Duality for Stieltjes Integral Equations

Milan Tvrdý

Institute of Mathematics of the Czech Academy of Sciences, Prague, Czech Republic

E-mail: tvrdy@math.cas.cz

1 Linear functionals and operators on the spaces of regulated functions

Let $-\infty < a < b < \infty$. $\mathbb{R}^{n\times n}$ is the space of real $n\times n$-matrices. Let BV^n and G^n be the spaces of n-vector valued functions with bounded variation on $[a,b]$ or regulated on $[a,b]$, respectively. (By regulated functions we understand functions having only discontinuities of the first kind.) Similarly, $\mathrm{BV}^{n\times n}$ and $\mathrm{G}^{n\times n}$ are spaces of of $n\times n$-matrix valued functions having the corresponding properties. The function $R:[a,b]\to\mathbb{R}^{n\times n}$ is said to be *summable* if it vanishes except for a countable set and $\sum\limits_{a\le t\le b}\|R(t)\|<\infty$.

Theorem 1.1. *If Φ is a continuous linear operator from G into $\mathbb{R}^n$ then there exist $K,\widetilde{K}\in BV^{n\times n}$, $M\in\mathbb{R}^{n\times n}$ and a summable function $R:[a,b]\to\mathbb{R}^{n\times n}$ such that*

$$\Phi(x)=M\,x(a)+\int\limits_a^b K\,\mathrm{d}x-\sum_{a\le t<b}R(t)\,\Delta^+x(t)\ \ \textit{for}\ \ x\in G^n$$

and

$$\Phi(x)=M\,x(a)+\int\limits_a^b \widetilde{K}\,\mathrm{d}x+\sum_{a<t\le b}R(t)\,\Delta^-x(t)\ \ \textit{for}\ \ x\in G^n.$$

Remark 1.1. $R(t)=\Phi(\chi_{[t]})$ and $\widetilde{K}(t)=K(t)-R(t)$ for $t\in[a,b]$.

The representation of linear bounded functionals in the space of left-continuous regulated functions is considerably simpler, as shown by the following older result from 1989, cf. [3]. ($\mathrm{G}^n_{\mathrm{L}}$ stands for the space of n-vector valued functions regulated on $[a,b]$, left-continuous on $(a,b]$ and right-continuous at a.)

Theorem 1.2. *Φ is a linear bounded operator from G^n_L into $\mathbb{R}^n$ if and only if there is $M\in\mathbb{R}^{n\times n}$ and an $n\times n$-matrix valued function K of bounded variation on $[a,b]$ such that*

$$\Phi(x)=M\,x(a)+\int\limits_a^b K\,\mathrm{d}[x]\ \ \textit{for}\ \ x\in G^n_L.$$

Later Š. Schwabik [7] generalized this result and described a general form of bounded linear operators on $\mathrm{G}^n_{\mathrm{L}}$.. In what follows $\mathcal{K}^{n\times n}_{\mathrm{L}}$ stands for the set of functions $K:[a,b]\times[a,b]\to\mathbb{R}^{n\times n}$ such that:

- $K(t,\cdot\,)\in\mathrm{BV}^{n\times n}$ for $t\in[a,b]$;

- the abstract function $t \in [a,b] \mapsto K(t,\cdot\,) \in \mathrm{BV}^{n\times n}$ is regulated on $[a,b]$ and left-continuous on $(a,b]$.

Theorem 1.3. *$\mathcal{L}$ is a linear compact operator on G^n_{L} if and only if there are a regulated function $A:[a,b]\to\mathbb{R}^{n\times n}$ and a function B from the class $\in \mathcal{K}^{n\times n}_{\mathrm{L}}$ such that*

$$(\mathcal{L}x)(t) = A(t)\,x(a) + \int_a^b B(t,s)\,\mathrm{d}[x(s)] \ \ \textit{for}\ \ x \in G^n_{\mathrm{L}} \ \ \textit{and}\ \ t \in [a,b].$$

2 Bray theorem

Remark 2.1. Let $\mathcal{K}^{n\times n}$ be the set of functions $K:[a,b]\times[a,b]\to\mathbb{R}^{n\times n}$ such that: $K(t,\cdot\,)\in\mathrm{BV}^{n\times n}$ for $t\in[a,b]$ and the mapping $t\in[a,b]\mapsto K(t,\cdot\,)\in\mathrm{BV}^{n\times n}$ is regulated on $[a,b]$. If $K\in\mathcal{K}^{n\times n}$, then

