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Mathematics

Categories with Partial Covers

Giorgi Arabidze

Department of Mathematical and Computer Sciences, Tbilisi Free University, Tbilisi, Georgia

(Presented by Academy Member Hvedri Inassaridze)

ABSTRACT. In this paper a category with partial covers is defined and the extra assumptions about stronger pretopology are considered. The examples of stronger pretopologies on different categories with partial covers are given. For each case covers are described and the fact whether they satisfy extra assumptions is checked. *© 2019 Bull. Georg. Natl. Acad. Sci.*

Key words: pretopology, partial cover, fibre product, groupoid, manifold, principal bundle, bibundle functor

A pretopology is an extra structure in general category that allows to develop various kinds of mathematical objects. For instance, groupoids, groupoid actions, principal bundles, groupoid fibrations, actors, Hilsum-Skandalis morphisms, Morita equivalences, and so on. A category with pretopology is equipped with a notion of "cover". There are many different kinds of groupoids. In each case, the range and source maps are assumed to be "covers". For instance, covers are surjective submersions in the context of Lie groupoids. The covers also influence the notion of principal bundle because their bundle projections are assumed to be covers; this is equivalent to "local triviality" in the sense of the pretopology. If our category is that of topological spaces and the covers are the continuous surjections with local continuous sections, then we get exactly the usual notion of local triviality for principal bundles; this is why many geometers prefer this pretopology on topological spaces. Many operator algebraists prefer the pretopology of open continuous surjections instead.

In the abstract setting of groupoids in a category with pretopology there occurred the importance to modify the categorical framework there to allow for "partial" notions. A category with partial covers is equipped with a notion of "partial cover", which allows to define partial sheaves, partial bibundle actors, partial Hilsum–Skandalis morphisms, partial Morita equivalences, and so on.

Stronger Pretopology

Definition 1.1. Let $\mathcal{C}$ be any category. We say there is defined a *stronger pretopology* on $\mathcal{C}$ if we have a collection $\mathcal{F}_p$ of arrows, called *partial covers*, with the following properties:

1. isomorphisms are partial covers;
2. the composite of two partial covers is a partial cover;

3. if $x : X \to B$ is an arrow in $\mathcal{C}$ and $f : A \xrightarrow{p} B$ is a partial cover, then the fibre product $A \times_{f,B,x} X$ exists in $\mathcal{C}$ and the coordinate projection $pr_2 : A \times_{f,B,x} X \xrightarrow{p} X$ is a partial cover.

Definition 1.2. A partial cover $f : A \xrightarrow{p} B$ is called a *cover* if it is the coequalizer of the coordinate projections $pr_1, pr_2 : A \times_{f,B,f} A \rightrightarrows A$. Let $\mathcal{F}$ be the collection of the covers in $\mathcal{C}$.

Definition 1.3. The arrow $f : A \xrightarrow{p} B$ is called a *split cover* if it is a cover and there is the arrow $g : B \to A$ such that $f \circ g = id_B$.

We use arrows $\xrightarrow{p}$ to denote partial covers and double-headed arrows $\twoheadrightarrow$ to denote covers. The following lemmas hold in any category $(\mathcal{C}, \mathcal{F}_p)$ with partial covers.

Lemma 1.4. If a partial cover splits, then it is a split cover.

Corollary 1.5. An isomorphism is a split cover.

Corollary 1.6. Let $f : A \xrightarrow{p} B$ be a partial cover. If f splits, then any pull-back of f is a split cover.

Corollary 1.7. Let $f : A \xrightarrow{p} B$ be a partial cover. The pull-back of f along itself is a split cover.

Corollary 1.8. The composition of split covers is a split cover.

Lemma 1.9. The composition of an isomorphism and a cover is a cover.

Lemma 1.10. If a cover is monic, then it is an isomorphism.

Lemma 1.11. Let $f : A \xrightarrow{p} B$ be a partial cover. f is a cover if and only if it is a coequaliser of some pair of parallel arrows $e_1, e_2 : E \rightrightarrows A$.

Lemma 1.12. Assume that the pull-back of an arrow $g : C \xrightarrow{p} B$ along a cover $f : A \twoheadrightarrow B$ is an isomorphism. If the coordinate projection $pr_2 : A \times_{f,B,g} C \xrightarrow{p} C$ is epic, then $g : C \xrightarrow{p} B$ is an isomorphism, too.

