On the Functional Integral Equation with the Two Types Controls

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In the paper, the nonlinear controlled functional integral equation corresponding to the quasi-linear neutral functional differential equation with two types controls is constructed. A structure and properties of the integral kernel are established. Equivalence of the functional integral equation and the neutral functional differential equation is established also. We note that theorems formulated below play a principal role in the study of well-posedness of Cauchy's problem for the quasi-linear neutral functional differential equations. In details, about of this investigations for the quasi-linear neutral functional differential equations without control are given in [1-3].

Let $\mathbb{R}^n_x$ be the n-dimensional vector space of points $x=(x^1,\dots,x^n)^T$, where T is the sign of transposition; let $I=[t_0,t_1]$ be a fixed interval and let $\tau>0$ be a given number, with $t_0+\tau<t_1$; the $n\times n$-dimensional matrix-function $A(t,x,y,v)$ and the n-dimensional vector-function $f(t,x,y,u)$ are continuous and bounded on the set $I\times\mathbb{R}^n_x\times\mathbb{R}^n_x\times\mathbb{R}^m_v$ and $I\times\mathbb{R}^n_x\times\mathbb{R}^n_x\times\mathbb{R}^r_u$, respectively, and satisfy Lipschptz's condition with respect to (x,y,v) and (x,y,u), i.e. there exist $L_A>0$ and $L_f>0$ such that

$$\big|A(t,x_1,y_1,v_1)-A(t,x_2,y_2,v_2)\big|\le L_A\big(|x_1-x_2|+|y_1-y_2|+|v_1-v_2|\big)$$
$$\forall\, t\in I,\ \ (x_i,y_i,v_i)\in\mathbb{R}^n_x\times\mathbb{R}^n_x\times\mathbb{R}^m_v,\ \ i=1,2,$$

and

$$\big|f(t,x_1,y_1,u_1)-f(t,x_2,y_2,u_2)\big|\le L_f\big(|x_1-x_2|+|y_1-y_2|+|u_1-u_2|\big)$$
$$\forall\, t\in I,\ \ (x_i,y_i,u_i)\in\mathbb{R}^n_x\times\mathbb{R}^n_x\times\mathbb{R}^r_u,\ \ i=1,2.$$

Further, denote by V and Ω the sets of piecewise-continuous control functions $v(t)\in\mathbb{R}^m_v$ with finitely many discontinuous of the first kind and bounded measurable control functions $u(t)\in\mathbb{R}^r_u$, respectively, equipped with the norm

$$\|v\|=\sup\big\{|v(t)|:\ t\in I\big\}\quad\Big(\|u\|=\sup\big\{|u(t)|:\ t\in I\big\}\Big);$$

$\varphi(t)\in\mathbb{R}^n_x$, $t\in[t_0-\tau,t_0]$ is a given continuously differentiable initial function; $x_0\in\mathbb{R}^n_x$ is a given initial vector.

Let us consider the quasi-linear controlled neutral functional differential equation

$$\dot x(t)=A\big(t,x(t),x(t-\tau),v(t)\big)\dot x(t-\tau)+f\big(t,x(t),x(t-\tau),u(t)\big),\ \ t\in I \tag{1}$$

with the initial condition

$$x(t)=\varphi(t),\ \ t\in[\widehat\tau,t_0),\ \ x(t_0)=x_0, \tag{2}$$

where $\widehat\tau=t_0-\tau$.

Definition 1. Let $w = (v(t), u(t)) \in W = V \times \Omega$. A function $x(t) = x(t; w)$, $t \in I_1 = [\widehat{\tau}, t_1]$, is called a solution of equation (1) with the initial condition (2), if it satisfies condition (2) and is absolutely continuous on the interval I and satisfies equation (1) almost everywhere on I.

