Asymptotic Representations of Some Classes of Solutions of Third Order Nonautonomous Ordinary Differential Equations

Nataliia Sharay

Odessa I. I. Mechnikov National University, Odessa, Ukraine
E-mail: rusnat36@gmail.com

Volodymyr Shinkarenko

Odessa National University of Economics, Odessa, Ukraine
E-mail: shinkar@te.net.ua

We consider the differential equation

$$y''' = \alpha_0 p(t) y |\ln |y||^\sigma, \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $\sigma \in \mathbb{R}$, $p : [a, \omega[\to]0 < +\infty[$ is a continuous function, $-\infty < a < \omega \leq +\infty$[1].

The solution y of equation (1), given and different from zero on the interval $[t_y, \omega[\subset [a, \omega[$, is called $P_\omega(\lambda_0)$-solution if it satisfies the following conditions:

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{if } \ 0, \\ \text{if } \ \pm\infty \end{cases} \qquad (k = 0, 1, 2), \quad \lim_{t\uparrow\omega} \frac{(y''(t))^2}{y'''(t) y'(t)} = \lambda_0.$$

In [6], for equation (1) the conditions for the existence of a $P_\omega(\lambda_0)$-solution were established in the non-singular case, when $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, 1\}$, asymptotic representations were also obtained for such solutions and their derivatives up to the second order inclusive. At the same time, the question of the number of solutions with the found asymptotic representations was also clarified.

In [5] Shinkarenko V., Sharay N. for the differential equation (1) investigated questions about the existence and asymptotics of the so-called $P_\omega(Y_0, \lambda_0)$– solutions at $\lambda_0 = +\infty$.

The purpose of this work is to establish necessary and sufficient conditions for the existence of y for the differential equation (1) $P_\omega(\frac{1}{2})$-solutions, as well as asymptotic representations at $t \uparrow \omega$ for all such solutions and their derivatives up to the second order inclusive.

In special cases, using the results from the work of Evtukhov V. [2, Ch. 3, § 10, pp. 142–144] it follows a corollary on the asymptotic properties of $P_\omega(\frac{1}{2})$-solutions of equation (1). To describe them, we need the following auxiliary notation

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \ \omega = +\infty, \\ t - \omega, & \text{if } \ \omega < +\infty. \end{cases}$$

Lemma. *For each $P_\omega(\frac{1}{2})$-solutions of the differential equation (1) when $t \uparrow \omega$ we have the asymptotic relations*

$$y'(t) = o\Big(\frac{y(t)}{\pi_\omega(t)}\Big), \quad y''(t) \sim -\frac{1}{\pi_\omega(t)}\, y'(t), \quad y'''(t) \sim \frac{2!}{[\pi_\omega(t)]^2}\, y'(t).$$

[1] We assume that $a > 1$ at $\omega = +\infty$ and $\omega - a < 1$ at $\omega < +\infty$.

Using this result and the work of Evtukhov V. and Samoylenko A. [3], the following result is established.

To formulate the main result, we need the auxiliary functions

$$J_A(t) = \int_A^t \pi_\omega(\tau)p(\tau)\,d\tau, \quad I_B(t) = \int_B^t J_A(\tau)\,d\tau,$$

where

$$A = \begin{cases} a, & \text{if } \displaystyle\int_a^\omega |\pi_\omega(\tau)|p(\tau)\,d\tau = +\infty, \\ \omega, & \text{if } \displaystyle\int_a^\omega |\pi_\omega(\tau)|p(\tau)\,d\tau < +\infty, \end{cases} \qquad B = \begin{cases} a, & \displaystyle\int_a^\omega |J_A(\tau)|\,d\tau = +\infty, \\ \omega, & \displaystyle\int_a^\omega |J_A(\tau)|\,d\tau < +\infty. \end{cases}$$

Theorem 1. *Let $\sigma \neq 1$. Then, for the existence of $P_\omega(\frac{1}{2})$-solutions of the differential equation* (1) *it is necessary and sufficient that the conditions*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)J_A'(t)}{J_A(t)} = -1, \quad \lim_{t\uparrow\omega} |I_B(t)|^{\frac{1}{1-\sigma}} = +\infty, \quad \lim_{t\uparrow\omega} \pi_\omega(t)J_A(t)|I_B(t)|^{\frac{\sigma}{1-\sigma}} = 0, \tag{2}$$

hold. Moreover, for each such solution, when $t \uparrow \omega$ the asymptotic representations

$$\ln|y(t)| = \nu_0\Big|\frac{1-\sigma}{2}I_B(t)\Big|^{\frac{1}{1-\sigma}}[1+o(1)], \tag{3}$$

$$\frac{y'(t)}{y(t)} = -\frac{\alpha_0 J_A(t)}{2}\Big|\frac{1-\sigma}{2}I_B(t)\Big|^{\frac{\sigma}{1-\sigma}}[1+o(1)], \tag{4}$$

