Characteristic Vectors for Normed Partitions of Cauchy Matrices

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For any map $y:\mathbb{R}_+\to\mathbb{R}^n$, where $\mathbb{R}_+=\{t\in\mathbb{R}:\ t\geq 0\}$, we can calculate the Lyapunov exponent $\lambda[y]$ as

$$\lambda[y]=\overline{\lim_{t\to+\infty}}\,\frac{1}{t}\,\ln\|y(t)\|. \tag{1}$$

It is well known that Lyapunov exponents play an important role in qualitative theory of differential equations and stability theory, see [2] or [8]. For maps defined on some subsets of $\mathbb{R}^m$ with $m>1$, such as solutions of total differential equations, we can not define the Lyapunov exponent by (1) without substantial improvements. Some appropriate definitions for the required analogs of Lyapunov exponents in multivariate case has been proposed by E. I. Grudo [5] and M. V. Kozhero [9].

Now the following asymptotic characteristics are used for solutions of total differential equations: strong exponents [9], (weak) characteristic exponents [9], [4, p. 115], and characteristic functionals (vectors) [5], [4, p. 108], [3, p. 82]. Each of these notions is a straightforward generalization of classical Lyapunov exponent and coincides with it when $m=1$.

The results concerning these asymptotic characteristics are summarized by I. V. Gaishun in monographs [3] and [4], where general and asymptotic theory of total differential equations are systematically presented. Some additional information on these issues can be found in [12].

Let $K\subset\mathbb{R}^n$ be a closed convex cone such that $K\cap(-K)=\{0\}$. A linear functional (in fact, a row vector) $\mu\in(\mathbb{R}^n)^*$ is said to be positive on K if $\mu(x)\geq 0$ for all $x\in K$. The set K^+ of all positive on K linear functionals is called the dual cone of K.

Take any $y:K\to\mathbb{R}^m$.

Definition 1. A linear functional $\lambda\in(\mathbb{R}^n)^*$ is said to be a characteristic functional of y with respect to the cone K if

$$\limsup_{\|x\|\to+\infty}\,\|x\|^{-1}\big(\lambda x+\ln\|y(x)\|\big)=0$$

and

$$\limsup_{\|x\|\to+\infty}\,\|x\|^{-1}\big(\lambda x+\mu x+\ln\|y(x)\|\big)>0$$

for all $\mu\in K^+$, $\mu\neq 0$.

The set of all characteristic functionals is called the characteristic set of y. We denote it by $\mathcal{M}[y]$.

Definition 2. The (weak) characteristic exponent of y is the function $\chi[y]:K\setminus\{0\}\to\mathbb{R}$ defined by

$$\chi[y](x):=\overline{\lim_{t\to+\infty}}\,\frac{1}{t\|x\|}\,\ln\|y(tx)\|.$$

There exist an interrelation between (weak) characteristic exponents and characteristic functionals. In [10] (see also [12]) it was proved that if $\ln\|y\|$ is a Lipshitz function, then $\mathcal{M}[y]=\mathcal{M}[\exp\psi[y]]$, where $\psi[y](x)=\|x\|\chi[y](x)$ is the modified characteristic exponent of y.

It occurs that the above asymptotic characteristics are useful not only in the study of total differential equations, but also in the theory of linear ordinary differential systems. To demonstrate this fact, consider a linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{2}$$

with piecewise continuous and bounded coefficient matrix A such that $\|A(t)\| \leq M < +\infty$ for all $t \geq 0$. We denote the Cauchy matrix of (2) by X_A and the highest Lyapunov exponent of (2) by $\lambda_n(A)$.

In [16], see also [15, p. 379] and [2, p. 236], I. G. Malkin has used estimations of the form

$$\|X_A(t,s)\| \leq D\exp(\alpha(t-s)+\beta s), \quad t \geq s \geq 0, \quad D > 0, \quad \alpha, \beta \in \mathbb{R}, \tag{3}$$

in order to investigate asymptotic stability of the trivial solution to a system

$$\dot{y} = A(t)y + f(t,y), \quad y \in \mathbb{R}^n, \quad t \geq 0,$$

with a nonlinear perturbation $f(t,y)$ of a higher order.

