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ON THE MULTIPOINT BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF LINEAR GENERALIZED DIFFERENTIAL EQUATIONS WITH SINGULARITIES

For the system of linear singular generalized ordinary differential equations

$$dx(t) = dA(t) \cdot x(t) + df(t) \ \text{ for } \ t \in]a, b[\tag{1}$$

we consider the multipoint boundary value problem

$$x_i(t_i+) = 0 \ \ (i = 1, \dots, n_0), \ \ x_i(t_i-) = 0 \ \ (i = n_0 + 1, \dots, n), \tag{2}$$

where $-\infty < a \leq t_i \leq t_{i+1} \leq b < +\infty$ $(i = 1, \dots, n-1)$, $n_0 \in \{0, \dots, n\}$, $x(t) = (x_i(t))_{i=1}^n$, $A = (a_{il})_{i,l=1}^n :]a, b[\to \mathbb{R}^{n\times n}$ and $f = (f_l)_{l=1}^n :]a, b[\to \mathbb{R}^n$ are, respectively, a matrix- and a vector-function with bounded total variation components on every closed segment from the interval $]a, b[$.

We investigate the question on the unique solvability of the problem (1), (2). On the basis of this theorem we have obtained effective criteria for the solvability of this problem.

Analogous and related questions are investigated in [9]–[11] (see also the references therein) for the singular boundary value problems linear and nonlinear systems of ordinary differential equations.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see, e.i., [1]–[8], [12], [13] and the references therein).

The following notation and definitions will be used.

$\mathbb{R} =]-\infty, +\infty[$ $R_+ = [0, +\infty[$; $I = [a, b]$ $(a, b \in \mathbb{R})$ is a closed interval.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{il})_{i,l=1}^{n,m}$ with the norm $\|X\| = \sum\limits_{i,l=1}^{n,m} |x_{il}|$;

$\mathbb{R}_+^{n\times m} = \Big\{(x_{il})_{i,l=1}^{n,m} : \ x_{il} \geq 0 \ (i = 1, \dots, n; \ l = 1, \dots, m)\Big\}$.

$O_{n\times m}$ (or O) is the zero $n \times m$ matrix.

If $X = (x_{il})_{i,l=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{il}|)_{i,l=1}^{n,m}$.

$\mathbb{R}^n = R^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.

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If $X \in \mathbb{R}^{n\times m}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n \times n$-matrix.

$\bigvee\limits_{c}^{b}(X)$ $(a < c < d < b)$ is the variation of the matrix-function $X = (x_{il})_{i,l=1}^{n,m} :]a,b[\to \mathbb{R}^{n\times m}$, i.e., the sum of variations of the latter's components; if $d < c$, then $\bigvee\limits_{c}^{d}(X) = -\bigvee\limits_{d}^{c}(X)$; $V(X)(t) = (v(x_{il})(t))_{i,l=1}^{n,m}$, where $v(x_{il})(c_0) = 0$, $v(x_{il})(t) = \bigvee\limits_{c_0}^{t}(x_{il})$ for $a < t \le b$, $c_0 = (a+b)/2$;

$X(t-)$ and $X(t+)$ are the left and the right limits of the matrix-function $X :]a,b[\to \mathbb{R}^{n\times m}$ at the point $t \in]a,b[$ (we will assume $X(t) = X(a+)$ for $t \le a$ and $X(t) = X(b-)$ for $t \ge b$, if necessary); $d_1X(t) = X(t) - X(t-)$, $\quad d_2X(t) = X(t+) - X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all matrix-functions of bounded variation $X : [a,b] \to \mathbb{R}^{n\times m}$ (i.e., such that $\bigvee\limits_{a}^{b}(X) < +\infty$);

$\mathrm{BV}_{loc}(]a,b[,\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X :]a,b[\to \mathbb{R}^{n\times m}$ such that $\bigvee\limits_{c}^{d}(X) < +\infty$ for every $a < c < d < b$;

