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Short Communications

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ON THE NONLOCAL NONLINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF GENERALIZED DIFFERENTIAL EQUATIONS WITH SINGULARITIES

Abstract. The general nonlocal boundary value problem is considered for systems of nonlinear generalized differential equations with singularities on a non-closed interval. Singularity is understood in a sense that the vector-function corresponding to the system may have unbounded variation with respect to the time variable on the whole interval. The sufficient conditions for the solvability of this problem are given.

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1. Statement of the Problem and Basic Notations

In the paper we investigate the question on the solvability of the system of generalized nonlinear differential equations

$$dx = dA(t) \cdot f(t,x) \tag{1.1}$$

under the general nonlinear boundary value problem

$$h(Hx) = 0, \tag{1.2}$$

where A and $H :]a,b[\to \mathbb{R}^{n\times n}$ are the matrix-functions with components of bounded variation on every closed interval from $]a,b[$, in addition, $\det H(t) \neq 0$ for $t \in]a,b[$; $f \in \mathrm{Car}_{loc}(]a,b[\times\mathbb{R}^n, \mathbb{R}^n; A)$, and $h : \mathrm{BV}_s([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ is a continuous operator.

The same question for the linear general and two-point boundary value problems for systems of generalized linear differential equations are investigated in [5]–[7].

The question on the existence of a solution of the problem (1.1), (1.2) when the matrix A and vector-function f are regular, i.e. $A \in \mathrm{BV}([a,b],\mathbb{R}^{n\times n})$ and $f \in \mathrm{Car}([a,b]\times\mathbb{R}^n,\mathbb{R}^n;A)$, is investigated in [1]–[3], where the Conti–Opial type theorems for the solvability of the problem (1.1), (1.2) are obtained.

Analogous and related questions are investigated in [11] (see also the references therein) for the singular boundary value problems for ordinary differential systems, and in [8], [12]–[14], [16] (see also the references therein) for the regular boundary value problems for ordinary differential systems and for functional differential equations.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see e.g. [4], [9], [10], [15], [17], [18] and the references therein).

Throughout the paper the following notation and definitions will be used.

$\mathbb{R} =]-\infty,+\infty[$, $R_+ = [0,+\infty[$; $[a,b]$ and $]a,b[$ $(a,b\in\mathbb{R})$ are the closed and open intervals, respectively.

$\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{il})_{i,l=1}^{n,m}$ with the norm

$$\|X\| = \sum_{i,l=1}^{n,m} |x_{il}|;$$

$$\mathbb{R}_+^{n\times m} = \Big\{ (x_{il})_{i,l=1}^{n,m} :\ x_{il}\geq 0 \ \ (i=1,\dots,n;\ l=1,\dots,m)\Big\}.$$

$O_{n\times m}$ (or O) is the zero $n\times m$-matrix.

If $X = (x_{il})_{i,l=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = \big(|x_{il}|\big)_{i,l=1}^{n,m}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x=(x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

If $X\in\mathbb{R}^{n\times n}$, then $\det X$ and X^{-1} are, respectively, the determinant of X and the matrix inverse to X; I_n is the identity $n\times n$-matrix.

$\bigvee\limits_c^d(X)$, where $a<c<d<b$, is the variation of the matrix-function $X:]a,b[\to\mathbb{R}^{n\times m}$ on the closed interval $[c,d]$, i.e., the sum of total variations of the latter components x_{il} $(i=1,\dots,n;\ l=1,\dots,m)$ on this interval; if $d<c$, then $\bigvee\limits_c^d(X) = -\bigvee\limits_d^c(X)$; $V(X)(t) = (v(x_{il})(t))_{i,l=1}^{n,m}$, where $v(x_{il})(t_0)=0$, $v(x_{il})(t) = \bigvee\limits_{t_0}^t(x_{il})$ for $a<t<b$, and $t_0=(a+b)/2$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X:]a,b[\to\mathbb{R}^{n\times m}$ at the point $t\in]a,b[$ (we assume $X(t)=X(a+)$ for $t\leq a$ and $X(t)=X(b-)$ for $t\geq b$, if necessary).

$d_1X(t) = X(t)-X(t-)$, $d_2X(t) = X(t+)-X(t)$.

$\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is the set of all matrix-functions of the bounded variation $X:[a,b]\to\mathbb{R}^{n\times m}$ (i.e., $\bigvee\limits_a^b(X)<+\infty$).