Let $(\mathcal{C}, \mathcal{F}_p)$ be a category with partial covers. Consider the following assumptions on the stronger pretopology.

Assumption 1.13. [1. Definition 2.1]. The composition of covers is a cover.

Assumption 1.14. [1.Definition 2.1]. Any pull-back of a cover is a cover.

Remark 1.15. Under Assumption 1.14 we do not need the requirement that the coordinate projection $pr_2 : A \times_{f,B,g} C \xrightarrow{p} C$ is epic in Lemma 1.12, because it automatically comes from Assumption 1.14. $pr_2 : A \times_{f,B,g} C \xrightarrow{p} C$ is a pull-back of the cover $f : A \twoheadrightarrow B$. Therefore, it is a cover. Thus, it is epic. Under Assumptions 1.13 and 1.14, a category $(\mathcal{C}, \mathcal{F}_p)$ with partial covers is a category $(\mathcal{C}, \mathcal{F})$ with pretopology as defined in [1].

The next assumptions are important for principal bundles and arrows between the orbit spaces. We need these assumptions for defining a composition of bibundle functors and for to composing bibundle actors.

Assumption 1.16. [1. Assumption 2.6]. If a pull-back of $f : A \to B$ along a cover is a cover, then $f : A \to B$ is a cover, too.

Assumption 1.17. [1. Assumption 2.7]. Let f and g be composable arrows. If $f \circ g$ and g are covers, then so is f.

The following assumption is about the final object. We know that the obvious example of a groupoid is a group. A group is a groupoid with only one object. So we need the following assumption to define groups in a category $(\mathcal{C}, \mathcal{F}_p)$ with partial covers.

Assumption 1.18. There is a final object in $(\mathcal{C}, \mathcal{F}_p)$ and arrows to it are covers.

Exapmles of Categories with Partial Covers

In this section, we discuss stronger pretopologies on different categories and check whether they satisfy our extra assumptions. For each case we check the conditions (1), (2) and (3) in Definition 1.1, and then we describe covers. We begin with a trivial examples on an arbitrary category with fibre products.

Example 2.1. Let $\mathcal{C}$ be any category with fibre products and let $\mathcal{F}_p$ be the class of arrows in $\mathcal{C}$ and let $\mathcal{F}$ be the class of coequalisers in $\mathcal{C}$. In this general case we have not any additional information about the extra assumptions.

Example 2.2. Let $\mathcal{C}$ be any category with fibre products and let $\mathcal{F}_p$ be the class of monics in $\mathcal{C}$. Then $(\mathcal{C}, \mathcal{F}_p)$ is a category with partial covers. The covers are the isomorphisms by Lemma 1.10. Assumptions 1.13, 1.14, 1.16 and 1.17 are satisfied. Generally, Assumption 1.18 does not hold.

Example 2.3. Let $(\mathcal{C}, \mathcal{F})$ be a category with a subcanonical pretopology as in [1. Definition 2.1]. If $\mathcal{F}_p = \mathcal{F}$ then we have a category $(\mathcal{C}, \mathcal{F}_p)$ with partial covers. In this case, Assumptions 1.16 and 1.17 hold by Definition in [1. Definition 2.1]. Generally, we have no additional information about the other extra assumptions.

Example 2.4. Let $\mathcal{S}$ be the category of sets and let $\mathcal{F}_p$ be the collection of maps in $\mathcal{S}$. Then $(\mathcal{S}, \mathcal{F}_p)$ is a category with partial covers. Also, it is clear that a map in $\mathcal{S}$ is a coequaliser if and only if it is a surjection. Therefore, the covers are the surjections. Assumptions 1.13, 1.14, 1.16 and 1.17 are satisfied. $\mathcal{S}$ does have a final object, but Assumption 1.18 does not hold. If we consider the subcategory of $\mathcal{S}$ without the empty set then Assumtion 1.18 is satisfied.

