Differential-Algebraic Boundary-Value Problems with Pulse Perturbations with Constant Rank of a Leading Coefficient Matrix

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We construct conditions for the existence of a solution of linear boundary-value problem for a system of differential-algebraic equations with pulse perturbations with constant rank of a leading coefficient matrix.

The problem of constructing solutions [2, 12]

$$z(t) \in \mathbb{C}^1\big\{[a,b] \setminus \{\tau_i\}_I\big\}, \ \ i = 1, 2, \dots, q$$

of the linear differential-algebraic system

$$A(t)z'(t) = B(t)z(t) + f(t), \ \ t \neq \tau_i, \tag{0.1}$$

subject to the boundary condition [5]

$$\ell z(\,\cdot\,) = \alpha, \ \ \alpha \in \mathbb{R}^k. \tag{0.2}$$

was studied. Here,

$$A(t), B(t) \in \mathbb{C}_{m\times n}[a,b]$$

are continuous matrices,

$$f(t) \in \mathbb{C}[a,b]$$

is a continuous vector function; $\ell z(\,\cdot\,)$ is a linear bounded vector functional

$$\ell z(\,\cdot\,) := \sum_{i=0}^{q} \ell_i z(\,\cdot\,) : \mathbb{C}^1\big\{[a,b] \setminus \{\tau_i\}_I\big\} \to \mathbb{R}^k,$$

in addition

$$\ell_i z(\,\cdot\,) : \mathbb{C}^1[\tau_i, \tau_{i+1}[\to \mathbb{R}^k, \ \ i = 0, \dots, p-1, \ \ \tau_0 := a,$$

and

$$\ell_q z(\,\cdot\,) : \mathbb{C}^1[\tau_p, b] \to \mathbb{R}^k$$

are linear bounded functionals. The differential-algebraic boundary-value problem (0.1), (0.2) generalizes the traditional formulation of Noetherian boundary-value problems for systems of differential equations with pulse perturbations [2, 5, 6, 11, 12]. The differential-algebraic boundary-value problem (0.1), (0.2) also generalizes the statements of various boundary-value problems for systems of differential-algebraic equations [3, 4].

1 Solvability conditions of a differential-algebraic system with impulse perturbations

Suppose that for the differential-algebraic system (0.1) with a matrix $A(t)$ of constant rank, the requirements of the theorem see, [7, p. 15] are fulfilled. We fix an arbitrary continuous vector function $\nu_p(t) \in \mathbb{C}_{\rho_p}[a,b]$. Substituting the general solution

$$z(t,c) := \begin{cases} X_p(t)\, c_0 + K\big[f(s), \nu_p(s)\big](t), & t \in [a; \tau_1[\,, \\ X_p(t)\, c_1 + K\big[f(s), \nu_p(s)\big](t), & t \in [\tau_1; \tau_2[\,, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ X_p(t)\, c_q + K\big[f(s), \nu_p(s)\big](t), & t \in [\tau_p; b] \end{cases}$$

of the Cauchy problem $z(a) = c$ for the differential-algebraic equation (0.1) into the boundary condition (0.2), we arrive at the linear algebraic equation

$$Q\, c = \alpha - \ell K f(\,\cdot\,). \tag{1.1}$$

Here, P_{Q^*} is orthoprojector

$$\mathbb{R}^k \to \mathbb{N}(Q^*)$$

and matrix $P_{Q_d^*}$ is formed from d independent lines of the orthoprojector P_{Q^*}, in addition,

$$Q := \big(\ell_0 X_p(\,\cdot\,) \ell_1 X_p(\,\cdot\,) \cdots \ell_q X_p(\,\cdot\,)\big) \in \mathbb{R}^{k \times \rho_p(q+1)}.$$

Equation (1.1) is solvable if and only if [1, 2]

$$P_{Q_d^*}\big\{\alpha - \ell K\big[f(s), \nu_p(s)\big](\,\cdot\,)\big\} = 0. \tag{1.2}$$

Under condition (1.2) and only under it, the general solution of equation (0.1)

$$c = Q^+\big\{\alpha - \ell K\big[f(s), \nu_p(s)\big](\,\cdot\,)\big\} + P_{Q_r}\, c_r, \quad c_r \in \mathbb{R}^r$$

determines the general solution of the boundary-value problem (0.1), (0.2)

$$z(t, c_r) = X_r(t) c_r + X(t) Q^+\big\{\alpha - \ell K\big[f(s), \nu_p(s)\big](\,\cdot\,)\big\} + K\big[f(s), \nu_p(s)\big](t), \quad c_r \in \mathbb{R}^r.$$

Here, P_Q is an orthoprojector matrix

$$\mathbb{R}^{\rho_p(q+1)} \to \mathbb{N}(Q);$$

the matrix $P_{Q_r} \in \mathbb{R}^{\rho_p(q+1) \times r}$ is composed of r linearly independent columns of the orthoprojector

$$P_Q := \begin{pmatrix} P_Q^{(0)} \\ P_Q^{(1)} \\ \dots\dots \\ P_Q^{(q)} \end{pmatrix} \in \mathbb{R}^{\rho_p(q+1) \times \rho_p(q+1)},$$

in addition, $c_0, c_1, \dots, c_q \in \mathbb{R}^{\rho_p}$ are constants

$$c := \operatorname{col}(c_0, \dots, c_q) := Q^+\big\{\alpha - \ell K\big[f(s), \nu_p(s)\big](\,\cdot\,)\big\} \in \mathbb{R}^{\rho_p(q+1)}.$$

Thus, the following lemma is proved.

Lemma. *Suppose that the differential-algebraic equation* (0.1) *satisfies the requirements of the theorem in the article* [7, p. 15]. *Under condition* (1.2) *and only under it, for a fixed continuous vector function*

$$\nu_p(t) \in \mathbb{C}_{\rho_p}[a,b],$$

general solution of the differential-algebraic boundary-value problem (0.1), (0.2)

$$z(t,c_r) = X_r(t)\,c_r + G\big[f(s);\nu_p(s);\alpha\big](t), \quad c_r \in \mathbb{R}^r$$

defines the generalized Green's operator of the differential-algebraic boundary-value problem (0.1), (0.2)

$$G\big[f(s);\nu_p(s);\alpha\big](t) := \begin{cases} X_p(t)c_0 + K\big[f(s),\nu_p(s)\big](t), & t \in [a,\tau_1[\,, \\ X_p(t)c_1 + K\big[f(s),\nu_p(s)\big](t), & t \in [\tau_1,\tau_2[\,, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ X_p(t)c_q + K\big[f(s),\nu_p(s)\big](t), & t \in [\tau_p,b]. \end{cases}$$

Here,

$$X_r(t) = \begin{cases} X_p(t)P_Q^{(0)}, & t \in [a,\tau_1[\,, \\ X_p(t)P_Q^{(1)}, & t \in [\tau_1,\tau_2[\,, \\ \dots\dots\dots\dots\dots\dots & \\ X_p(t)P_Q^{(q)}, & t \in [\tau_p,b]. \end{cases}$$

Note that the matrix differential-algebraic boundary-value problem with pulse perturbations, studied in the article [10], is reduced to the form (0.1), (0.2), while in the articles [9–11] the case of a non-degenerate system of the form (0.1) was studied. We also note the essentiality of the requirement of constancy of the rank of the matrix under the derivative [7, 8].

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