Quaternionic Exponentially Dichotomous Operators

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1 Introduction

The notion of quaternions that is a noncommutative extension of complex numbers is a mathematical concept introduced by Irish mathematician Hamilton in 1843 and it has been widely applied to both pure and applied mathematics and physics. For instance, Adler studied the quaternionic quantum mechanics and quantum fields in 1995 (see [1]). Since the quaternionic algebra has a significant feature that its multiplication does not follow commutative law and it refers to applied dynamic equations (see [5–8]) and many mathematical and physical research fields, many momentous studies based on quaternionic theory have been hot topics.

In 1998, Colombo and Sabadini studied the quaternionic functional calculus of Fueter-regular function based on Cauchy formula (see [4]). In [2], by using the theory of S-spectrum, Cerejeiras et al. studied the slice hyper-holomorphism of S-resolvent operator for the perturbation problem of quaternionic normal operator in Hilbert space and the conditions to ensure the existence of nontrivial hyper-invariant subspace of quaternionic linear operator were given. In the book [3], Colombo et al. systematically presented the discovery of the S-spectrum and of the S-functional calculus in the introduction and how hypercomplex analysis methods were used to identify the appropriate notion of quaternionic spectrum whose existence was suggested by quaternionic quantum mechanics. In 2022, based on S-spectral theory, Wang, Qin and Agarwal introduced the notion of the quaternionic exponentially dichotomous operator and obtained its integral representation formula.

2 Quaternionic exponentially dichotomous operators

In this section, we will present a notion of the quaternionic exponentially dichotomous operators of the quaternionic version and some fundamental results which are important to discuss the quaternionic evolution equations. For more details, one may consult [9].

2.1 Quaternionic bisemigroups and direct sum decomposition of quaternionic Banach space

Definition 2.1 ([9])**.** Let X be a quaternionic Banach space, by a (strongly continuous) bisemigroup we mean a function $E(\,\cdot\,) : \mathbb{R} \setminus \{0\} \to \mathcal{B}(X)$ having the following properties:

(1) If $t,s>0$, we have $E(t)E(s)=E(t+s)$ and for $t,s<0$ we have $E(t)E(s)=-E(t+s)$.

(2) For every $x\in X$ the function $E(\,\cdot\,)x:\mathbb{R}\setminus\{0\}\to X$ is continuous, apart from a jump discontinuity in $t=0$. That is,

$$\lim_{t\to 0^\pm}\|E(t)x-E(0^\pm)x\|_X=0,\ \ x\in X.$$

(3) $E(0^+)x-E(0^-)x=x$ for every $x\in X$.

(4) There exist $M,\lambda>0$ such that $\|E(t)\|_{\mathcal{B}(X)}\leqslant Me^{-\lambda|t|}$ for $t\in\mathbb{R}\setminus\{0\}$.

From Definition 2.1, any quaternionic strongly continuous semigroup $\{E(t)\}_{t\geqslant 0}$ having a negative exponential growth bound extends to a uniformly continuous bisemigroup when defining $E(t)=0_{\mathcal{B}(X)}$ for $t<0$. Notice properties (1) and (3) in Definition 2.1, we can obtain the following proposition.

Proposition 2.1 ([9])**.** *Let $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ be a strongly continuous bisemigroup and $P=-E(0^-)$, then the following holds*

$$\begin{cases}E(t)\big[\operatorname{Ker}P\big]\subset\operatorname{Ker}P, & t>0,\\ E(t)\big[\operatorname{Im}P\big]\subset\operatorname{Im}P, & t<0.\end{cases}$$

Proposition 2.1 implies that $E(0^+)$ and $-E(0^-)$ are bounded complementary, we may introduce the concept of the constituent semigroup of a bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ as follows.

Definition 2.2 ([9])**.** Let $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ be a strongly continuous bisemigroup, then we call the operator $P=-E(0^-)$ the separating projection of the bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$. The restriction of $E(t)$ to $\operatorname{Ker}P$ is a quaternionic strongly continuous semigroup on $\operatorname{Ker}P$, while the restriction of $-E(-t)$ to $\operatorname{Im}P$ is a strongly continuous semigroup on $\operatorname{Im}P$. These two semigroups are called the constituent semigroups of the bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$.

Definition 2.2 indicates that we can describe the exponential growth bounds of $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ through the exponential growth bounds of its corresponding constituent semigroups, hence we introduce the following notion.