Let $\mathcal{T}$ be the category of topological spaces and continuous maps. This category is complete and cocomplete. In particular, fibre products and coequalisers exist. We have different kinds of stronger pretopologies in the category of topological spaces. We begin with biquotient maps as covers. First of all, consider the main working lemma for this example, which is proved in [2]. We need the following definitions:

Definition 2.5. A map $f: X \to Y$ is *limit lifting* if every convergent net in Y lifts to a convergent net in X. More precisely, let $(I, \leq)$ be a directed set and let $(y_i)_{i \in I}$ be a net in Y converging to some $y \in Y$. A *lifting* of this convergent net is a directed set $(J, \leq)$ with a surjective order-preserving map $\varphi: J \to I$ and a net $(x_j)_{j \in J}$ in X with $f(x_j) = y_{\varphi(j)}$ for all $j \in J$, converging to some $x \in X$ with $f(x) = y$.

Definition 2.6. Let $f: X \to Y$ be a continuous surjection. It is a *biquotient* map if for every $y \in Y$ and every open covering $\mathfrak{U}$ of $f^{-1}(y)$ in X, there are finitely many $U \in \mathfrak{U}$ for which the subsets $f(U)$ cover some neighbourhood of y in Y.

Lemma 2.7. Biquotient maps are the same as limit lifting maps.

Example 2.8. Let $\mathcal{F}_p$ be the collection of maps in $\mathcal{T}$ which are biquotient on its image with the subspace topology. Then $(\mathcal{T}, \mathcal{F}_p)$ is a category with partial covers. We know that limit lifting maps are quotient maps and quotient maps are coequalisers, so the biquotient maps onto the image are coequalisers if and only if they are surjective biquotient maps. So covers are surjective biquotient maps. In this category with such stronger pretopology, Assumptions 1.13, 1.14, 1.16 and 1.17 are satisfied. Also, such stronger pretopology satisfies Assumption 1.18 if we remove the empty space from the category.

The following three examples are given by continuous sections. Let $f: A \to B$ be a continuous map in $\mathcal{T}$. We call a continuous map $\sigma_b: U_b \to A$ a *local continuous section* for f at $b \in B$ if U_b is a neighbourhood of b and $f \circ \sigma_b = id_{U_b}$.

Definition 2.9. We call $f: A \to B$ *locally split* if local continuous sections $\sigma_b: U_b \to A$ exist at all $b \in B$.

Example 2.10. Let $\mathcal{F}_p$ be the collection of maps in $\mathcal{T}$ which are locally split onto their image (assume image to be open). Then $(\mathcal{T},\mathcal{F}_p)$ is a category with partial covers. Any locally split map is a biquotient map because any convergent net can be lifted by a local continuous section. A map biquotient on its image is a coequaliser if and only if it is surjective. Therefore, a locally split map to its image is a coequaliser if and only if it is surjective. So the covers are the locally split surjections. Asssumptions 1.13, 1.14, 1.16 and 1.17 hold. Assumption 1.18 holds if we exclude the empty space.

As in the previous case, the next example is defined using continuous sections. Let $f : A \to B$ be a continuous map.

Definition 2.11. $f : A \to B$ *has many local continuous sections* if for all $a \in A$ there is an open neighbourhood $U_a \subseteq B$ of $f(a)$ and a continuous map $\sigma_a : U_a \to A$ with $\sigma_a(f(a)) = a$ and $f \circ \sigma_a = id_{U_a}$.

Example 2.12. Let $\mathcal{F}_p$ be the collection of continuous maps with many local continuous sections. Then $(\mathcal{T},\mathcal{F}_p)$ is a category with partial covers. Any continuous map with many local continuous sections is a biquotient map on its image because any convergent net in the image can be lifted by a local continuous section. A biquotient map on its image is a coequaliser if and only if it is surjective. Therefore, a continuous map with many local continuous sections is a coequaliser if and only if it is surjective. So the covers are the surjections with many local continuous sections. Asssumptions 1.13, 1.14, 1.16 and 1.17 hold. Assumption 1.18 holds if we exclude the empty space from the category.

The next example is defined by global continuous sections. Let f be a continuous map.

Definition 2.13. $f : A \to B$ is a *splitting* map if there is a continuous section $\sigma_f : B \to A$ with $f \circ \sigma_f = id_B$.

