On the Optimization Problem for One Class of Controlled Functional Differential Equation with the Mixed Initial Condition

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In the paper, necessary conditions of optimality of the initial and final moments, delay parameters, the initial vector and initial functions, the control function are obtained for the optimization problem containing the nonlinear functional differential equation with constant delays in the phase coordinates and controls.

Let $\mathbb{R}^n$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$. Suppose that $P\subset\mathbb{R}^k$, $Q\subset\mathbb{R}^m$, $V\subset\mathbb{R}^r$ are convex and open sets, with $k+m=n$, $x=(p,q)^T\in O=(P,Q)^T$. Let $a_{11}<a_{12}<a_{21}<a_{22}$, $\tau_2>\tau_1>0$, $\sigma_2>\sigma_1>0$, $\theta_2>\theta_1>0$ be given numbers, with $a_{21}-a_{12}>\tau_2$; let $I=[a_{11},a_{22}]$, $I_1=[\widehat{\tau},a_{12}]$ and $I_2=[a_{11}-\theta_2,a_{22}]$, where $\widehat{\tau}=a_{11}-\max\{\tau_2,\sigma_2\}$. Furthermore, let the n-dimensional function $f(t,x,p,q,u,v)$ be continuous on $I\times O\times P\times Q\times V^2$, and continuously differentiable with respect to (x,p,q,u,v). Denote by $AC_\varphi(I_1,P)$ the space of absolutely continuous functions $\varphi:I_1\to\mathbb{R}^k$, with $|\dot\varphi(t)|\le const$. Let us introduce the sets:

$$\Phi=AC_\varphi(I_1,K),\ \ G=AC_g(I_1,M),\ \ \Omega=AC_u(I_2,U),$$

where $K\subset P$, $M\subset Q$ and $U\subset V$ are convex and compact sets. To any element

$$w=(t_0,t_1,\tau,\sigma,\theta,p_0,\varphi,g,u)\in W\\ =(a_{11},a_{12})\times(a_{21},a_{22})\times(\tau_1,\tau_2)\times(\sigma_1,\sigma_2)\times(\theta_1,\theta_2)\times P_0\times\Phi\times G\times\Omega,$$

where $P_0\subset P$ is a convex and compact set, we assign the nonlinear controlled functional differential equation with delays in the phase coordinates and controls

$$\dot x(t)=f\big(t,x(t),p(t-\tau),q(t-\sigma),u(t),u(t-\theta)\big),\ \ t\in[t_0,t_1] \tag{1}$$

with the mixed initial condition

$$\begin{cases}x(t)=(p(t),q(t))^T=(\varphi(t),g(t))^T, & t\in[\widehat{\tau},t_0),\\ x(t_0)=(p_0,g(t_0))^T.\end{cases} \tag{2}$$

Condition (2) is said to be the mixed initial condition, because it consists of two parts: the first part is $p(t)=\varphi(t)$, $t\in[\widehat{\tau},t_0)$, $p(t_0)=p_0$, the discontinuous part, since in general $p(t_0)\neq\varphi(t_0)$; discontinuity at the initial moment may be related to the instant change in a dynamic process, for example, changes of investment and environment etc; the second part is $q(t)=g(t)$, $t\in[\widehat{\tau},t_0]$, the continuous part, since always $q(t_0)=g(t_0)$.

Definition 1. Let $w=(t_0,t_1,\tau,\sigma,\theta,p_0,\varphi,g,u)\in W$. A function $x(t)=x(t;w)\in O, t\in[\widehat{\tau},t_1]$, is called a solution of equation (1) with the initial condition (2) or a solution corresponding to the element w, if it satisfies condition (2) and is absolutely continuous on the interval $[t_0,t_1]$ and satisfies equation (1) almost everywhere on $[t_0,t_1]$.

Let the scalar-valued functions $z^i(t_0,t_1,\tau,\sigma,\theta,p,x)$, $i=\overline{0,l}$, be continuously differentiable on $[a_{11},a_{12}]\times[a_{21},a_{22}]\times[\tau_1,\tau_2]\times[\sigma_1,\sigma_2]\times[\theta_1,\theta_2]\times P\times O$.

Definition 2. An element $w=(t_0,t_1,\tau,\sigma,\theta,p_0,\varphi,g,u)\in W$ is said to be admissible if the corresponding solution $x(t)=x(t;w)$ satisfies the boundary conditions

$$z^i(t_0,t_1,\tau,\sigma,\theta,p_0,x(t_1))=0,\ \ i=\overline{1,l}. \tag{3}$$

By W_0 we denote the set of admissible elements.