$\|X\|_s=\sup\big\{\|X(t)\|:\ t\in[a,b]\big\}$.

$\mathrm{BV}_s([a,b],\mathbb{R}^{n\times m})$ is the normed space $(\mathrm{BV}([a,b],\mathbb{R}^{n\times m}),\|\cdot\|_s)$.

$\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:]a,b[\,\to\mathbb{R}^{n\times m}$ such that $\bigvee\limits_c^d(X)<+\infty$ for every $a<c<d<b$.

If $a<\alpha<\beta<b$ and $X\in\mathrm{BV}([\alpha,\beta],\mathbb{R}^{n\times m})$, then $X_{\alpha,\beta}\in\mathrm{BV}([a,b],\mathbb{R}^{n\times m})$ is a matrix-function defined by

$$X_{\alpha,\beta}(t)=\begin{cases}X(\alpha-) & \text{for } \ a\le t<\alpha,\\ X(t) & \text{for } \ \alpha\le t\le\beta,\\ X(\beta+) & \text{for } \ \beta<t\le b.\end{cases}$$

Let $G\in\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$. By $\mathrm{BV}_G([a,b],\mathbb{R}^n)$ we denote the set of all vector-functions $x\in\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ for which there exist the finite limits $\lim\limits_{t\to a+}G(t)x(t)$ and $\lim\limits_{t\to b-}G(t)x(t)$. It is evident that $x_G\in\mathrm{BV}([a,b],\mathbb{R}^n)$ for every $x\in\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$, where the vector-function $x_G:[a,b]\to\mathbb{R}^n$ is defined by

$$x_G(t)=\begin{cases}G(t)x(t) & \text{for } \ a<t<b,\\ \lim\limits_{t\to a+}G(t)x(t) & \text{for } \ t=a,\\ \lim\limits_{t\to b-}G(t)x(t) & \text{for } \ t=b.\end{cases}$$

A matrix-function is said to be continuous, nondecreasing, integrable, etc., if each of its components is such.

If $I\subset\mathbb{R}$ is an interval, then $C(I,\mathbb{R}^{n\times m})$ is the set of all continuous matrix-functions $X:I\to\mathbb{R}^{n\times m}$.

If B_1 and B_2 are normed spaces, then an operator $g:B_1\to B_2$ (nonlinear, in general) is positive homogeneous if

$$g(\lambda x)=\lambda g(x)$$

for every $\lambda\in\mathbb{R}_+$ and $x\in B_1$.

$s_1,s_2,s_c:\mathrm{BV}([a,b],\mathbb{R})\to\mathrm{BV}([a,b],\mathbb{R})$ are the operators defined, respectively, by

$$s_1(x)(a)=s_2(x)(a)=0,$$
$$s_1(x)(t)=\sum_{a<\tau\le t}d_1x(\tau)\ \ \text{and}\ \ s_2(x)(t)=\sum_{a\le\tau<t}d_2x(\tau)\ \text{ for } \ a<t\le b,$$

and

$$s_c(x)(t)=x(t)-s_1(x)(t)-s_2(x)(t)\ \text{ for } \ t\in[a,b].$$

If $g : [a,b] \to \mathbb{R}$ is a nondecreasing function, $x : [a,b] \to \mathbb{R}$ and $a \le s < t \le b$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_{]s,t[} x(\tau)\,ds_c(g)(\tau) + \sum_{s<\tau\le t} x(\tau)d_1g(\tau) + \sum_{s\le\tau<t} x(\tau)d_2g(\tau),$$

where $\int_{]s,t[} x(\tau)\,ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(s_c(g))$ corresponding to the function $s_c(g)$. If $a = b$, then we assume $\int_a^b x(t)\,dg(t) = 0$; so that $\int_s^t x(\tau)\,dg(\tau)$ is the Kurzweil–Stieltjes integral (see [20], [22], [24]). Moreover, we put

$$\int_{s+}^t x(\tau)\,dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int_{s+\varepsilon}^t x(\tau)\,dg(\tau)$$

and

$$\int_s^{t-} x(\tau)\,dg(\tau) = \lim_{\varepsilon\to 0,\ \varepsilon>0} \int_s^{t-\varepsilon} x(\tau)\,dg(\tau).$$