Example 2.14. Let $\mathcal{F}_p$ be the collection of continuous maps in $\mathcal{T}$ which split on the image (assume image to be open). Then $(\mathcal{T},\mathcal{F}_p)$ is a category with partial covers. Any splitting map is a biquotient map because any convergent net can be lifted by a continuous section. A biquotient map to its image is a coequaliser if and only if it is surjective. Therefore, a splitting map to its image is a coequaliser if and only if it is surjective. So the covers are the splitting surjections. Asssumptions 1.13, 1.14, 1.16 and 1.17 hold. Assumption 1.18 holds if we exclude the empty space.

The next example is given by using proper maps. Let $f : A \to B$ be a continuous map. f is closed if it maps closed subsets to closed subsets. It is *proper* if and only if it is closed and $f^{-1}(b)$ is quasi-compact for all $b \in B$, (see [3], I.10.2).

Lemma 2.15. (see [3], I.10.1) A continuous map $f : A \to B$ is proper if and only if the map $(f \times id_X) : A \times X \to B \times X$ is closed for any topological space X.

Example 2.16. Let $\mathcal{F}_p$ be the collection of proper maps in $\mathcal{T}$. Then $(\mathcal{T},\mathcal{F}_p)$ is a category with partial covers. Hence the proper maps are coequalisers if and only if they are proper surjections. Therefore, the covers are the surjective proper maps. Asssumptions 1.13 and 1.14 are clearly satisfied. Let $f : A \to B$ and $g : B \to C$ be composable continuous maps. If $g \circ f$ and f are surjective proper maps, then the map g is so. For any topological space X the maps $(g \circ f \times id_X) : A \times X \to C \times X$ and $(f \times id_X) : A \times X \to B \times X$ are closed. Let U be any closed subset in $B \times X$. Since $(f \times id_X)$ is a continuous, the subset $(f \times id_X)^{-1}(U)$ is closed in $A \times X$. Since $g \circ f$ is proper, $(g \circ f \times id_X)(f \times id_X)^{-1}(U)$ is closed in $C \times X$. This closed subset equals $(g \times id_X)(U)$ because $(f \times id_X)$ is surjective. Therefore, Assumptions 1.16 and 1.17 are satisfied by Lemma 2.15. The map from a space A to the one-point space is proper if and only if A is quasi-compact. Hence Assumption 1.18 fails even if we exclude the empty space.

The next example is defined using open maps. Let $f : A \to B$ be a continuous map.

Definition 2.17. $f : A \to B$ is *open* if it maps open subsets in A to open subsets in B.

Lemma 2.18. [2, Proposition 1.15] A continuous surjection $f : A \to B$ between topological spaces is open if and only if, for any $a \in A$, a convergent net $(b_i)_{i\in I}$ in B with $lim_{i\in I} b_i = f(a)$ lifts to a net in A converging to a.

Example 2.19. Let $\mathcal{F}_p$ be the collection of open maps in $\mathcal{T}$. Then $(\mathcal{T}, \mathcal{F}_p)$ is a category with partial covers. An open map is a limit lifting map to the image by 2.18. Hence the open surjections are biquotient maps, and therefore, an open map is a coequaliser if and only if it is surjective. So the covers are the open surjections. Asssumptions 1.13, 1.14, 1,19 and 1.17 hold. Assumption 1.18 is satisfied if we remove the empty space from the category.

The next example is defined using étale maps. Let $f : A \to B$ be a continuous map.

Definition 2.20. $f : A \to B$ is *étale* if for all $a \in A$ there is an open neighbourhood U_a such that $f(U_a)$ is open and $f|_{U_a} : U_a \xrightarrow{\sim} f(U_a)$ is a homeomorphism for the subspace topologies on U_a and $f(U_a)$ from A and B, respectively.

Example 2.21. Let $\mathcal{F}_p$ be the collection of étale maps in $\mathcal{T}$. Then $(\mathcal{T}, \mathcal{F}_p)$ is a category with partial covers. It is clear that ĕtale surjections are limit lifting, and therefore, they are biquotient maps. Hence étale maps are coequalisers if and only if they are étale surjections. Therefore, the covers are the étale surjections.

Asssumptions 1.13 and 1.14 hold because the composition of surjections is a surjection and a pull-back of a surjection is a surjection.

