On the Continuous Dependence of Solutions to Linear Boundary Value Problems on Boundary Conditions

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1 Introduction

The general questions of the continuous dependence of solutions to boundary value problems on parameters as applied to functional differential equations are studied in [1, 3, 4, 8], see also the references to Section 1.5 in [1].

We consider a quite broad class of functional differential systems with aftereffect and follow the notation and basic statements of the general theory of functional differential equations in the part concerning linear systems with aftereffect [1, 4].

Let $L^n = L^n[0,T]$ be the Lebesgue space of all summable functions $z : [0,T] \to R^n$ defined on a finite segment $[0,T]$ with the norm

$$\|z\|_{L^n} = \int\limits_0^T |z(t)|\,dt,$$

where $|\cdot|$ is a norm in R^n. Below we use $\|\cdot\|$ for the matrix norm agreed with $|\cdot|$.

Denote by $AC^n = AC^n[0,T]$ the space of absolutely continuous functions $x : [0;T] \to R^n$ with the norm

$$\|x\|_{AC^n} = |x(0)| + \|\dot{x}\|_{L^n}.$$

In the sequel we will use some results from [1, 4].

The system

$$\mathcal{L}x = f \tag{1.1}$$

with a linear bounded Volterra operator $\mathcal{L} : AC^n \to L^n$ is considered under the assumption that the general solution of equation (1.1) has the form

$$x(t) = X(t)x(0) + \int\limits_0^t C(t,s)f(s)\,ds, \tag{1.2}$$

where $X(t)$ is the fundamental matrix to the homogeneous equation $\mathcal{L}x = 0$, $C(t,s)$ is the Cauchy matrix. A broad class of operators $\mathcal{L}$ with property (1.2) is described, for instance, in [5].

We consider the boundary value problems (BVPs)

$$\mathcal{L}x = f, \quad \ell_0 x = 0, \tag{1.3}$$

and

$$\mathcal{L}x = f, \quad \ell x = 0, \tag{1.4}$$

where $\ell_0, \ell : AC^n \to R^n$ are linear bounded vector-functional, assuming (1.3) to be uniquely solvable, i.e. $\det \ell_0 X \neq 0$. We will consider the question of the continuous dependence of solutions on the boundary conditions in terms of the proximity of ℓ to ℓ_0 and the proximity of the solution x of BVP (1.4) to the solution x_0 of BVP (1.3).

2 Two theorems

First we give a theorem that follows from the theorem on the invertible operator (see, for instance, Theorem 3.6.3 [2]).

Theorem 2.1. *Let the inequality*

$$\Delta = \|\ell_0 X - \ell X\| \cdot \|(\ell_0 X)^{-1}\| < 1 \tag{2.1}$$

be fulfilled. Then BVP (1.4) *is uniquely solvable and the estimate*

$$\|x_0 - x\|_{AC^n} \le \|X\|_{AC^{n\times n}} \cdot \frac{\Delta}{1-\Delta} \cdot \|(\ell_0 X)^{-1}\| \cdot \|\ell\|_{AC^n \to R^n} \cdot \|Cf\|_{AC^n} + \|X\|_{AC^{n\times n}} \cdot \|(\ell_0 X)^{-1}\| \cdot \|\ell_0 - \ell\|_{AC^n \to R^n} \cdot \|Cf\|_{AC^n}$$

holds.

Results of the constructive study of boundary value problems, based on conditions like (2.1), are presented systematically in [1,7], see also [6]. Condition (2.1) often turns out to be quite rigid. To formulate the next theorem based on another approach, we introduce additional notation:

$$\ell_0 X = \Gamma^0 = (\gamma^0_{ij})_{i,j=1,\dots,n}; \quad \ell X = \Gamma = \big([\gamma^b_{ij},\, \gamma^u_{ij}]\big)_{i,j=1,\dots,n}; \quad \gamma^0_{ij} \in [\gamma^b_{ij}, \gamma^u_{ij}];$$
$$(\ell_0 X)^{-1} = B^0 = (\beta^0_{ij})_{i,j=1,\dots,n}; \quad (\ell X)^{-1} = B = \big([\beta^b_{ij},\, \beta^u_{ij}]\big)_{i,j=1,\dots,n};$$
$$M = \max\big(\det\Gamma : \ \gamma_{ij} \in [\gamma^b_{ij}, \gamma^u_{ij}], \ i,j = 1,\dots,n\big);$$
$$\mu = \min\big(\det\Gamma : \ \gamma_{ij} \in [\gamma^b_{ij}, \gamma^u_{ij}], \ i,j = 1,\dots,n\big).$$

For an $(n \times n)$-matrix $\mathcal{A}$ with interval-valued elements $[a_{ij}, b_{ij}]$ we define $\|\mathcal{A}\|_I$ by the equality

$$\|\mathcal{A}\|_I = \big\|(\alpha_{ij})_{i,j=1,\dots,n}\big\|,$$

where $\alpha_{ij} = \max(|a_{ij}|, |b_{ij}|)$.

Theorem 2.2. *Let the inequality*

$$M \cdot \mu > 0$$

be fulfilled. Then BVP (1.4) *is uniquely solvable and the estimate*

$$\|x_0 - x\|_{AC^n} \le \|X\|_{AC^{n\times n}} \cdot \|B_0 - B\|_I \cdot \|\ell\|_{AC^n \to R^n} \cdot \|Cf\|_{AC^n} + \|X\|_{AC^{n\times n}} \cdot \|B_0\| \cdot \|\ell_0 - \ell\|_{AC^n \to R^n} \cdot \|Cf\|_{AC^n}$$

holds.