$L([a,b],\mathbb{R};g)$ is the space of all functions $x : [a,b] \to \mathbb{R}$, measurable and integrable with respect to the measure $\mu(g_c(g))$ for which

$$\sum_{a<\tau\le b} |x(t)|d_1g(\tau) + \sum_{a\le\tau<b} |x(t)|d_2g(t) < +\infty,$$

with the norm

$$\|x\|_{L,g} = \int_a^b |x(t)|\,dg(t).$$

If $g_j : [a,b] \to \mathbb{R}$ $(j = 1,2)$ are nondecreasing functions, $g(t) \equiv g_1(t) - g_2(t)$, and $x : [a,b] \to \mathbb{R}$, then

$$\int_s^t x(\tau)\,dg(\tau) = \int_s^t x(\tau)\,dg_1(\tau) - \int_s^t x(\tau)\,dg_2(\tau) \quad \text{for} \quad a \le s \le t \le b.$$

If $G = (g_{ik})_{i,k=1}^{l,n} : [a,b] \to \mathbb{R}^{l\times n}$ is a nondecreasing matrix-function and $D \subset \mathbb{R}^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X = (x_{kj})_{k,j=1}^{n,m} : [a,b] \to D$ such that $x_{kj} \in L([a,b],R;g_{ik})$ $(i = 1,\dots,l;\ k = 1,\dots,n;\ j = 1,\dots,m)$;

$$\int_s^t dG(\tau)\cdot X(\tau) = \left(\sum_{k=1}^n \int_s^t x_{kj}(\tau)dg_{ik}(\tau)\right)_{i,j=1}^{l,m} \quad \text{for} \quad a \le s \le t \le b,$$

$$S_j(G)(t) \equiv \big(s_j(g_{ik})(t)\big)_{i,k=1}^{l,n} \ \ (j = 1,2) \ \text{ and } \ S_c(G)(t) \equiv \big(s_c(g_{ik})(t)\big)_{i,k=1}^{l,n}.$$

If $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^{n\times m}$, then $\operatorname{Car}([a,b]\times D_1, D_2; G)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} : [a,b]\times D_1 \to D_2$ such that for each $i \in \{1,\dots,l\}$, $j \in \{1,\dots,m\}$ and $k \in \{1,\dots,n\}$:

(i) the function $f_{kj}(\,\cdot\,,x) : [a,b] \to D_2$ is $\mu(s_c(g_{ik}))$-measurable for every $x \in D_1$;

(ii) the function $f_{kj}(t,\,\cdot\,) : D_1 \to D_2$ is continuous for $\mu(s_c(g_{ik}))$-almost every $t \in [a,b]$ and for every $t \in D_{g_{ik}}$, and

$$\sup\big\{|f_{kj}(\,\cdot\,,x)| :\ x \in D_0\big\} \in L([a,b],R;g_{ik})$$

for every compact $D_0 \subset D_1$;

$\operatorname{Car}_{loc}(]a,b[\times D_1, D_2; G)$ is the local Carathéodory class, i.e., the set of all mappings $F = (f_{kj})_{k,j=1}^{n,m} :]a,b[\times D_1 \to D_2$ the restriction of which on every closed interval $[\alpha,\beta]$ belongs to $\operatorname{Car}([\alpha,\beta]\times D_1, D_2; G)$ for every $a < \alpha < \beta < b$.

If $G_j : [a,b] \to \mathbb{R}^{l\times n}$ $(j = 1,2)$ are nondecreasing matrix-functions, $G(t) \equiv G_1(t) - G_2(t)$, and $X : [a,b] \to \mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \int_s^t dG_1(\tau)\cdot X(\tau) - \int_s^t dG_2(\tau)\cdot X(\tau) \ \text{ for } \ a \le s \le t \le b,$$

$$S_k(G)(t) \equiv S_k(G_1)(t) - S_k(G_2)(t) \ \ (k = 1,2),$$
$$S_c(G)(t) \equiv S_c(G_1)(t) - S_c(G_2)(t).$$

If $G_1(t) \equiv V(G)(t)$ and $G_2(t) \equiv V(G)(t) - G(t)$, then

$$L([a,b],D;G) = \bigcap_{j=1}^{2} L([a,b],D;G_j),$$

$$\operatorname{Car}([a,b]\times D_1, D_2; G) = \bigcap_{j=1}^{2} \operatorname{Car}([a,b]\times D_1, D_2; G_j),$$

$$\operatorname{Car}_{loc}(]a,b[\times D_1, D_2; G) = \bigcap_{j=1}^{2} \operatorname{Car}_{loc}(]a,b[\times D_1, D_2; G_j).$$