Let the composition $g \circ f$ of an étale surjection $f : A \twoheadrightarrow B$ and a continuous map $g : B \to C$ be an étale surjection. For any element b in B we have an element a in A and an open neighbourhood U_a of a such that $f(a) = b$ and the restriction $f|_{U_a} : U_a \xrightarrow{\sim} f(U_a)$ is a homeomorphism. Since $g \circ f$ is étale, we have an open neighbourhood V_a of a such that the restriction $(g \circ f)|_{V_a} : V_a \xrightarrow{\sim} g(f(U_a))$ is a homeomorphism. The map $\left(f|_{U_a}\right)^{-1}$

$$(g \circ f)|_{U_a \cap V_a} \circ \left(f|_{U_a}\right)^{-1}\Big|_{f(U_a)\cap f(V_a)} :$$

$$f(U_a) \cap f(V_a) \to g(f(U_a) \cap f(V_a))$$

is a restriction of g and it is a homeomorphism because it is a composition of homeomorphisms. Therefore, $g : B \to C$ is étale. It clearly is a surjection. So Assumptions 1.16 and 1.17 are satisfied. Unless A is discrete, the constant map from A to a point is not étale, so Assumption 1.18 fails even if we exclude the empty space.

The examples of a stronger pretopology can be defined in the categories of finite-dimensional manifolds $\boldsymbol{Mfd}_{fin}$; Hilbert manifolds $\boldsymbol{Mfd}_{Hil}$; Banach manifolds $\boldsymbol{Mfd}_{Ban}$; Fréchet manifolds $\boldsymbol{Mfd}_{Fré}$ and locally convex manifolds $\boldsymbol{Mfd}_{lcs}$. Such manifolds are Hausdorff topological spaces that are locally homeomorphic to finite-dimensional vector spaces, Hilbert spaces, Banach spaces, Fréchet spaces, or locally convex topological vector spaces, respectively. The morphisms between all these types of manifolds are smooth maps. In each case, a stronger pretopology is defined by submersions.

Definition 2.22. [4, Definition 4.4.8], [5, Appendix A] Let X and Y be locally convex manifolds. A smooth map $f : X \to Y$ is a *submersion* if for each $x \in X$, there is an open neighbourhood V of x in X such that $U = f(V)$ is open in Y, and there are a smooth manifold W and a diffeomorphism $V \cong U \times W$ that intertwines f and the coordinate projection $pr_1 : U \times W \to U$.

Example 2.23. Let $\mathcal{C}$ be one of the categories $\mathbf{Mfd}_{fin}$, $\mathbf{Mfd}_{Hil}$, $\mathbf{Mfd}_{Ban}$, $\mathbf{Mfd}_{Fré}$, $\mathbf{Mfd}_{lcs}$ considered above. Let $\mathcal{F}_p$ be the collection of submersions in $\mathcal{C}$. Then $(\mathcal{C}, \mathcal{F}_p)$ is a category with partial covers. In the proof of Proposition 9.40 in [1] it is shown that the pretopology defined by surjective submersions is subcanonical. That is, surjective submersions are coequalisers. It is clear that a coequaliser is surjective. Therefore, a submersion is surjective if and only if it is a coequaliser. So covers are surjective submersions.

In all categories described in Example 2.23, Assumptions 1.13 and 1.14 hold by Proposition 9.40 in [1]. In the categories $\mathbf{Mfd}_{fin}$, $\mathbf{Mfd}_{Hil}$ and $\mathbf{Mfd}_{Ban}$ with such stronger pretopology, Assumption 1.16 holds by Proposition 9.42 in [1]. We have no information about these assumptions in other categories.

მათემატიკა

კატეგორიები ნაწილობრივი დაფარვებით

გ. არაბიძე

თბილისის თავისუფალი უნივერსიტეტი, მათემატიკისა და კომპიუტერული მეცნიერებების დეპარტამენტი, თბილისი, საქართველო

(წარმოდგენილია აკადემიის წევრის ხ. ინასარიძის მიერ)

სტატიაში განხილულია კატეგორია ნაწილობრივი დაფარვებით და მოყვანილია დამატებითი დაშვებები გაძლიერებული პრეტოპოლოგიის შესახებ. მოყვანილია გაძლიერებული პრეტოპოლოგიების მაგალითები სხვადასხვა სახის კატეგორიებში. თითოეულ შემთხვევაში აღწერილია დაფარვები და შემოწმებულია აკმაყოფილებენ თუ არა ისინი დამატებით დაშვებებს.

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