- $K(\,\cdot\,,s)\in\mathrm{G}^{n\times n}$ for all $s\in[a,b]$ and
$$g(t) := \int_a^b \mathrm{d}_s K(t,s)\,x(s) \in \mathrm{G}^n \ \text{ for all }\ x\in\mathrm{G}^n;$$
- $\operatorname{var}_a^b K(t,\cdot\,) \le \varkappa < \infty$ for all $t\in[a,b]$ and
$$h^*(s) := \int_a^b y^*(t)\,\mathrm{d}_s K(t,s) \in \mathrm{BV}^n \ \text{ for all }\ y\in\mathrm{BV}^n;$$
- $K(\,\cdot\,,s)$ is left-continuous for all $s\in[a,b]$ and $g\in\mathrm{G}^n_{\mathrm{L}}$ for all $x\in\mathrm{G}^n$ whenever $K\in\mathcal{K}^{n\times n}_{\mathrm{L}}$.

A crucial tool for deriving the explicit form of the dual operator $\mathcal{L}^*$ to $\mathcal{L}$ is the next Fubini type theorem called usually the Bray theorem, cf. [5].

Theorem 2.1. *If $K\in\mathcal{K}^{n\times n}$, then*

$$\int_a^b y^*(t)\,\mathrm{d}_t\Big[\int_a^b K(t,s)\,\mathrm{d}[x(s)]\Big] = \int_a^b \Big(\int_a^b y^*(t)\,\mathrm{d}_t[K(t,s)]\Big)\,\mathrm{d}[x(s)]$$

holds for any $x\in G^n$ and any $y\in BV^n$.

3 Linear integral equations in $\mathrm{G}^n_{\mathrm{L}}$

If $\mathcal{L}:\mathrm{G}^n_{\mathrm{L}}\to\mathrm{G}^n_{\mathrm{L}}$ is linear compact operator and $f\in\mathrm{G}^n_{\mathrm{L}}$, then $x-\mathcal{L}\,x=f$ can be rewritten as

$$x(t) - A(t)\,x(a) - \int_a^b B(t,s)\,\mathrm{d}[x(s)] = f(t), \ \ t\in[a,b],$$

where $A\in\mathrm{G}^{n\times n}_{\mathrm{L}}$ and $B\in\mathcal{K}^{n\times n}_{\mathrm{L}}$. Obviously, Fredholm–Stieltjes integral equations, Volterra–Stieltjes integral equations, and generalized linear differential equations are special cases. Adjoint operator $\mathcal{L}^*$ maps $\mathrm{BV}^n\times\mathbb{R}^n$ into $\mathrm{BV}^n\times\mathbb{R}^n$. In view of Bray Theorem we have, cf. [5].

Theorem 3.1. $\mathcal{L}^* : (y,\gamma) \in BV^n \times \mathbb{R}^n \to (\mathcal{L}_1^*(y,\gamma), \mathcal{L}_2^*(y,\gamma)) \in BV^n \times \mathbb{R}^n$, *where*

$$\big(\mathcal{L}_1^*(y,\gamma)\big)(t) = B^*(a,t)\,\gamma + \int_a^b \mathrm{d}_s[B^*(s,t)]\,y(s) \ \ \textit{for} \ \ t \in [a,b],$$

$$\mathcal{L}_2^*(y,\gamma) = A^*(a)\,\gamma + \int_a^b \mathrm{d}[A^*(s)]\,y(s).$$

Analogously. cf. [4], we can treat the boundary value problem

$$\left.\begin{aligned} & x(t) - x(a) - \int_a^t \mathrm{d}[A]\,x = f(t) - f(a) \ \text{ on } [a,b], \\ & M\,x(a) + \int_a^b K\,\mathrm{d}[x] = r, \end{aligned}\right\} \tag{BVP}$$

where $A \in \mathrm{BV}_{\mathrm{L}}^{n\times n}$, $f \in \mathrm{G}_{\mathrm{L}}^n$ and $r \in \mathbb{R}^n$, and corresponding operator $\mathcal{L}\colon \mathrm{G}_{\mathrm{L}}^n \to \mathrm{G}_{\mathrm{L}}^n \times \mathbb{R}^n$. The adjoint $\mathcal{L}^*$ of $\mathcal{L}$ maps $(\mathrm{BV}^n \times \mathbb{R}^n) \times \mathbb{R}^n$ into $\mathrm{BV}^n \times \mathbb{R}^n$. Next theorem has been proved in [4].

