text{and} \quad z_2(k) = y(k+1), \qquad k = 0, 1, \dots,$$

is a solution of the $2n \times 2n$-difference system

$$\Delta z(k-1) = G(k)z(k) + g(k), \qquad k = 1, 2, \dots, \tag{3.19}$$

where $G(k) = (G_{ij}(k))_{i,j=1}^2$ is defined by (1.22), and $g(k) = (g_i(k))_{i=1}^2$, where $g_1(k) \equiv g_0(k)$, $g_2(k) \equiv 0$.

Conversely, if $z(k) = (z_i(k))_{i=1}^2$ $(k = 0, 1, \dots)$ is a solution of the $2n \times 2n$ system (3.19), then due to (1.20), $y(k) = (I_n + G_1(k))^{-1} z_1(k)$ $(k = 0, 1, \dots)$ is a solution of system (1.4). Indeed, by (3.19) we have

$$z_2(k) = (I_n + G_1(k+1))^{-1} z_1(k+1) = y(k+1), \qquad k = 0, 1, \dots,$$

and

$$\begin{aligned} &(I_n + G_1(k))y(k-1) - (I_n + G_1(k-1))y(k-1) \\ &\qquad = (G_1(k) + G_2(k))\,y(k) + G_3(k)z_2(k) + g_1(k), \qquad k = 0, 1, \dots, \end{aligned}$$

i.e., y satisfies system (1.4).

On the other hand, the vector-function $z(k)$ $(k = 0, 1, \dots)$ is a solution of system (3.19) if and only if the vector-function $x(t) = z([t])$ for $t \in \mathbf{R}_+$ ($[t]$ is the integral part of t) is a solution of the $2n \times 2n$ system (1.1), where

$$\begin{aligned} &A(t) = O_{2n\times 2n}, \quad \text{and} \quad f(t) = O_{2n}, && \text{for } 0 \le t < 1, \\ &A(t) = \sum_{i=1}^{[t]} G(i), \quad \text{and} \quad f(t) = \sum_{i=1}^{[t]} g(i), && \text{for } t \ge 1. \end{aligned}$$

It is evident that $d_2A(t) = O_{2n\times 2n}$ for $t \in \mathbb{R}_+$, $d_1A(t) = O_{2n\times 2n}$ for $t \in \mathbb{R}_+ \setminus \mathbb{N}$, and $d_1A(k) = G(k)$ for $k \in \mathbb{N}$. Therefore, $\det(I_{2n}+d_2A(t)) = 1$ for $t \in \mathbb{R}_+$, $\det(I_{2n}-d_1A(t)) = 1$ for $t \in \mathbb{R}_+ \setminus \mathbb{N}$, and by (1.21),

$$\begin{aligned}\det(I_{2n} - d_1A(k)) &= \det(I_{2n} - G(k)) \\ &= \det\left(\exp\left(-\sum_{l=1}^{m}\Delta\beta_l(k-1)\cdot B_l\right)\right) \neq 0, \qquad k = 1,2,\dots.\end{aligned}$$

Thus (1.21) guarantees condition (1.5).

Finally, if we assume $\alpha_l(t) \equiv \beta_l([t])$ $(l = 1,\dots,m)$, then the conditions of Theorem 1.1 are fulfilled. Consequently, Theorem 1.5 follows from Theorem 1.1 if we take into account that

$$\|x(k)\| = \|(I_n + G_1(k))y(k)\| + \|y(k+1)\|, \qquad k = 0,1,\dots. \qquad ∎$$

Corollaries 1.7 and 1.8 follow from Corollaries 1.1 and 1.2, respectively, or from Theorem 1.5.

PROOF OF THEOREM 1.6. As above we construct a system of the form (1.1) in order to apply Theorem 1.2. This system differs from the system constructed in the proof of Theorem 1.5, since Theorem 1.2 cannot be applied to the last system.

By (1.23), (1.25), and (1.27),

$$\det(I_n + G_{01}) \neq 0 \quad \text{and} \quad \det G_{03} \neq 0. \tag{3.20}$$

It is easy to verify that the vector-function $y \in E(\mathbb{N}_0;\mathbb{R}^n)$ is a solution of the homogeneous difference system

$$\Delta y(k-1) = G_{01}y(k-1) + G_{02}y(k) + G_{03}y(k+1), \qquad k = 1,2,\dots,$$

if and only if the vector-function $z = (z_i)_{i=1}^2 \in E(\mathbb{N}_0;\mathbb{R}^n)$, where

$$z_1(k) = (I_n + G_{01})y(k) \quad \text{and} \quad z_2(k) = (I_n + G_{01})y(k) - Sy(k+1), \qquad k = 0,1,\dots,$$

is a solution of the $2n \times 2n$-difference system

$$\Delta z(k-1) = G_0 z(k), \qquad k = 1,2,\dots, \tag{3.21}$$

where $G_0 = (G_{ij})_{i,j=1}^2$, $G_{i1} = (G_{01}+G_{02}+\delta_{i2}S)(I_n+G_{01})^{-1}-G_{i2}$ $(i = 1,2)$, $G_{i2} = \delta_{i2}I_n - G_{03}S^{-1}$ $(i = 1,2)$.