Theorem 1. *For any $w \in W$ there exists the unique solution $x(t) = x(t; w)$, $t \in I_1$.*

Theorem 2. *The solution $x(t)$, $t \in I_1$ of problem (1), (2) can be represented on the interval I in the following form*

$$\begin{aligned} x(t) = x_0 + \int\limits_{t_0-\tau}^{t_0} & Y\big(\xi+\tau; t, x(\,\cdot\,), v(\,\cdot\,)\big) A\big(\xi+\tau, x(\xi+\tau), x(\xi), v(\xi+\tau)\big)\dot{\varphi}(\xi)\, d\xi \\ & + \int\limits_{t_0}^{t} Y\big(\xi; t, x(\,\cdot\,), v(\,\cdot\,)\big) f\big(\xi, x(\xi), x(\xi-\tau), u(\xi)\big)\, d\xi, \end{aligned}$$

where

$$x(\xi) = \varphi(\xi), \ \ \xi \in [\widehat{\tau}, t_0)$$

and $Y(\xi, t, x(\,\cdot\,), v(\,\cdot\,))$ is the matrix-function satisfying the difference equation

$$Y(\xi; t, x(\,\cdot\,), v(\,\cdot\,)) = E + Y\big(\xi+\tau; t, x(\,\cdot\,), v(\,\cdot\,)\big) \cdot A\big(\xi+\tau, x(\xi+\tau), x(\xi), v(\xi+\tau)\big) \tag{3}$$

on (t_0, t) for any fixed $t \in (t_0, t_1]$ and the condition

$$Y(\xi; t, x(\,\cdot\,), v(\,\cdot\,)) = \begin{cases} E, & \xi = t, \\ \Theta, & \xi > t. \end{cases}$$

Here, E is the identity matrix and Θ is the zero matrix.

The expression

$$\begin{aligned} y(t) = x_0 + \int\limits_{t_0}^{t_0+\tau} & Y\big(\xi; t, y(\,\cdot\,), v(\,\cdot\,)\big) A\big(\xi, y(\xi), y(\xi-\tau), v(\xi)\big)\dot{\varphi}(\xi-\tau)\, d\xi \\ & + \int\limits_{t_0}^{t} Y\big(\xi; t, y(\,\cdot\,), v(\,\cdot\,)\big) f\big(\xi, y(\xi), y(\xi-\tau), u(\xi)\big)\, d\xi \end{aligned} \tag{4}$$

with the condition

$$y(\xi) = \varphi(\xi), \ \ \xi \in [\widehat{\tau}, t_0) \tag{5}$$

is called the functional integral equation corresponding to problem (1), (2).

Definition 2. Let $w \in W$. A function $y(t) = y(t; w)$, $t \in I_1$, is called a solution of equation (4) with condition (5), if it satisfies condition (5) and is continuous on the interval I and satisfies equation (4) everywhere on I.

Theorem 3. *Let $t \in (t_0, t_1]$ be a fixed point. The solution of the difference equation (3) can be represented by the following formula*

$$Y(\xi; t, x(\,\cdot\,), v(\,\cdot\,)) = \chi(\xi; t)E + \sum_{i=1}^{k} \chi(\xi+i\tau; t) \prod_{q=i}^{1} A\big(\xi+q\tau, x(\xi+q\tau), x(\xi+(q-1)\tau), v(\xi+q\tau)\big),$$

where

$$\chi(\xi;t)=\begin{cases}1, & t_0\le\xi\le t,\\ 0, & \xi>t\end{cases}$$

and k is a minimal natural number satisfying the condition

$$t_1-k\tau<t_0.$$

Theorem 4. *Let $s_1,s_2\in(t_0,t_1]$ and $0<s_2-s_1<\tau$. Let $y(t)$, $t\in I$ be a continuous function. Then there exist subintervals $I_1(s_1,s_2)\subset I$ and $I_2(s_1,s_2)\subset I$ such that*