Theorem 3.2. *Let $B(a) = A(a)$, $B(b) = A(b)$ and $B(t) = A(t+)$ on (a,b). Then $(y,\gamma,\delta) \in \mathcal{N}(\mathcal{L}^*)$ if and only if*

$$\left.\begin{aligned} & y^*(t) - y^*(b) - \int_t^b y^*(s)\,\mathrm{d}[B(s)] = \delta^*\,(K(t) - K(b)) \ \ \textit{on } [a,b], \\ & y^*(a) + \delta^*\,(K(a) - M) = 0, \quad y^*(b) + \delta^*\,K(b) = 0, \end{aligned}\right\} \tag{BVP*}$$

Moreover, (BVP) *has a solution if and only if*

$$\int_a^b y^*\,\mathrm{d}[f] + \delta^*\,r = 0 \ \ \textit{for all solutions} \ \ (y,\delta) \ \ \textit{of } (\mathrm{BVP}^*).$$

Remark 3.1. Let $t_0 \in [a,b]$, $A \in \mathrm{BV}^{n\times n}$, $\det[I + \Delta^+ A(t)] \neq 0$ for $t \in [a,t_0)$ and $\det[I - \Delta^- A(t)] \neq 0$ for $t \in (t_0,b]$. Then, there is a unique $X : [a,b] \to \mathbb{R}^{n\times n}$ such that

$$X(t) = I + \int_{t_0}^t \mathrm{d}[A]\,X \ \text{ for } \ t \in [a,b].$$

This X is then called the *generalized exponential* and denoted $X(t) = \exp_{dA}(t,t_0)$.

4 Alternative approach based on the Lagrange identity

Besides the functional analytical tool, there is an alternative way to obtain the duality theory. This approach is based on the Lagrange identity. It is well known, cf. [6], that the classical Lagrange identity can be extended to generalized linear differential systems as follows: Let $A \in \mathrm{BV}_{\mathrm{L}}^{n\times n}$, $B(a) = A(a)$, $B(b) = A(b)$ and $B(t) = A(t+)$ on (a,b). Then

$$\int_a^b y^*(t)\,\mathrm{d}\Big[x(t) - \int_a^t \mathrm{d}[A]\,x\Big] + \int_a^b \mathrm{d}\Big[y^*(s) - \int_s^b y^*\,\mathrm{d}[B]\Big]x(s) = y^*(b)\,x(b) - y^*(a)\,x(a)$$

for all $x \in \mathrm{G}_{\mathrm{L}}^n$ and $y \in \mathrm{BV}^n$ right-continuous on $[a,b)$. The proof easily follows from the integration-by-parts theorem for Kurzweil–Stieltjes integrals. Notice that this theorem can be slightly modified as follows, cf. [1].

Theorem 4.1 (Integration by parts)**.** *Let $f, g \in G^n$ and let at least one of them has a bounded variation on $[a,b]$. Then*

$$\int_a^b f^*(t-)\,\mathrm{d}[g(t)] + \int_a^b \mathrm{d}[f^*(t)]\,g(t+) = f^*(b)\,g(b) - f^*(a)\,g(a),$$

where $f(a-) = f(a)$ and $g(b+) = g(b)$.

As a result, we can reformulate the Lagrange formula under less restrictive continuity requirements. To this aim consider operators

$$(L\,x)(t) := x(t) - x(t_0) - \int_{t_0}^t \mathrm{d}[A(s)]\,x(s-) \ \text{ and } \ (L^*\,y)(t) := y^*(t) - y^*(t_0) + \int_{t_0}^t y^*(s+)\,\mathrm{d}[A(s)]$$

under the conventions

$$x(s-) = x(s) \ \text{ if } \ s = \min\{t, t_0\} \ \text{ and } \ y(s+) = y(s) \ \text{ if } \ s = \max\{t, t_0\}$$

in the integrals. More exactly:

$$(L\,x)(t) := \begin{cases} x(t) - x(t_0) + \displaystyle\int_t^{t_0} \mathrm{d}[A(s)]\,\big(x(t)\,\chi_{[t]}(s) + x(s-)\chi_{(t,t_0]}(s)\big) & \text{if } t \le t_0, \\ x(t) - x(t_0) - \displaystyle\int_{t_0}^{t} \mathrm{d}[A(s)]\,\big(x(t_0)\,\chi_{[t_0]}(s) + x(s-)\chi_{(t_0,t]}(s)\big) & \text{if } t \ge t_0, \end{cases}$$