Definition 2.3 ([9])**.** Let $E_j:[0,\infty)\to X_j(j=1,2)$ be the quaternionic strongly continuous semigroups, and both have a negative exponential growth bound, we define the strongly continuous bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ on $X=X_1\oplus X_2$ by

$$E(t)=\begin{cases}E_1(t)\oplus 0_{X_2}, & t>0,\\ 0_{X_1}\oplus\big(-E_2(-t)\big), & t<0,\end{cases}$$

which has $\{E_1(t)\}_{t\geqslant 0}$ and $\{E_2(t)\}_{t\geqslant 0}$ as its constituent semigroups. For the pair of exponential growth bounds of a bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$, we denote the pair of (necessarily negative) exponential growth bounds of its constituent semigroups by:

$$\big\{\lambda_+(E),\lambda_-(E)\big\}.$$

For the exponential growth bound $\lambda(E)$ of a bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$, we denote it by

$$\lambda(E)\overset{\text{def}}{=}\max\big\{\lambda_-(E),\lambda_+(E)\big\}<0.$$

Definition 2.4 ([9]). Let $T_+(\operatorname{Ker} P \to \operatorname{Ker} P)$ and $-T_-(\operatorname{Im} P \to \operatorname{Im} P)$ stand for the infinitesimal generators of the constituent semigroups of the bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ on X, then the linear quaternionic operator $T(X \to X)$ defined by

$$\mathcal{D}(T) = \big\{x_+ \oplus x_- : \ \ x_+ \in \mathcal{D}(T_+),\ x_- \in \mathcal{D}(T_-)\big\},$$
$$T(x_+ \oplus x_-) = T_+(x_+) - T_-(x_-)$$

is called the (infinitesimal) generator of the bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$, since $T(X \to X)$ is closed and densely defined, then we define the constituent Laplace transform formulas as follows:

$$S_R^{-1}(s,T_+)x_+ = \int_0^\infty e^{-st}E(t)x_+\,dt,\ \ x_+ \in \operatorname{Ker} P,\ \ \operatorname{Re}(s) > \lambda_+(E),$$
$$S_R^{-1}(-s,-T_-)x_- = -\int_0^\infty e^{st}E(-t)x_-\,dt,\ \ x_- \in \operatorname{Im} P,\ \ \operatorname{Re}(-s) > \lambda_-(E),$$

where both of $\lambda_\pm(E) < 0$, which imply the Laplace transform formula

$$S_R^{-1}(s,T)x = \int_{-\infty}^\infty e^{-st}E(t)x\,dt,\ \ \lambda_+(E) < \operatorname{Re}(s) < -\lambda_-(E), \tag{1}$$

where the (Bochner) integral converges absolutely in the norm of X. Now we will write $E(t,T)$ for the strongly continuous bisemigroup with infinitesimal generator T.

Remark 2.1. From Definition 2.4, there exists a quaternionic district in the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ about the 2-dimensional sphere $\mathbb{S}$ contained in the S-resolvent set of the infinitesimal generator T of $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$.

2.2 Quaternionic exponentially dichotomous operators and integral representation

We will present the concept of a quaternionic exponentially dichotomous operator.

Definition 2.5 ([9]). A closed and densely defined linear quaternionic operator $T(X \to X)$ on a quaternionic Banach space X is called exponentially dichotomous if it is the infinitesimal generator of a strongly continuous bisemigroup $\{E(t)\}_{t\in\mathbb{R}\setminus\{0\}}$ on X.

Proposition 2.2 ([9]). *Let X be an quaternionic Banach space, $T(X \to X)$ be an exponentially dichotomous quaternionic operator. Then T has precisely one separating projection P of the bisemigroup $E(t,T)$.*

Definition 2.6 ([3]). Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \cap \mathbb{R} \neq \varnothing$ and suppose that $f \in \mathcal{R}^L_{\overline{\sigma}_S(T)}$ (resp. $f \in \mathcal{R}^R_{\overline{\sigma}_S(T)}$). Let us consider $k \in \mathbb{R}$ and the function $\Phi : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ defined by $p = \Phi(s) = (s-k)^{-1}$, $\Phi(\infty) = 0$, $\Phi(k) = \infty$. Now consider

$$\phi(p) := f(\Phi^{-1}(p))$$

and the bounded linear operator defined by

$$A := (T - k\mathcal{I})^{-1} \ \text{ for some } \ k \in \rho_S(T) \cap \mathbb{R}.$$

We define, in both cases, the operator $f(T)$ as

$$f(T) = \phi(A). \tag{2}$$

Now we introduce the following slice symmetric domain of the quaternionic Banach space.

Definition 2.7 ([9])**.** We define a slice symmetric domain $D_{\eta-I\xi}^{\eta+I\xi}$ as follows:

$$D_{\eta-I\xi}^{\eta+I\xi} := \Big\{ s\in\mathbb{C}_I:\ \ \mathrm{Re}(s)=\eta,\ \ I\in\mathbb{S},\ \ |\mathrm{Im}(s)|\leqslant\xi\Big\}.$$

Moreover, if $\xi=\infty$, we denote the ∞-symmetric domain by $D_{\eta-I\infty}^{\eta+I\infty}$.