In addition,

$$\lim_{k\to+\infty}\|y(k)\| = 0, \qquad \text{iff } \lim_{k\to+\infty}\|z(k)\| = 0. \tag{3.22}$$

Moreover, the vector-function $z(k)$ $(k = 0,1,\dots)$ is a solution of system (3.21) if and only if the vector-function $x(t) = z([t])(t \in \mathbb{R}_+)$ is a solution of system (1.1_0), where

$$A(t) = [t]G_0, \qquad \text{for } t \in \mathbb{R}_+.$$

In addition,

$$\lim_{k\to+\infty}\|z(k)\| = 0, \qquad \text{iff } \lim_{t\to+\infty}\|x(t)\| = 0. \tag{3.23}$$

Clearly, $d_2A(t) = O_{2n\times 2n}$ for $t \in \mathbb{R}_+$, $d_1A(t) = O_{2n\times 2n}$ for $t \in \mathbb{R}_+ \setminus \mathbb{N}$, and $d_1A(k) = G_0$ for $k \in \mathbb{N}$. On the other hand, by (3.20),

$$\det(I_{2n} - d_1A(k)) = \pm\det S \cdot \det(I_n + G_{01})^{-1}\det G_{03} \neq 0, \qquad k = 1,2,\dots.$$

We assume $\mu_i(t) = \mu_{1i}[t]$ and $\mu_{n+i}(t) = \mu_{2i}[t]$ for $t \in \mathbb{R}_+$ $(i = 1, \dots, n)$. Then, due to (1.26), μ_i $(i = 1, \dots, 2n)$ are nondecreasing functions such that $s_0(\mu_i)(t) = 0$ and $d_2\mu_i(t) = 0$ for $t \in \mathbb{R}_+$ $(i = 1, \dots, 2n)$, $d_1\mu_i(t) = 0$ for $t \in \mathbb{R}_+ \setminus \mathbb{N}$ $(i = 1, \dots, 2n)$, and $d_1\mu_{i+n(j-1)}(k) = \mu_{ji}$ $(j = 1, 2;$ $i = 1, \dots, n;$ $k = 0, 1, \dots)$. Hence, $\eta_0(t) = \eta_2(t) = 0$ for $t \in \mathbb{R}_+$, $\sigma_i = 0$ $(i = 1, \dots, 2n)$, $\eta_1(t) = 0$ for $t \in \mathbb{R}_+ \setminus \mathbb{N}$, and $\eta_1(k) = \max\{\alpha_{jii}\mu_{ji} : j = 1, 2;\ i = 1, \dots, n\} = \text{const} < 1$ for $k \in \mathbb{N}$ if $\alpha_{jii} < 0$ $(j = 1, 2;\ i = 1, \dots, n)$. Thus condition (1.12) is fulfilled.

Assume now $A_{jj} = A_j$ $(j = 1, 2)$, $A_{21} = A_3$, and $A_{12} = M_1^{-1}(M_2A_2 - I_n) = (\mu_{1i}^{-1}(\mu_{2i}\alpha_{2il} - \delta_{il}))_{i,l=1}^n$. Then, by (1.23)–(1.25),

$$A(t) = [t](M_mA_{mj})_{m,j=1}^2 = (\alpha_{il}\mu_i(t))_{i,l=1}^{2n}, \qquad \text{for } t \in \mathbb{R}_+,$$

where $\alpha_{il} = \alpha_{1il}$ $(i, l = 1, \dots, n)$, $\alpha_{in+l} = \mu_{1i}^{-1}(\alpha_{2il}\mu_{2i} - \delta_{il})$ $(i, l = 1, \dots, n)$, $\alpha_{n+il} = \alpha_{3il}$ $(i, l = 1, \dots, n)$, and $\alpha_{n+in+l} = \alpha_{2il}$ $(i, l = 1, \dots, n)$.

Moreover,

$$(H_{mj})_{m,j=1}^2 = \left((1 - \delta_{il})(1 + |\sigma_i|)^{-1}|\alpha_{il}||\alpha_{ii}|^{-1}\right)_{i,l=1}^n.$$

Therefore, conditions (1.13)–(1.15) are equivalent to conditions (1.28)–(1.30). In view of conditions (3.22), (3.23), and Remarks 1.2, 1.3 we conclude the validity of the theorem. ∎

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