Definition 3. An element $w_0=(t_{00},t_{10},\tau_0,\sigma_0,\theta_0,p_{00},\varphi_0,g_0,u_0)\in W_0$ is said to be optimal if for an arbitrary element $w\in W_0$ the inequality

$$z^0\big(t_{00},t_{10},\tau_0,\sigma_0,\theta_0,p_{00},x_0(t_{10})\big)\le z^0\big(t_0,t_1,\tau,\sigma,\theta,p_0,x(t_1)\big), \tag{4}$$

where $x_0(t)=x(t;w_0)$, holds.

(1)–(4) is called the optimization problem for the functional differential equation (1) with the mixed initial condition (2).

Theorem 1. *Let w_0 be an optimal element and let $x_0(t)=(p_0(t),q_0(t))^T$, $t\in[\widehat{\tau},t_{10}]$ be the corresponding solution. The function $\dot g_0(t)$ is continuous at the point t_{00}. Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\ne 0$, with $\pi_0\le 0$, and a solution $\psi(t)=(\psi_1(t),\dots,\psi_n(t))$ of the equation*

$$\dot\psi(t)=-\psi(t)f_{0x}[t]-\psi(t+\tau_0)\big(f_{0p}[t+\tau_0],\Theta_{n\times m}\big)-\psi(t+\sigma_0)\big(\Theta_{n\times k},f_{0q}[t+\sigma_0]\big),\ \ t\in(t_{00},t_{10})$$

with the initial condition

$$\psi(t_{10})=\pi Z_{0x},\ \ \psi(t)=0,\ \ t>t_{10},$$

where $\Theta_{n\times m}$ is the $n\times m$ zero matrix and $Z=(z^0,\dots,z^l)^T$,

$$Z_{0x}=\frac{\partial Z(t_{00},t_{10},\tau_0,\sigma_0,\theta_0,p_{00},x_0(t_{10}))}{\partial x}\,,$$
$$f_{0x}[t]=f_x\big(t,x_0(t),p_0(t-\tau_0),q_0(t-\sigma_0),u_0(t),u_0(t-\theta_0)\big),$$

such that the following conditions hold:

1) *the condition for the initial moment t_{00}*

$$\pi Z_{0t_0}+\big(\pi Z_{0q}+\big(\psi_{k+1}(t_{00}),\dots,\psi_n(t_{00})\big)\big)\dot q_0(t_{00})=\psi(t_{00})f_0[t_{00}]+\psi(t_{00}+\tau_0)f_1,$$

where

$$f_0[t]=f\big(t,x_0(t),p_0(t-\tau_0),q_0(t-\sigma_0),u_0(t),u_0(t-\theta_0)\big),$$
$$f_1=f\Big(t_{00}+\tau_0,x_0(t_{00}+\tau_0),p_{00},q_0(t_{00}+\tau_0-\sigma_0),u_0(t_{00}+\tau_0),u_0(t_{00}+\tau_0-\theta_0)\Big)$$
$$-f\Big(t_{00}+\tau_0,x_0(t_{00}+\tau_0),\varphi_0(t_{00}),q_0(t_{00}+\tau_0-\sigma_0),u_0(t_{00}+\tau_0),u_0(t_{00}+\tau_0-\theta_0)\Big);$$

2) *the condition for the final moment t_{10}*

$$\pi Z_{0t_1}=-\psi(t_{10})f_0[t_{10}];$$

3) *the condition for the delay* τ_0

$$\pi Z_{0\tau} = \psi(t_{00}+\tau_0)f_1 + \int\limits_{t_{00}}^{t_{10}} \psi(t) f_{0p}[t]\dot{p}_0(t-\tau_0)\,dt;$$

4) *the condition for the delay* σ_0

$$\pi Z_{0\sigma} = \int\limits_{t_{00}}^{t_{10}} \psi(t) f_{0q}[t]\dot{q}_0(t-\sigma_0)\,dt;$$

5) *the condition for the delay* θ_0

$$\pi Z_{0\theta} = \int\limits_{t_{00}}^{t_{10}} \psi(t) f_{0v}[t]\dot{u}_0(t-\theta_0)\,dt;$$

6) *the condition for the vector* p_{00}

$$\big(\pi Z_{0p} + (\psi_1(t_{00}),\dots,\psi_k(t_{00}))\big)p_{00} = \max_{p_0\in P_0}\big(\pi Z_{0p} + (\psi_1(t_{00}),\dots,\psi_k(t_{00}))\big)p_0;$$

7) *the condition for the initial function* $\varphi_0(t)$

$$\int\limits_{t_{00}-\tau_0}^{t_{00}} \psi(t+\tau_0) f_{0p}[t+\tau_0]\varphi_0(t)\,dt = \max_{\varphi\in\Phi}\int\limits_{t_{00}-\tau_0}^{t_{00}} \psi(t+\tau_0) f_{0p}[t+\tau_0]\varphi(t)\,dt;$$

