On Limit Theorems for Solutions of Boundary-Value Problems

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The question of finding the conditions for the convergence of solutions of systems of ordinary differential equations arises in many problems of modern analysis and its applications. It were deeply investigated in the case of the solutions of Cauchy's problems for the system of first-order differential equations. More complicated case of linear boundary-value problems was studied by I. T. Kiguradze [2, 3] and his followers [1, 4–8].

On a finite interval $(a,b)\subset\mathbb{R}$, we consider the systems of $m\in\mathbb{N}$ linear differential equations of the first order

$$y'(t,n)+A(t,n)y(t,n)=f(t,n) \tag{1}$$

with inhomogeneous boundary conditions

$$B(n)y(\,\cdot\,,n)=c(n), \tag{2}$$

where

$$B(n):C([a,b];\mathbb{C}^m)\to\mathbb{C}^m,\ \ n\in\mathbb{N}\cup\{0\}$$

is a linear continuous operator.

We suppose that the matrix-valued functions $A(\,\cdot\,,n)\in L_1([a,b];\mathbb{C}^{m\times m})$, the vector-valued functions $f(\,\cdot\,,n)\in L_1([a,b];\mathbb{C}^m)$, and the vectors $c(n)\in\mathbb{C}^m$.

The solution of the system of differential equations (1) is understood as a vector-valued function $y(\,\cdot\,)\in W_1^1([a,b];\mathbb{C}^m)$ absolutely continuous on the compact interval $[a,b]$ satisfying the vector equation (1) almost everywhere. The inhomogeneous boundary condition (2) is correctly defined on the solutions of system (1) and cover all classical types of boundary condition. It was shown (see, e.g., [7]) that the boundary-value problem (1), (2) is a Fredholm problem with zero index. For the unique solvability of this problem everywhere, it is necessary and sufficient to guarantee that the corresponding homogeneous boundary-value problem has only a trivial solution.

Assume that the solution of problem (1), (2), with $n=0$, is uniquely defined. Then the following problems are of high importance:

Under what conditions imposed on the left-hand sides of problems (1), (2) their solutions $y(\,\cdot\,,n)$ exist and are unique for sufficiently large $n\in\mathbb{N}$? What additional conditions imposed on the left- and right-hand sides of problems (1), (2) guarantee the limit equality

$$\|y(\,\cdot\,,n)-y(\,\cdot\,,0)\|_\infty\to 0,\ \ n\to\infty, \tag{3}$$

where $\|\cdot\|_\infty$ – sup-norm on the compact interval $[a,b]$.

For the first time, these problems were investigated by Kiguradze [3] in the case of real-valued functions.

We introduce the notation:

$$R_A(\,\cdot\,,n) := A(\,\cdot\,,n) - A(\,\cdot\,,0) \in L_1([a,b];\mathbb{C}^{m\times m}),$$

$$F(\,\cdot\,,n) := \begin{pmatrix} f_1(\,\cdot\,,n) & 0 & \dots & 0 \\ f_2(\,\cdot\,,n) & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_m(\,\cdot\,,n) & 0 & \dots & 0 \end{pmatrix} \in L_1([a,b];\mathbb{C}^{m\times m}),$$

$$R_F(\,\cdot\,,n) = F(\,\cdot\,,n) - F(\,\cdot\,,0),$$

$$R_F^{\vee}(t,n) := \int_a^t R_F(s,n)\,ds, \quad R_A^{\vee}(t,n) := \int_a^t R_A(s,n)\,ds.$$

Put also $\|\cdot\|_1$ is the norm in Lebesgue space of vector-valued functions (matrix-valued functions) on the interval $[a,b]$.

Further we assume that all asymptotic relations are considered as $n \to \infty$.

Theorem (Kiguradze [3])**.** *Suppose that*

(0) *the homogeneous boundary-value problem* (1), (2)*, with* $n = 0$*, has only the trivial solution;*

(I) $\|R_A^{\vee}(\,\cdot\,,n)\|_\infty \to 0$*;*

(II) $\|R_A(\,\cdot\,,n)\|_1 = O(1)$*;*

(III) $B(n)y \to B(0)y$*,* $y(\,\cdot\,) \in C([a,b];\mathbb{C}^m)$*.*

Then, for sufficiently large n*, problem* (1), (2) *possesses a unique solution. In addition, if the right-hand sides of problems satisfy the following conditions*

(IV) $c(n) \to c(0)$*;*

(V) $\|R_F^{\vee}(\,\cdot\,,n)\|_\infty \to 0$,

then the unique solutions of problems (1), (2) *satisfy the limit equality* (3)*.*

The examples show that all the conditions of Kiguradze's Theorem are essential and none of them can be omitted. However, some conditions can be weakened.

Denote by $\mathcal{M}^m := \mathcal{M}(a,b;m)$, $m \in \mathbb{N}$ class of sequences of the matrix functions $R(\,\cdot\,,n) : \mathbb{N} \to L_1([a,b];\mathbb{C}^{m\times m})$ such that solution $Z(\,\cdot\,,n)$ of the Cauchy problem

$$Z'(\,\cdot\,,n) + R(\,\cdot\,,n)Z(\,\cdot\,,n) = O, \quad Z(a,n) = I_m$$

satisfies the limit equality

$$\|Z(\,\cdot\,,n) - I_m\|_\infty \to 0,$$

where I_m is an identity $(m\times m)$-matrix.