To present an integral representation for the separating projection P, we established the following lemma.

Lemma 2.1 ([9])**.** *Let $\Psi(X\to X)$ be a closed linear quaternionic operator on the two-side quaternionic Banach space X such that $\rho_S(\Psi)\cap\mathbb{R}\neq\varnothing$ and assume that $\sigma_S(\Psi)\subset\{s\in\mathbb{H}:\ \mathrm{Re}(s)>\eta\}$, where η is some positive real number. Let $f\in\mathcal{R}^R_{\sigma_S(\Psi)}$ and $\mathbb{C}_I=\mathbb{R}+I\mathbb{R}$ for any $I\in\mathbb{S}$, Φ and ϕ are the same as in Definition 2.6. Then*

$$f(\Psi)x=\frac{1}{2\pi}\oint\limits_{\gamma_\eta}\phi(p)\,dp_I S_R^{-1}(p,A)x,\ \ x\in\mathcal{D}(\Psi^2),$$

where $A=(\Psi-k\mathcal{I})^{-1}$ with $k\in\rho_S(\Psi)\cap\mathbb{R}$ such that $|k|<\eta$, $p=\Phi(s)$, $\gamma_\eta=\{s\in\mathbb{C}_I:\ |s-(\eta-k)^{-1}/2|=(\eta-k)^{-1}/2\}$ and whenever $x\in\mathcal{D}(\Psi^2)$ and $S_R^{-1}(s,\Psi)$ is bounded on $\mathrm{Re}(s)\leqslant\eta$.

Theorem 2.1 ([9])**.** *Let $T(X\to X)$ be a quaternionic exponentially dichotomous operator such that $\rho_S(T)\cap\mathbb{R}\neq\varnothing$ and suppose the exponential growth bound $\lambda(E)<0$, and let P be its separating projection, Φ, p and A are as in Definition 2.6. Then*

$$Px=\frac{1}{2\pi}\oint\limits_{\gamma_\eta}dp_I S_R^{-1}(p,A)x,\ \ x\in\mathcal{D}(T^2),\tag{3}$$

where $dp_I=dp/I$, $|k|<\eta<-\lambda(E)$ and γ_η, k, η are as in Lemma 2.1.

Remark 2.2. Let $E(t,T)$ be a bisemigroup, for any $k\in\rho_S(T)\cap\mathbb{R}$ and $|\eta|<\lambda(E)$, noticing that $\Phi\big(D_{\eta-I\infty}^{\eta+I\infty}\big)$ is a circle with the center $p_0=\Phi(\eta)/2$ and radius $(\eta-k)^{-1}/2$, we denote it by γ_η.

References

[1] S. L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields.* International Series of Monographs on Physics, 88. The Clarendon Press, Oxford University Press, New York, 1995.

[2] P. Cerejeiras, F. Colombo, U. Kähler and I. Sabadini, Perturbation of normal quaternionic operators. *Trans. Amer. Math. Soc.* **372** (2019), no. 5, 3257–3281.

[3] F. Colombo, J. Gantner and D. P. Kimsey, *Spectral Theory on the S-Spectrum for Quaternionic Operators.* Operator Theory: Advances and Applications, 270. Birkhäuser/Springer, Cham, 2018.

[4] F. Colombo and I. Sabadini, The quaternionic symbolic calculus. *Quaderni di Dipartimento, Università degli Studi di Milano*, 1998, no. 18.

[5] Z. Li, C. Wang, R. P. Agarwal and D. O'Regan, Commutativity of quaternion-matrix-valued functions and quaternion matrix dynamic equations on time scales. *Stud. Appl. Math.* **146** (2021), no. 1, 139–210.

[6] C. Wang, D. Chen and Z. Li, General theory of the higher-order quaternion linear difference equations via the complex adjoint matrix and the quaternion characteristic polynomial. *J. Difference Equ. Appl.* **27** (2021), no. 6, 787–857.

[7] C. Wang, Z. Li and R. P. Agarwal, A new quaternion hyper-complex space with hyper argument and basic functions via quaternion dynamic equations. *J. Geom. Anal.* **32** (2022), no. 2, Paper no. 67, 83 pp.

[8] C. Wang, Z. Li and R. P. Agarwal, Hyers–Ulam–Rassias stability of high-dimensional quaternion impulsive fuzzy dynamic equations on time scales. *Discrete Contin. Dyn. Syst. Ser. S* **15** (2022), no. 2, 359–386.

[9] C. Wang, G. Qin and R. P. Agarwal, Quaternionic exponentially dichotomous operators through S-spectral splitting and applications to Cauchy problem. *Adv. Math.* **410** (2022), Paper no. 108747.