This theorem allows to cover a set of boundary value problems (1.4) for which condition (2.1) is not fulfilled.

3 An example

Consider the boundary value problem

$$\dot{x}(t) = Fx(t) + f(t), \ \ t \in [0,1], \ \ \ell_{01}x \equiv ax_1(0) + bx_2(1) = 0, \ \ \ell_{02}x \equiv cx_1(1) + dx_2(0) = 0. \tag{3.1}$$

Here
$$F=\begin{pmatrix}0.5 & -0.1\\ -0.2 & 0.6\end{pmatrix},\quad A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}.$$

For definiteness, let the norm in R^2 be defined by the equality $|x|=\max(|x_1|,|x_2|)$, hence for $B=(b_{ij})$ we have
$$\|B\|=\max\big(|b_{11}|+|b_{12}|,|b_{21}|+|b_{22}|\big).$$

For the case the matrix ℓ_0X is defined by the equality
$$\ell_0X=\begin{pmatrix}0.304 & 3.680\\ 4.997 & 3.478\end{pmatrix},\quad (\ell_0X)^{-1}=\begin{pmatrix}-0.200 & 0.212\\ 0.288 & -0.018\end{pmatrix},\quad \|(\ell_0X)^{-1}\|=0.413.$$

Thus by virtue of Theorem 2.1 problem (3.1) is uniquely solvable and, together with it, any problem
$$\dot{x}=Fx+f,\ \ \ell x=0 \tag{3.2}$$
with ℓ such that $\|\ell X-\ell_0X\|<2.421$ is uniquely solvable too.

Let us show that Theorem 2.2 makes it possible to go beyond this inequality. Immerse the matrix ℓ_0X into the family $\Gamma=\begin{pmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{pmatrix}$ with $\gamma_{11}\in[0.2,0.4]$, $\gamma_{12}\in[3.5,3.8]$, $\gamma_{21}\in[4.5,5.5]$, $\gamma_{22}\in[3.4,8]$.

Further
$$\max\Big(\det\Gamma:\ \gamma_{11}\in[0.2,0.4],\ \gamma_{12}\in[3.5,3.8],\ \gamma_{21}\in[4.5,5.5],\ \gamma_{22}\in[3.4,8]\Big)=-12.55,$$
$$\min\Big(\det\Gamma:\ \gamma_{11}\in[0.2,0.4],\ \gamma_{12}\in[3.5,3.8],\ \gamma_{21}\in[4.5,5.5],\ \gamma_{22}\in[3.4,8]\Big)=-20.22,$$

therefore, the determinant of any matrix from the family Γ differs from zero. It should be noted that in terms of the parameters a, b, c, d of ℓ_0 it means the unique solvability for all the problems (3.2) with $a\in[0.862,1.119]$, $b\in[1.902,2.065]$, $c\in[2.701,3.301]$, $d\in[3.870,8.574]$.

Let us take the element $\Gamma_1=\begin{pmatrix}0.2 & 3.5\\ 5.5 & 8\end{pmatrix}$ from Γ and calculate
$$\|\ell_0X-\Gamma_1\|=5.025>\frac{1}{\|(\ell_0X)^{-1}\|}=2.421.$$

As for estimating difference of a solution x_0 to (3.1) and a solution x to an arbitrary problem from (3.2) with $\ell X\in\Gamma$, first we calculate
$$\Gamma^{-1}=\begin{pmatrix}[-0.637,-0.168] & [0.188,0.279]\\ [0.272,0.359] & [-0.032,-0.010]\end{pmatrix}$$
with $\|\Gamma^{-1}\|_I\le 0.805$ and
$$\ell_0X-\Gamma^{-1}=\begin{pmatrix}[0.032,0.437,] & [-0.067,0.024]\\ [-0.016,0.071] & [-0.008,0.014]\end{pmatrix},$$
hence $\|(\ell_0X)^{-1}-\Gamma^{-1}\|_I\le 0.504$. Having in mind the representation
$$x_0-x=X\big[(\ell_0X)^{-1}\ell-(\ell X)^{-1}\ell_0\big]Cf=X\big[(\ell_0X)^{-1}-(\ell X)^{-1}\big]\ell Cf+X\big[(\ell_0X)^{-1}(\ell_0-\ell)\big]Cf,$$
we obtain
$$\|x_0-x\|_{AC_2}\le 0.504\|X\|_{AC^{2\times 2}}\cdot\|\ell\|_{AC^2\to R^2}\cdot\|Cf\|_{AC^2}+0.414\|X\|\cdot\|\ell_0-\ell\|_{AC^2\to R^2}\cdot\|Cf\|_{AC^2}$$

and, taking into account the estimate $\|X\|_{AC^{2\times 2}} \le 2.188$,

$$\|x_0 - x\|_{AC^2} \le 1.103\,\|\ell\|_{AC^2\to R^2} \cdot \|Cf\|_{AC^2} + 0.906\,\|\ell_0 - \ell\|_{AC^2\to R^2} \cdot \|Cf\|_{AC^2}.$$

Note again that, in this example, the statements of Theorem 2.2 cover the set of problems including those that do not belong to the set defined by Theorem 2.1.

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