$$(L^*\,y)(t) := \begin{cases} y^*(t) - y^*(t_0) + \displaystyle\int_{t_0}^{t} \big(y^*(s+)\,\chi_{[t,t_0)}(s) + y^*(t_0)\,\chi_{[t_0]}(s)\big)\,\mathrm{d}[A(s)] & \text{if } t \le t_0, \\ y^*(t) - y^*(t_0) - \displaystyle\int_{t}^{t_0} \big(y^*(s+)\,\chi_{[t_0,t)}(s) + y^*(t)\,\chi_{[t](s)}\big)\,\mathrm{d}[A(s)] & \text{if } t \ge t_0. \end{cases}$$

The related equations $L\,x = 0$ and $L^*\,y = 0$ are, of course, no longer generalized ODEs, but special cases of Stieltjes integral equations. The modified version of the Lagrange identity, cf. [1], then reads as follows:

Theorem 4.2 (Lagrange Identity)**.** *Let $A \in BV^{n\times n}$, $x, y \in G^n$, $x(t-) = x(t)$ if $t = \min\{t_0, T\}$ and $y(t+) = y(t)$ if $t = \max\{t_0, T\}$. Then for each $t_0 \in [a,b]$ and $T \in [a,b]$ we have*

$$\int_{t_0}^T y^*(t+)\,\mathrm{d}[(L\,x)(t)] + \int_{t_0}^T \mathrm{d}[(L^*\,y)(t)]\,x(t-) = y^*(T)\,x(T) - y^*(t_0)\,x(t_0).$$

Remark 4.1. The above result no longer holds if we abandon the convention concerning the endpoints.

Corollary. *If* $L\,x = 0$ *and* $L^*\,y = 0$ *on* $[a,b]$, *then* $y^*\,x$ *is constant on* $[a,b]$.

In other words, the equations

$$x(t) = x(t_0) + \int_{t_0}^{t} \mathrm{d}[A(s)]\,x(s-) \quad \text{on } [a,b], \tag{E}$$

$$y^*(t) = y^*(t_0) - \int_{t_0}^{t} y^*(s+)\,\mathrm{d}[A(s)] \ \text{ on } [a,b] \tag{E^*}$$

can be considered to be mutually dual.

Remark 4.2. If we restrict to $t_0 = a$, everything becomes considerably simpler. In particular, in such a case we get

$$\begin{aligned} L\,x = 0 \ \text{ on } [a,b] &\implies x(t-) = \big[I + \Delta^- A(t)\big]^{-1} x(t) \ \text{ if } \ t \in (a,b] \ \text{ and } \ \det[I + \Delta^- A(t)] \neq 0, \\ L^*\,y = 0 \ \text{ on } [a,b] &\implies y^*(t+) {=} y^*(t)\,\big[I + \Delta A(t)\big]^{-1}(I + \Delta^- A(t)) \\ &\qquad\qquad\qquad\qquad \text{if } \ t \in [a,b) \ \text{ and } \ \det\big[I + \Delta A(t)\big] {\neq} 0. \end{aligned}$$

Therefore, if $\det[I + \Delta^- A(t)] \neq 0$ and $\det[I + \Delta A(t)] \neq 0$, then $L\,x = 0$ if and only if

$$x(t) = x(a) + \int_a^t \mathrm{d}[K]\,x \ \text{ on } [a,b], \ \text{ where } \ K(s) = \int_a^s \mathrm{d}[A(\tau)]\,\big[I + \Delta^- A(\tau)\big]^{-1}.$$

Analogously, $L^*\,y = 0$ if and only if

$$y^*(t) = y^*(a) - \int_a^t y^*\,\mathrm{d}[L] \ \text{ on } [a,b], \ \text{ where } \ L(s) = \int_a^s [I + \Delta A(\tau)]^{-1}\big[I + \Delta^- A(\tau)\big]\,\mathrm{d}[A(\tau)].$$

Concluding comments

The present contribution is closely related to the recent paper [1]. Some of its results have been here extended from the scalar case to the n-dimensional case and functional analytical background has been recalled. On the other hand, in [1] Stieltjes differential equations and dynamical equations on time scale were considered. For more details, see [1]. To a large extent, the properties of the Kurzweil-Stieltjes integral are utilized. For more details, see the monograph [2].

References

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