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If $\alpha \in \mathrm{BV}([a,b],\mathbb{R})$ has no more than a finite number of points of discontinuity, and $m \in \{1,2\}$, then $D_{\alpha m} = \{t_{\alpha m1}, \dots, t_{\alpha m n_{\alpha m}}\}$ $(t_{\alpha m1} < \dots < t_{\alpha m n_{\alpha m}})$ is the set of all points from $[a,b]$ for which $d_m\alpha(t) \ne 0$, and $\mu_{\alpha m} = \max\{d_m\alpha(t) : t \in D_{\alpha m}\}$ $(m = 1,2)$.

If $\beta \in \mathrm{BV}([a,b],\mathbb{R})$, then

$$\nu_{\alpha m\beta j} = \max\left\{ d_j\beta(t_{\alpha ml}) + \sum_{t_{\alpha m\, l+1-m} < \tau < t_{\alpha m\, l+2-m}} d_j\beta(\tau) : l = 1,\dots,n_{\alpha m} \right\}$$

$(j,m = 1,2)$; here $t_{\alpha 20} = a - 1$, $t_{\alpha 1 n_{\alpha 1}+1} = b + 1$.

$s_j : \mathrm{BV}([a,b],\mathbb{R}) \to \mathrm{BV}([a,b],\mathbb{R})$ $(j = 0,1,2)$ are the operators defined, respectively, by

$$s_1(x)(a) = s_2(x)(a) = 0,$$

$$s_1(x)(t) = \sum_{a<\tau\le t} d_1x(\tau) \quad \text{and} \quad s_2(x)(t) = \sum_{a\le\tau<t} d_2x(\tau) \ \text{ for } \ a < t \le b,$$

$$s_0(x)(t) = x(t) - s_1(x)(t) - s_2(x)(t) \ \text{ for } \ t \in [a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$, then

$$\int_s^t x(\tau)\, dg(\tau) = \int_{]s,t[} x(\tau)\, ds_0(g)(\tau) + \sum_{s<\tau\le t} x(\tau) d_1g(\tau) +$$

$$+ \sum_{s\le\tau<t} x(\tau) d_2g(\tau) \ \text{ for } \ a \le s < t \le b,$$

where $\int\limits_{]s,t[} x(\tau)\,ds_0(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu_0(s_0(g))$ corresponding to the function $s_0(g)$;

$L([a,b],\mathbb{R};g)$ is the space of all functions $x : [a,b] \to \mathbb{R}$ measurable and integrable with respect to the measures $\mu(g)$ with the norm $\|x\|_{L,g} = \int_a^b |x(t)|\,dg(t)$.

If $g(t) \equiv g_1(t) - g_2(t)$, where g_1 and g_2 are nondecreasing functions, then

$$\int\limits_s^t x(\tau)\,dg(\tau) = \int\limits_s^t x(\tau)\,dg_1(\tau) - \int\limits_s^t x(\tau)\,dg_2(\tau) \quad \text{for} \quad s \le t.$$

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to R^{l\times n}$ is a nondecreasing matrix-function and $D \subset \mathbb{R}^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to D$ such that $x_{kj} \in L([a,b],R;g_{ik})$ $(i = 1,\dots,l;\ k = 1,\dots,n;\ j = 1,\dots,m)$;

$$\int\limits_s^t dG(\tau)\cdot X(\tau) = \left(\sum_{k=1}^n \int\limits_s^t x_{kj}(\tau)dg_{ik}(\tau)\right)_{i,j=1}^{l,m} \quad \text{for} \quad a \le s \le t \le b,$$

$$S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \quad (j = 0,1,2).$$

If $G_j : [a,b] \to \mathbb{R}^{l\times n}$ $(j = 1,2)$ are nondecreasing matrix-functions, $G = G_1 - G_2$ and $X : [a,b] \to \mathbb{R}^{n\times m}$, then

$$\int\limits_s^t dG(\tau)\cdot X(\tau) = \int\limits_s^t dG_1(\tau)\cdot X(\tau) - \int\limits_s^t dG_2(\tau)\cdot X(\tau) \ \text{ for } \ s \le t,$$

$$S_k(G) = S_k(G_1) - S_k(G_2) \quad (k = 0,1,2),$$

$$L([a,b],D;G) = \bigcap_{j=1}^2 L([a,b],D;G_j).$$

The inequalities between the vectors and between the matrices are understood componentwise.