$$\begin{cases}Y\big(\xi;s_1,y(\,\cdot\,),v(\,\cdot\,)\big)=Y\big(\xi;s_2,y(\,\cdot\,),v(\,\cdot\,)\big), & \xi\in I_1(s_1;s_2),\\ Y\big(\xi;s_1,y(\,\cdot\,),v(\,\cdot\,)\big)\neq Y\big(\xi;s_2,y(\,\cdot\,),v(\,\cdot\,)\big), & \xi\in I_2(s_1;s_2),\end{cases}$$

with

$$\lim_{s_2-s_1\to0}\operatorname{mes}I_2(s_1,s_2)\to0.$$

Theorem 5. *Let $y(t)\in R^n$, $t\in[\widehat{\tau},t_1]$ be a given piecewise-continuous function, with $y(\xi)=\varphi(\xi)$, $\xi\in[\widehat{\tau},t_0)$; $v(t)\in V$ and $u(t)\in\Omega$. Then the function*

$$z(t)=x_0+\int\limits_{t_0}^{t_0+\tau}Y\big(\xi;t,y(\,\cdot\,),v(\,\cdot\,)\big)A\big(\xi,y(\xi),y(\xi-\tau),v(\xi)\big)\dot{\varphi}(\xi-\tau)\,d\xi$$
$$+\int\limits_{t_0}^{t}Y\big(\xi;t,y(\,\cdot\,),v(\,\cdot\,)\big)f\big(\xi,y(\xi),y(\xi-\tau),u(\xi)\big)\,d\xi$$

is continuous on the interval I.

Theorem 6. *Let $y_i(t)\in\mathbb{R}^n_x$, $t\in I$, $i=1,2$ be continuous functions and $v_i(t)\in V$, $i=1,2$. Then for a fixed $(\xi,t)\in I^2$,*

$$\Big|Y\big(\xi;t,y_1(\,\cdot\,),v_1(\,\cdot\,)\big)-Y\big(\xi;t,y_2(\,\cdot\,),v_2(\,\cdot\,)\big)\Big|$$
$$\le L_A\sum_{i=1}^{k}\chi(\xi+i\tau;t)\|A\|^{i-1}\bigg(\sum_{q=i}^{1}\Big[\big|y_1(\xi+q\tau)-y_2(\xi+q\tau)\big|$$
$$+\big|y_1(\xi+(q-1)\tau)-y_2(\xi+(q-1)\tau)\big|+\big|v_1(\xi+q\tau)-v_2(\xi+q\tau)\big|\Big]\bigg),$$

where

$$\|A\|=\sup\Big\{|A(t,x,y,v)|:\ (t,x,y,v)\in I\times R^n_x\times R^n_x\times R^m_v\Big\}.$$

Theorem 7. *Let $y_i(t)\in\mathbb{R}^n_x$, $t\in I$, $i=0,1,\dots$ be continuous functions and $v_i(t)\in V$, $i=0,1,\dots$, with*

$$\|y_i-y_0\|\to0,\quad \|v_i-v_0\|\to0,$$

Then

$$\int\limits_{t_0}^{t}Y\big(\xi;t,y_i(\,\cdot\,),v_i(\,\cdot\,)\big)\,d\xi\longrightarrow\int\limits_{t_0}^{t}Y\big(\xi;t,y_0(\,\cdot\,),v_0(\,\cdot\,)\big)\,d\xi$$

uniformly for $t\in I$.

Theorem 8. *The functional integral equation* (4) *with condition* (5) *has the unique solution.*

Theorem 9. *The quasi-linear neutral functional differential equation* (1) *and the functional integral equations* (4) *are equivalent.*

Remark. The analogous theorems for the case, where $A(t,x,y,v) \equiv A(t)$ and functional integral equation (3) depends on the one control function, are proved in [1, 3, 4] and [2], respectively.

Conclusion

On the basis of the given theorems, it can be investigated continuous dependence of a solution of the quasi-linear controlled neutral functional differential equation (1) with respect to perturbations of the initial data. In future work the case, where a controlled functional integral equation contains several variable delays, will be considered.

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