If $G \in \operatorname{BV}([a,b];\mathbb{R}^{n\times n})$ and $X : [a,b] \to \mathbb{R}^{n\times m}$), then

$$\mathcal{B}(G,X)(t) \equiv G(t)X(t) - G(a)X(a) - \int_{t_0}^{t} dG(\tau)\cdot X(\tau).$$

The inequalities between the matrices are understood componentwise.
Below we assume that

$$A_1(t) \equiv V(A)(t) \ \text{ and } \ A_2(t) \equiv V(A)(t) - A(t).$$

A vector-function $x \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ is said to be a solution of the system (1.1) if

$$x(t) = x(s) + \int\limits_s^t dA(\tau)\cdot f(\tau, x(\tau)) \ \text{ for } \ a < s \le t < b.$$

Under a solution of the problem (1.1), (1.2) we mean solutions x of the system (1.1) such that $x \in \mathrm{BV}_H([a,b],\mathbb{R}^n)$ and the equality $h(x_H) = 0$ holds.

We say that the operator $g : \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n) \to \mathbb{R}^n)$ has some property in the set $\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ if the operator $g_{\alpha,\beta} : \mathrm{BV}([\alpha,\beta],\mathbb{R}^n) \to \mathbb{R}^n$, defined by $g_{\alpha,\beta}(x) = g(x_{\alpha,\beta})$, has the same property for every $\alpha,\beta \in]a,b[$ $(\alpha < \beta)$; If, moreover, $B \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$, then we say that the problem

$$dx = dB(t)\cdot x \ \text{ for } \ t \in]a,b[\,, \ \ g(x) \le 0$$

has some property in $\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$, if the problem

$$dx = dB_{\alpha,\beta}(t)\cdot x \ \text{ for } \ t \in [\alpha,\beta], \ \ g_{\alpha,\beta}(x) \le 0$$

has the same property for every $\alpha,\beta \in]a,b[$ $(\alpha < \beta)$.

In particular, we say that the operator $g : \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n) \to \mathbb{R}^n)$ is continuous in the set $\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ if

$$\lim_{k\to+\infty} g(x_{k;\alpha,\beta}) = g(x_{0;\alpha,\beta}) \ \text{ for every } \ a < \alpha < \beta < b,$$

where $x_0 \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ and $x_k \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$ $(k = 1,2,\dots)$ is an arbitrary sequence such that

$$\lim_{k\to+\infty} x_{k;\alpha,\beta}(t) = x_0(t) \ \text{ uniformly on } \ [\alpha,\beta] \ \text{ for } \ a < \alpha < \beta < b.$$

Definition 1.1. Let a matrix-function $H \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$ be such that $\det H(t) \ne 0$ for $t \in]a,b[$. Let, moreover, $l : \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n) \to \mathbb{R}^n$ and $l_0 : \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n) \to \mathbb{R}^n_+$ be, respectively, linear continuous and positive homogeneous continuous operators in the set $\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$. Then by $\mathcal{O}(]a,b[\,,l,l_0;A,H)$ we denote the set of all matrix-functions $P \in \mathrm{Car}_{loc}(]a,b[\,\times\mathbb{R}^n,\mathbb{R}^{n\times n})$ satisfying the Opial condition with respect to the set of four $(l,l_0;A;H)$, i.e.,

(i) there exists $\Phi \in L_{loc}(]a,b[\,,\mathbb{R}^{n\times n}_+;A)$ such that

$$|P(t,x)| \le \Phi(t) \ \text{ on the set } \ ]a,b[\,\times\mathbb{R}^n;$$

(ii)
$$\det\Big(I_n + (-1)^j\big(d_jB(t) + d_jH(t)\cdot H^{-1}(t)\big)\Big) \ne 0 \tag{1.3}$$
$$\text{for } \ a < t < b \ \ (j = 1,2)$$

and the problem

$$dx = \big(dB(t) + dH(t)\cdot H^{-1}(t)\big)\cdot x, \ \ |l(x)| \le l_0(x)$$

has only the trivial solution in $]a,b[$ for every $B \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$ for which there exists a sequence $z_k \in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n$ $(k=1,2,\dots)$ such that

$$\lim_{k\to+\infty}\int_c^t d\mathcal{B}(H,A)(\tau)\cdot P(\tau,z_k(\tau)) = B(t) \ \text{ uniformly into } \ ]a,b[\,.$$

Remark 1.1. In particular, the condition (1.4) holds if

$$\big\|d_j\mathcal{B}(H,A)(t)\cdot\Phi(t)\big\| < 1 \ \text{ for } \ t\in\,]a,b[\ \ (j=1,2).$$

guarantees the condition (1.3).