A vector-function $x \in \mathrm{BV}_{loc}(]a,b[,\mathbb{R}^n)$ is said to be a solution of the system (1.1) if

$$x(t) = x(s) + \int\limits_s^t dA(\tau)\cdot x(\tau) + f(t) - f(s) \quad \text{for} \ \ a < s \le t < b.$$

Under a solution of the problem (1),(2) we mean a solution $x(t) = (x_i(t))_{i=1}^n$ of the system (1) such that the onesided limits $x_i(t_i+)$ $(i = 1,\dots,n_0)$ and $x_i(t_i+)$ $(i = n_0+1,\dots,n)$ exist and the equalities (2) are fulfilled.

A vector-function $x \in \mathrm{BV}_{loc}(]a,b[,\mathbb{R}^n)$ is said to be a solution of the system of generalized differential inequalities

$$dx(t) - dB(t)\cdot x(t) - dq(t) \le 0 \ (\ge 0) \ \text{ for } \ t \in]a,b[,$$

where $B \in \mathrm{BV}_{loc}(]a,b[,\mathbb{R}^{n\times n})$, $q \in \mathrm{BV}_{loc}(]a,b[,\mathbb{R}^n)$, if

$$x(t) - x(s) + \int\limits_s^t dB(\tau)\cdot x(\tau) - q(t) + q(s) \le 0 (\ge 0) \quad \text{for } \ a < s \le t < b.$$

With out of generality we assume that $A(0) = O_{n\times n}$, $f(0) = 0$. Moreover, we assume

$$\det(I_n + (-1)^j d_j A(t)) \ne 0 \ \text{ for } \ t \in]a,b[\quad (j = 1,2).$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding systems (see [13, Theorem III.1.4]).

If $s \in]a,b[$ and $\alpha \in \mathrm{BV}_{loc}(]a,b[,\mathbb{R})$ are such that

$$1 + (-1)^j d_j \alpha(t) \ne 0 \quad \text{for} \quad t \in]a,b[,$$

then by $\gamma_\alpha(\cdot,s)$ we denote the unique solution of the Cauchy problem

$$d\gamma(t) = \gamma(t)\, d\alpha(t), \quad \gamma(s) = 1.$$

It is known (see, [8]) that

$$\gamma_\alpha(t,s) = \begin{cases} \exp(s_0(\alpha)(t) - s_0(\alpha)(s)) \prod\limits_{s<\tau\le t} (1 - d_1\alpha(\tau))^{-1} \times \\ \qquad \times \prod\limits_{s\le\tau<t} (1 + d_2\alpha(\tau)) \ \text{ for } \ t > s, \\ \exp(s_0(\alpha)(t) - s_0(\alpha)(s)) \prod\limits_{t<\tau\le s} (1 - d_1\alpha(\tau)) \times \\ \qquad \prod\limits_{t\le\tau<s} (1 + d_2\alpha(\tau))^{-1} \ \text{ for } \ t < s, \\ \qquad\qquad 1 \qquad\qquad \text{for } \ t = s. \end{cases}$$

Definition 0.1. Let $n_0 \in \{0,\dots,n\}$. We say that a matrix-function $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ belongs to the set $\mathcal{U}(a,b,t_1,\dots,t_n;n_0)$ if the functions c_{il} $(i \ne l;\ i,l = 1,\dots,n)$ are nondecreasing on $[a,b]$ and the system

$$\mathrm{sgn}(t - t_i)\cdot dx_i(t) \le \sum_{l=1}^n x_l(t)\, dc_{il}(t) \ \text{ for } \ t \in]a,b[\ \ (i = 1,\dots,n)$$

has no nontrivial, nonnegative solution satisfying the condition (1.2).