Put

$$A_F(\,\cdot\,,n) := \begin{pmatrix} A(\,\cdot\,,n) & F(\,\cdot\,,n) \\ O_m & O_m \end{pmatrix} \in L_1([a,b];\mathbb{C}^{2m\times 2m}),$$

$$R_{A_F}(\,\cdot\,,n) := A_F(\,\cdot\,,n) - A_F(\,\cdot\,,0) \in L_1([a,b];\mathbb{C}^{2m\times 2m}),$$

where O_m is a zero $(m\times m)$–matrix.

Theorem 1. *In Kiguradze's Theorem, conditions* (I), (II) can be replaced by one condition

$$R_A(\,\cdot\,,n) \in \mathcal{M}^m, \tag{4}$$

if condition (V) *is replaced by the following*

$$R_{A_F}(\,\cdot\,,n) \in \mathcal{M}^{2m}. \tag{5}$$

Conditions (4), (5) *are very general but not constructive because there are no explicit descriptions of the classes* $\mathcal{M}^m$ *and* $\mathcal{M}^{2m}$*.*

However, the results of [4] contain explicit sufficient conditions that the sequence of matrix-valued functions belongs to the class $\mathcal{M}^m$ or $\mathcal{M}^{2m}$. These sufficient conditions are more convenient to use. Therefore, from Theorem 1 follows a number of constructive statements that generalize or complement Kiguradze's Theorem.

Theorem 2. *In Kiguradze's Theorem, condition* (II) *can be replaced by the one more general condition*

(II*) $\|R_A(\,\cdot\,,n)R^\vee_A(\,\cdot\,,n)\|_1 \to 0$,

with the additional condition

(VI*) $\|R_A(\,\cdot\,,n)R^\vee_F(\,\cdot\,,n)\|_1 \to 0$.

This theorem generalizes Kiguradze's result, since it does not contain the requirement of boundedness of the norms of coefficients of systems.

The advantages of Theorem 2 over Kyguradze's Theorem become more noticeable if we consider their applications to systems of linear differential equations of the higher order of the form

$$y^{(r)}(t,n) + A_{r-1}(t,n)y^{(r-1)}(t,n) + \ldots + A_0(t,n)y(t,n) = f(t,n) \tag{6}$$

with inhomogeneous boundary conditions

$$B_j(n)y(\,\cdot\,,n) = c_j(n), \;\; j \in \{1,\ldots,r\} := [r], \;\; n \in \mathbb{N} \cup \{0\}, \tag{7}$$

where $B_j(n) : C^{(r-1)}([a,b];\mathbb{C}^m) \to \mathbb{C}^m$ are linear continuous operators with $j \in [r]$.

Assume that the matrix-valued functions $A_{j-1}(\,\cdot\,,n)$, the vector-valued functions $f(\,\cdot\,,n)$ and the vectors $c_j(n)$ satisfy the conditions presented above for problem (1), (2).

A solution of the system of differential equations (6), (7) is understood as a vector-valued function $y(\,\cdot\,,n) \in W^r_1([a,b];\mathbb{C}^m)$ satisfying the equation almost everywhere. The inhomogeneous boundary conditions (7) are correctly defined on the solutions of system (6) and cover all classical types of boundary conditions.

Each of these problems can be reduced to the general inhomogeneous boundary-value problem for the system of equations of the first order. For applied to these problems, Kiguradze's Theorem takes the following form.

Theorem 3. *Suppose that the solutions of problem* (6), (7) *are uniquely defined and*

(I$'$) $\|R^\vee_{A_{j-1}}(\,\cdot\,,n)\|_\infty \to 0$;

(II$'$) $\|R_{A_{j-1}}(\,\cdot\,,n)\|_1 = O(1)$;

(III$'$) $B_j(n)y \to B_j(0)y$, $y \in C^{(r-1)}([a,b];\mathbb{C}^m)$.

Then, for sufficiently large n problems (6), (7) *possess the unique solutions. Moreover, if*

(IV′) $c_j(n) \to c_j(0)$,

(V′) $\|R_F^{\vee}(\,\cdot\,,n)\|_\infty \to 0$,

then the unique solutions of problems (6), (7) *satisfy the limit equality*

$$\left\|y^{(j-1)}(\,\cdot\,,0) - y^{(j-1)}(\,\cdot\,,n)\right\|_\infty \to 0.$$

In this case, from Theorem 2 follows the next result.

Theorem 4. *In Theorem 3, condition* (II′) can be replaced by the condition

(II**) $\|R_{A_{r-1}}(\,\cdot\,,n)R^{\vee}_{A_{j-1}}(\,\cdot\,,n)\|_1 \to 0$,

if the additional condition is fulfilled

(VI**) $\|R_{A_{r-1}}(\,\cdot\,,n)R_F^{\vee}(\,\cdot\,,n)\|_1 \to 0$, $n \to \infty$.

The condition (VI**) *is fulfilled if conditions* (II′), (V′) *hold.*

Note also that conditions (II**), (VI**) are obviously fulfilled if

$$\|R_{A_{r-1}}(\,\cdot\,,n)\|_1 = O(1).$$

At the same time, there are no restrictions on the sequence $\{\|R_{A_{j-1}}(\,\cdot\,,n)\|_1 \,:\, n \geq 1\}$, with $j \in [r-1]$.

These and other results are presented in more detail in [6–8].

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