Remark 1.2. If $H(t)\equiv I_n$, then Definition 1.1 coincides with the Opial class definition for the regular case on every closed interval $[\alpha,\beta]$ (see [2]).

We will assume that $H\in \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^{n\times n})$ is a matrix-function such that $\det H(t)\neq 0$ for $t\in\,]a,b[\,$. Note that we can consider the case in which the matrix function H is regular only in the right and left neighborhood of the points a and b, respectively. In this case we assume that $H(t)=I_n$ if the point t does not belong to these neighborhoods.

2. Formulation of the Main Results

Theorem 2.1. *Let* $f=(f_l)_{l=1}^n$ *and* $f_k=(f_{kl})_{l=1}^n \in \mathrm{Car}_{loc}(]a,b[\,\times\mathbb{R}^n,\mathbb{R}^n;A)$ $(k=1,2,\dots)$,

$$\big|f_{kl}(t,x)\big| \le f_{0l}(t,x) \ \textit{ for } \ \mu(v(a_{il})) - \textit{for almost all } \ t\in\,]a,b[\,, \ \ x\in\mathbb{R}^n$$
$$(i,l=1,\dots,n;\ k=1,2,\dots)$$

and

$$\lim_{k\to+\infty} f_{kl}(t,x) = f_l(t,x) \ \textit{ for } \ \mu(v(a_{jil})) \ \textit{ for almost all } \ t\in\,]a,b[\,, \ \ x\in\mathbb{R}^n$$
$$(j=1,2;\ \ i,l=1,\dots,n;\ \ k=1,2,\dots),$$

where $f_l \in \mathrm{Car}_{loc}(]a,b[\,\times\mathbb{R}^n,\mathbb{R}^n;a_{il})$ $(i,l=1,\dots,n)$. *Let, moreover, for every natural* k, *the system*

$$dx = dA(t)\cdot f_k(t,x)$$

under the condition (1.2) *has a solution* x_k *such that*

$$\lim_{t\to a+}\sup\Big\{\big\|x_{k,H}(a+)-x_{k,H}(t)\big\| :\ k=1,2,\dots\Big\}=0,$$
$$\lim_{t\to b-}\sup\Big\{\big\|x_{k,H}(b-)-x_{k,H}(t)\big\| :\ k=1,2,\dots\Big\}=0$$

and

$$\sup\big\{\|x_k(t)\| :\ k=1,2,\dots\big\} \le \psi(t) \ \textit{ for } \ a<t<b,$$

where $\psi\in \mathrm{BV}_G([a,b],\mathbb{R}^n)$. *Then the sequence* x_k $(k=1,2,\dots)$ *contains a subsequence, convergent in the open interval* $]a,b[\,$, *and its limit is a solution of the problem* (1.1), (1.2).

Theorem 2.2. *Let the conditions*

$$\big|f(t,H^{-1}(t)x)-P(t,x)x\big|\leq\alpha\big(t,\|x\|\big)\ \text{ for } t\in\,]a,b[\,,\ x\in\mathbb{R}^n,$$

and

$$\big|h(x)-l(x)\big|\leq l_0(x)+l_1\big(\|x\|_v\big)\ \text{ in } \mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$$

be fulfilled, where $l:\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)\to\mathbb{R}^n$ *and* $l_0:\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)\to\mathbb{R}^n_+$ *are, respectively, linear continuous and positive homogeneous continuous operators in* $\mathrm{BV}_{loc}(]a,b[\,,\mathbb{R}^n)$; $P\in\mathcal{O}(]a,b[\,,l,l_0;A,H)$ *and a nondecreasing in the second variable matrix– and vector-functions, respectively,* $\alpha\in\mathrm{Car}_{loc}(]a,b[\,\times\mathbb{R}_+,\mathbb{R}^n_+;A)$ *and* $l_1\in C(\mathbb{R},\mathbb{R}^n_+)$ *are such that*

$$\lim_{\rho\to+\infty}\frac{1}{\rho}\int\limits_{a+}^{b-}dV(A)(t)\cdot\alpha(t,\rho)<1\ \text{ for } a<\alpha<\beta<b,$$

and

$$\lim_{\rho\to+\infty}\frac{l_1(\rho)}{\rho}<1.$$

Then the problem (1.1), (1.2) *is solvable.*

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