The similarly definition of set $\mathcal{U}$ has been introduced by I. Kiguradze for ordinary differential equations (see [14]).

Theorem 1. *Let the vector-function* $f=(f_l)_{l=1}^n$ *belong to* $\mathrm{BV}([a,b],\mathbb{R}^n$, *and the components of the matrix-function* $A=(a_{il})_{i,l=1}^n\in\mathrm{BV}_{loc}(]a,b[,\mathbb{R}^{n\times n})$ *satisfies the conditions*

$$\big(s_0(a_{ii})(t)-s_0(a_{ii})(s)\big)\operatorname{sgn}(t-t_i)\le s_0(c_{ii}-\alpha_i)(t)-s_0(c_{ii}-\alpha_i)(s)$$

$$\text{for } a\le s<t<t_i \text{ or } t_i<s<t\le b \ (i=1,\dots,n), \tag{3}$$

$$(-1)^j\big(\big|1+(-1)^jd_ja_{ii}(t)\big|-1\big)\operatorname{sgn}(t-t_i)\le d_j\big(c_{ii}(t)-\alpha_i(t)\big)$$

$$\text{for } t\in[a,t_i[\cup]t_i,b] \ (j=1,2;\ i=1,\dots,n) \tag{4}$$

$$\big|s_0(a_{il})(t)-s_0(a_{il})(s)\big|\le s_0(c_{il})(t)-s_0(c_{il})(s)$$

$$\text{for } a\le s<t<t_i \text{ or } t_i<s<t\le b \ (i\ne l;\ i,l=1,\dots,n), \tag{5}$$

$$|d_ja_{il}(t)|\le d_jc_{il}(t) \text{ for } t\in[a,t_i[\cup]t_i,b] \ (i\ne l;\ i,l=1,\dots,n), \tag{6}$$

where

$$C=(c_{il})_{i,l=1}^n\in\mathcal{U}(a,b,t_1,\dots,t_n;n_0),$$

$\alpha_i:[a,b]\setminus\{t_i\}\to\mathbb{R}$ $(i=1,\dots,n)$ *are nondecreasing on the intervals* $[a,t_i[$ *and* $]t_i,b]$ *functions such that*

$$\lim_{t\to t_i+}d_1\alpha_i(t)<1 \ (i=1,\dots,n_0),$$
$$\lim_{t\to t_i-}d_2\alpha_i(t)<1 \ (i=n_0+1,\dots,n), \tag{7}$$

$$\begin{aligned}\lim_{t\to t_i+}\sup\{\gamma_{\alpha_i}(t,t_i+1/k):k=1,2,\dots\}=0 \ (i=1,\dots,n_0),\\ \lim_{t\to t_i-}\sup\{\gamma_{\alpha_i}(t,t_i-1/k):k=1,2,\dots\}=0 \ (i=n_0+1,\dots,n).\end{aligned} \tag{8}$$

Then the problem (1), (2) *has one and only one solution.*

Corollary 1. *Let the vector-function* $f=(f_l)_{l=1}^n$ *belong to* $\mathrm{BV}([a,b],\mathbb{R}^n$, *and the components of the matrix-function* $A=(a_{il})_{i,l=1}^n\in\mathrm{BV}_{loc}(]a,b[,\mathbb{R}^{n\times n})$ *satisfies the conditions*

$$\big(s_0(a_{ii})(t)-s_0(a_{ii})(s)\big)\operatorname{sgn}(t-t_i)\le-\big(s_0(\alpha_i)(t)-s_0(\alpha_i)(s)\big)+$$