8) *the condition for the initial function* $g_0(t)$

$$\begin{aligned}\big(\psi_{k+1}(t_{00}),\dots,\psi_n(t_{00})\big)g_0(t_{00}) &+ \int\limits_{t_{00}-\sigma_0}^{t_{00}} \psi[t+\sigma_0] f_{0q}[t+\sigma_0]g_0(t)\,dt \\ &= \max_{g\in G}\Bigg[(\psi_{k+1}(t_{00}),\dots,\psi_n(t_{00}))g(t_0) + \int\limits_{t_{00}-\sigma_0}^{t_{00}} \psi(t+\sigma_0) f_{0q}[t+\sigma_0]g(t)\,dt\Bigg];\end{aligned}$$

9) *the condition for the control function* $u_0(t)$

$$\int\limits_{t_{00}}^{t_{10}} \psi(t)\big[f_{0u}[t]u_0(t) + f_{0v}[t]u_0(t-\theta_0)\big]\,dt = \max_{u\in\Omega}\int\limits_{t_{00}}^{t_{10}} \psi(t)\big[f_{0u}[t]u(t) + f_{0v}[t]u(t-\theta_0)\big]\,dt.$$

Theorem 1 is proved by the scheme given in [2]. A problem with the mixed initial without optimization of delay parameters was considered in [1]. Now we consider a particular case of problem (1)–(4):

$$\begin{aligned}\dot{x}(t) &= (\dot{p}(t),\dot{q}(t))^T \\ &= A(t)x(t) + B(t)p(t-\tau) + C(t)q(t-\sigma) + D(t)u(t) + E(t)u(t-\theta),\quad t\in[t_0,t_1],\end{aligned} \tag{5}$$

$$\begin{cases} x(t) = (p(t),q(t))^T = (\varphi(t),g(t))^T, & t\in[\widehat{\tau},t_0),\\ x(t_0) = (p_0,g(t_0))^T. \end{cases} \tag{6}$$

$$z^i(\tau,\sigma,\theta,x(t_1)) = 0,\quad i=\overline{1,l} \tag{7}$$

$$z^0(\tau,\sigma,\theta,x(t_1)) \to \min. \tag{8}$$

Here $A(t)$, $B(t)$, $C(t)$, $D(t)$ and $E(t)$ are the continuous matrix functions with dimensions $n\times n$, $n\times k$, $n\times m$, $n\times r$ and $n\times r$, respectively; t_0, t_1 are fixed moments; $\varphi(t)$, $g(t)$ are fixed initial functions; p_0 is a fixed initial function. In this case we have

$$w=(\tau,\sigma,\theta,u)\in W=(\tau_1,\tau_2)\times(\sigma_1,\sigma_2)\times(\theta_1,\theta_2)\times\Omega \ \text{ and } \ w_0=(\tau_0,\sigma_0,\theta_0,u_0);$$
$$Z(\tau,\sigma,\theta,x)=(z^0(\tau,\sigma,\theta,x),\dots,z^l(\tau,\sigma,\theta,x))^T, \quad Z_{0x}=\frac{\partial Z(\tau_0,\sigma_0,\theta_0,x_0(t_1))}{\partial x}\,.$$

Theorem 2. *Let w_0 be an optimal element for problem* (5)–(8)*. Then there exist a vector $\pi=(\pi_0,\dots,\pi_l)\neq 0$, with $\pi_0\leq 0$, and a solution $\psi(t)=(\psi_1(t),\dots,\psi_n(t))$ of the equation*

$$\dot\psi(t)=-\psi(t)A(t)-\psi(t+\tau_0)(B(t+\tau_0),\Theta_{n\times m})-\psi(t+\sigma_0)(\Theta_{n\times k},C(t+\sigma_0)), \ \ t\in(t_0,t_1)$$

with the initial condition

$$\psi(t_1)=\pi Z_{0x}, \ \ \psi(t)=0, \ \ t>t_1,$$

such that the following conditions hold:

10) *the condition for the delay τ_0*

$$\pi Z_{0\tau}=\psi(t_0+\tau_0)[p_0-\varphi(t_0)]+\int\limits_{t_0}^{t_{10}}\psi(t)B[t]\dot p_0(t-\tau_0)\,dt;$$

11) *the condition for the delay σ_0*

$$\pi Z_{0\sigma}=\int\limits_{t_0}^{t_1}\psi(t)C(t)\dot q_0(t-\sigma_0)\,dt;$$

12) *the condition for the delay θ_0*

$$\pi Z_{0\theta}=\int\limits_{t_0}^{t_1}\psi(t)E(t)\dot u_0(t-\theta_0)\,dt;$$

13) *the condition for the control function $u_0(t)$*

$$\int\limits_{t_0}^{t_1}\psi(t)\big[D(t)u_0(t)+E(t)[t]u_0(t-\theta_0)\big]\,dt=\max_{u\in\Omega}\int\limits_{t_0}^{t_1}\psi(t)\big[D(t)u(t)+E(t)u(t-\theta_0)\big]\,dt.$$

References

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