$$\frac{y''(t)}{y(t)} = \frac{\alpha_0}{2}\,\frac{J_A(t)}{\pi_\omega(t)}\Big|\frac{1-\sigma}{2}I_B(t)\Big|^{\frac{\sigma}{1-\sigma}}[1+o(1)] \tag{5}$$

hold, where

$$\nu_0 = -\alpha_0 \operatorname{sign}[(1-\sigma)I_B(t)].$$

Furthermore, if conditions (2) *are met, the differential equation* (1) *in the case when $\omega = +\infty$ has a one-parameter family of solutions with representations* (3)–(5)*, if $\sigma < 1$, and in the case when $\omega < \infty$ solutions there is a two-parameter family if $\sigma > 1$ and three-parameter family if $\sigma < 1$.*

Remark 1. It is also shown that under conditions (2) it can be proved that for $\omega = +\infty$ and $\sigma > 1$ the differential equation (1) has a unique solution that admits for $t \uparrow \omega$ the asymptotic representations (3)–(5).

Remark 2. When checking the fulfillment of (2), we can take into account the fact that by virtue of the first of them the second and third ones are equivalent to the conditions

$$\lim_{t\uparrow\omega}\Big|\int_B^t \pi_\omega^2(\tau)p(\tau)\,d\tau\Big|^{\frac{1}{1-\sigma}} = +\infty, \quad \lim_{t\uparrow\omega}\pi_\omega^3(t)p(t)\Big|\int_B^t \pi_\omega^2(\tau)p(\tau)\,d\tau\Big|^{\frac{\sigma}{1-\sigma}} = 0.$$

In conclusion, we pay attention to the fact that Theorem 1 covers the case $\sigma = 0$, i.e. when equation (1) is a linear differential equation of the form

$$y''' = \alpha_0 p(t)y. \tag{6}$$

For this equation, by virtue of Theorem 1, the following assertion holds.

Corollary. *For the existence of $P_\omega(\frac{1}{2})$-solutions of* (6)*, it is necessary and sufficient that the conditions*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_A'(t)}{J_A(t)}=-1,\quad \lim_{t\uparrow\omega}|I_B(t)|=+\infty,\quad \lim_{t\uparrow\omega}\pi_\omega(t)J_A(t)=0 \tag{7}$$

hold. Moreover, for each such solution, $t\uparrow\omega$ the asymptotic representations

$$\ln|y(t)|=\nu_0\Big|\frac{1-\sigma}{2}\,I_B(t)\Big|[1+o(1)], \tag{8}$$

$$\frac{y'(t)}{y(t)}=-\frac{\alpha_0 J_A(t)}{2}\,[1+o(1)], \tag{9}$$

$$\frac{y''(t)}{y(t)}=\frac{\alpha_0}{2}\,\frac{J_A(t)}{\pi_\omega(t)}\,[1+o(1)] \tag{10}$$

hold, where

$$\nu_0=-\alpha_0\,\mathrm{sign}[(1-\sigma)I_B(t)].$$

Furthermore, when conditions (7) *are met, the differential equation* (6) *has a one-parameter family of solutions, and in the case $\omega=+\infty$, a two-parameter family of solutions with representations* (8)*–*(10)*.*

The obtained asymptotics are consistent with the already known results for linear differential equations (see [4]).

References

[1] M. J. Abu Elshour and V. Evtukhov, Asymptotic representations for solutions of a class of second order nonlinear differential equations. *Miskolc Math. Notes* **10** (2009), no. 2, 119–127.

[2] V. M. Evtukhov, Asymptotic representations of solutions of non-autonomous ordinary differential equations. *Diss. D-ra Fiz.-Mat.Nauk, Kiev, Ukraine*, 1998.

[3] V. M. Evtukhov and A. M. Samoilenko, Conditions for the existence of solutions of real nonautonomous systems of quasilinear differential equations vanishing at a singular point. *Ukrainian Math. J.* **62** (2010), no. 1, 56–86.

[4] I. T. Kiguradze and T. A. Chanturiya, *Asymptotic Properties of Solutions of Non-Autonomous Ordinary Differential Equations.* (Russian) Nauka, Moscow, 1990.

[5] N. V. Sharai and V. N. Shinkarenko, Asymptotic representations for the solutions of third-order nonlinear differential equations. *J. Math. Sci. (N.Y.)* **215** (2016), no. 3, 408–420.

[6] N. V. Sharay and V. N. Shinkarenko, Asymptotic behavior of solutions for one class of third order nonlinear differential equations. *Abstracts of the International Workshop on the Qualitative Theory of Differential Equations – QUALITDE-2018*, Tbilisi, Georgia, December 1-3, pp. 165–169;
`http://www.rmi.ge/eng/QUALITDE-2018/Sharay_Shinkarenko_workshop_2018.pdf`.