$$+\int_s^t h_{ii}(\tau)\,ds_0(\beta_i)(\tau) \text{ for } a\le s<t<t_i \text{ or } t_i<s<t\le b \ (i=1,\dots,n),$$

$$(-1)^j\big(\big|1+(-1)^jd_ja_{ii}(t)\big|-1\big)\operatorname{sgn}(t-t_i)\le h_{ii}(t)d_j\beta_i(t)-d_j\alpha_i(t)\big)$$

$$\text{for } t\in[a,t_i[\cup]t_i,b] \ (j=1,2;\ i=1,\dots,n),$$

$$\big|s_0(a_{il})(t)-s_0(a_{il})(s)\big|\le\int_s^t h_{il}(\tau)\,ds_0(\beta_l)(\tau)$$

$$\textit{for } a \le s < t < t_i \ \textit{ or } \ t_i < s < t \le b \ \ (i \ne l; \ \ i,l = 1,\dots,n),$$

$$|d_j a_{il}(t)| \le h_{il}(t) d_j \beta_l(t) \ \textit{ for } \ t \in [a,t_i[\cup]t_i,b] \ \ (i \ne l; \ i,l = 1,\dots,n)$$

hold, where $\alpha_i : [a,b] \setminus \{t_i\} \to \mathbb{R}$ $(i = 1,\dots,n)$ *are nondecreasing on the intervals* $[a,t_i[$ *and* $]t_i,b]$ *functions, satisfying the conditions* (7) *and* (8), β_l $(l = 1,\dots,n)$ *are functions nondecreasing on* $[a,b]$ *and having not more than a finite number of points of discontinuity,* $h_{ii} \in L^\mu([a,b],\mathbb{R};\beta_i)$, $h_{il} \in L^\mu([a,b],\mathbb{R}_+;\beta_l)$ $(i \ne l; \, i,l = 1,\dots,n)$, $1 \le \mu \le +\infty$. *Let, moreover,*

$$r(\mathcal{H}) < 1,$$

where the $3n \times 3n$*-matrix* $\mathcal{H} = (\mathcal{H}_{j+1\,m+1})_{j,m=0}^2$ *is defined by*

$$\mathcal{H}_{j+1\,m+1} = \big(\lambda_{kmij} \|h_{ik}\|_{\mu,s_m(\beta_i)}\big)_{i,k=1}^n \quad (j,m = 0,1,2),$$

$$\xi_{ij} = \big(s_j(\beta_i)(b) - s_j(\beta_i)(a)\big)^{\frac{1}{\nu}} \quad (j = 0,1,2,; \ \ i = 1,\dots,n);$$

$$\lambda_{k0i0} = \begin{cases} \big(\frac{4}{\pi^2}\big)^{\frac{1}{\nu}} \xi_{k0}^2 & \textit{if } \ s_0(\beta_i)(t) \equiv s_0(\beta_k)(t), \\ \xi_{k0}\xi_{i0} & \textit{if } \ s_0(\beta_i)(t) \not\equiv s_0(\beta_k)(t) \ \ (i,k = 1,\dots,n); \end{cases}$$

$$\lambda_{kmij} = \xi_{km}\xi_{ij} \quad \textit{if} \quad m^2 + j^2 > 0, \ \ mj = 0$$
$$(j,m = 0,1,2; \ \ i,k = 1,\dots,n),$$

$$\lambda_{kmij} = \bigg(\frac{1}{4}\,\mu_{\alpha_k m}\nu_{\alpha_k m \alpha_i j} \sin^{-2} \frac{\pi}{4n_{\alpha_k m}+2}\bigg)^{\frac{1}{\nu}}$$
$$(j,m=1,2; \ \ i,k = 1,\dots,n),$$

and $\frac{1}{\mu} + \frac{2}{\nu} = 1$. *Then the problem* (1), (2) *has one and only one solution.*

Remark 1. The $3n \times 3n$-matrix $\mathcal{H}$, appearing in Corollary 1 can be replaced by the $n \times n$-matrix

$$\bigg(\max\bigg\{\sum_{j=0}^2 \lambda_{kmij} \|h_{ik}\|_{\mu,s_m(\alpha_k)} \ : \ m = 0,1,2\bigg\}\bigg)_{i,k=1}^n.$$

By Remark 1, Corollary 1 has the following form for $h_{il}(t) \equiv h_{il} = const$, $\alpha_i(t) \equiv \alpha(t)$ $(i = 1,\dots,n)$ and $\beta_i(t) \equiv \beta(t)$ $(i,l = 1,\dots,n)$ and $\mu = +\infty$.

Corollary 2. *Let the vector-function* $f = (f_l)_{l=1}^n$ *belong to* $\mathrm{BV}([a,b],\mathbb{R}^n$, *and the components of the matrix-function* $A = (a_{il})_{i,l=1}^n \in \mathrm{BV}_{loc}(]a,b[, \mathbb{R}^{n\times n})$ *satisfies the conditions*

$$\big(s_0(a_{ii})(t) - s_0(a_{ii})(s)\big)\,\mathrm{sgn}(t - t_i) \le h_{ii}\big(s_0(\beta)(t) - s_0(\beta)(s)\big)$$
$$-\big(s_0(\alpha)(t) - s_0(\alpha)(s)\big) \ \textit{ for } \ a \le s < t < t_i \ \textit{ or } \ t_i < s < t \le b \ \ (i = 1,\dots,n),$$

$$(-1)^j\big(\big|1 + (-1)^j d_j a_{ii}(t)\big| - 1\big)\,\mathrm{sgn}(t - t_i) \le h_{ii} d_j \beta(t) - d_j \alpha(t)\big)$$

$$\textit{for } \ t \in [a,t_i[\cup]t_i,b] \ \ (j = 1,2; \ i = 1,\dots,n),$$

$$\big|s_0(a_{il})(t) - s_0(a_{il})(s)\big| \le h_{il}\big(s_0(\beta)(t) - s_0(\beta)(s)\big)$$

$$for\ \ a \le s < t < t_i \ \ or\ \ t_i < s < t \le b \ \ (i = 1, \dots, n),$$

$$|d_j a_{il}(t)| \le h_{il} d_j \beta(t) \ \ for\ \ t \in [a, t_i[\cup]t_i, b] \ \ (j = 1, 2;\ i = 1, \dots, n)$$

hold, where $\alpha : [a, b] \setminus \{t_i\} \to \mathbb{R}$ *is a nondecreasing on the intervals* $[a, t_i[$ *and* $]t_i, b]$ *function, satisfying the conditions* (7) *and* (8), β *is a nondecreasing on* $[a, b]$ *function and having not more than a finite number of points of discontinuity,* $h_{ii} \in \mathbb{R}$, $h_{il} \in \mathbb{R}_+$; $(i \ne l;\ i, l = 1, \dots, n)$. *Let, moreover,*

$$\rho_0\, r(\mathcal{H}) < 1,$$

where

$$\mathcal{H} = (h_{ik})_{i,k=1}^n, \quad \rho_0 = \max\Big\{ \sum_{j=0}^{2} \lambda_{mj} : m = 0, 1, 2 \Big\},$$

$$\lambda_{00} = \frac{2}{\pi} \big(s_0(\beta)(b) - s_0(\beta)(a) \big),$$

$$\lambda_{0j} = \lambda_{j0} = \big(s_0(\beta)(b) - s_0(\alpha)(a) \big)^{\frac{1}{2}} \big(s_j(\beta)(b) - s_j(\beta)(a) \big)^{\frac{1}{2}} \quad (j = 1, 2),$$

$$\lambda_{mj} = \frac{1}{2} \big(\mu_{\alpha_m} \nu_{\alpha_m alpha_j} \big)^{\frac{1}{2}} \sin^{-1} \frac{\pi}{4 n_{\alpha_m + 2} + 2} \quad (m, j = 1, 2).$$

Then the problem (1), (2) *has one and only one solution.*

Theorem 2. *Let the vector-function* $f = (f_l)_{l=1}^n$ *belong to* $\mathrm{BV}([a, b], \mathbb{R}^n)$, *and the components of the matrix-function* $A = (a_{il})_{i,l=1}^n \in \mathrm{BV}_{loc}(]a, b[, \mathbb{R}^{n \times n})$ *satisfies the conditions* (3)–(6), *where* $c_{il}(t) \equiv h_{il} \beta_i(t) + \beta_{il}(t)$ $(i, l = 1, \dots, n)$,

$$d_1 \beta_i(t_i) \le 0 \ \ and\ \ 0 \le d_2 \beta_i(t) < |\eta_i|^{-1} \ \ for\ \ a \le t < t_i \ \ (i = n_0 + 1, \dots, n),$$

$$d_2 \beta_i(t_i) \le 0 \ \ and\ \ 0 \le d_1 \beta_i(t) < |\eta_i|^{-1} \ \ for\ \ t_i < t \le b \ \ (i = 1, \dots, n_0),$$

where $\alpha_i : [a, b] \setminus \{t_i\} \to \mathbb{R}$ $(i = 1, \dots, n)$ *are nondecreasing on the intervals* $[a, t_i[$ *and* $]t_i, b]$ *functions, satisfying the conditions* (7) *and* (8), $h_{ii} < 0$, $h_{il} \ge 0$, $\eta_i < 0$ $(i \ne l;\ i, l = 1, \dots, n)$; β_{ii} $(i = 1, \dots, n)$ *are functions nondecreasing on* $[a, b]$; $\beta_{il}, \beta_i \in \mathrm{BV}([a, b], \mathbb{R})$ $(i \ne l; i, l = 1, \dots, n)$ *are functions nondecreasing on every interval* $[a, t_i[$ *and* $]t_i, b]$. *Let, moreover, the condition (9) hold, where* $\mathcal{H} = (\xi_{il})_{i,l=1}^n$,

$$\xi_{ii} = \eta_i, \ \ \xi_{il} = \frac{h_{il}}{|h_{ii}|} \ \ (i \ne l;\ i, l = 1, \dots, n),$$

$$\eta_i = V\big(\mathcal{A}(\zeta_i, a_i)\big)(b) - V\big(\mathcal{A}(\zeta_i, a_i)\big)(t_i+) \ \ (i = 1, \dots, n_0),$$

$$\eta_i = V\big(\mathcal{A}(\zeta_i, a_i)\big)(t_i-) - V\big(\mathcal{A}(\zeta_i, a_i)\big)(a) \ \ (i = n_0 + 1, \dots, n);$$

$$\zeta_i(t) \equiv \sum_{k=l}^{n} \beta_{il}(t) \ \ (i = 1, \dots, n),$$

$$a_i(t) \equiv h_{ii} \cdot \big(\beta_i(t) - \beta_i(t_i+)\big) \;\; for \;\; t_i < t \le b \;\; (i = 1, \dots, n_0),$$

$$a_i(t) \equiv h_{ii} \cdot \big(\beta_i(t_i-) - \beta_i(t)\big) \;\; for \;\; a \le t < t_i \;\; (i = n_0 + 1, \dots, n).$$

Then the problem (1), (2) *has one and only one solution.*

Remark 2. If $\eta_i < 1 \;\; (i = 1, \dots, n)$, then, in Theorem 1.2, we can assume that

$$\xi_{ii} = 0, \;\; \xi_{il} = \frac{h_{il}}{(1 - \eta_i)|h_{ii}|} \;\; (i \ne l; \; i, l = 1, \dots, n).$$

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