An ordered pair $(\alpha,\beta) \in \mathbb{R}^2$ is called a Malkin estimation for system (2) if there exists a number $D = D(\alpha,\beta) > 0$ such that (3) holds. A pair $(\alpha,\beta) \in \mathbb{R}^2$ is said to be a minimal Malkin estimation [11] if $(\alpha+\xi,\beta+\eta) \in E(A)$ for all $\xi > 0$, $\eta > 0$, and $(\alpha+\xi,\beta+\eta) \notin E(A)$ for all $\xi \leq 0$, $\eta \leq 0$, $\xi^2+\eta^2 \neq 0$.

It can be easily seen that the set of minimal Malkin estimations for system (2) coincides with the set of Grudo characteristic vectors for the function $\|X_A(t,s)\|$ with respect to the cone $C = \{(t,s) \in \mathbb{R}^2 : \ t \geq s \geq 0\}$. Using this fact, in [11] we have given an alternative description for the set of minimal Malkin estimations in terms of the function

$$\overline{\lim_{s\to+\infty}} \frac{1}{(\theta-1)s} \ln \|X_A(\theta s, s)\|. \tag{4}$$

Definition 3. Let τ be an increasing sequence $t_0 < t_1 < \cdots < t_{s+1}$ of $s+2$ real numbers. The expression

$$P_A(\tau) = \prod_{i=0}^{s} \|X_A(t_{i+1}, t_i)\|$$

is said to be a normed partition of the Cauchy matrix for system (2).

Normed partitions are common in Lyapunov exponents theory. Formulae for calculating the central (see [2, p. 99], [8, p. 43])

$$\Omega(A) = \lim_{T\to+\infty} \overline{\lim_{m\to\infty}} \frac{1}{mT} \sum_{k=1}^{m} \ln \|X_A(kT, kT-T)\|$$

as well as the exponential exponent (see [7], [8, p. 52])

$$\nabla_0(A) = \lim_{\theta\to1+0} \overline{\lim_{m\to\infty}} \frac{1}{\theta^m} \sum_{k=1}^{m} \ln \|X_A(\theta^k, \theta^{k-1})\|, \tag{5}$$

contain the expressions of the form

$$\Xi_A(\tau) = \sum_{i=0}^{s} \ln \|X_A(t_{i+1}, t_i)\| = \ln P_A(\tau)$$

with some appropriate τ. The highest sigma-exponent (or the Izobov exponent) of system (2) (see [6], [8, p. 225])

$$\nabla_\sigma(A) = \overline{\lim_{m\to\infty}} \frac{\xi_m(\sigma)}{m}\,,$$
$$\xi_m(\sigma) = \max_{i<m} \big(\ln\|X_A(m,i)\| + \xi_i(\sigma) - \sigma i\big), \;\; \xi_1 = 0, \;\; i \in \mathbb{N},$$

can be represented in an equivalent form [1] (see also [14]) as

$$\nabla_\sigma(A) = \overline{\lim_{m\to\infty}}\, m^{-1} \max_{\tau\in\mathcal{D}_0(m)} \big(\Xi_A(\tau) - \sigma\|\tau\|_{\rm i}\big), \tag{6}$$

where $\mathcal{D}_0(m)$ is the set of all increasing sequences $0 = t_0 < t_1 < \ldots < t_{s+1} = m$ of integer numbers with at least two terms and $\|\tau\|_{\rm i} = t_1 + \cdots + t_s$. Note that $\tau \in \mathcal{D}_0(m)$ may have different numbers of elements.

Let $t_0 = 0$. Fix some $k \in \mathbb{N}$ and consider sequences $0 < t_1 < \cdots < t_{k+1}$ of real numbers with $k+1$ elements as vectors $(t_1,\ldots,t_{k+1}) \in \mathbb{R}^{k+1}$. Taking $K = \{\tau = (t_1,\ldots,t_{k+1}) \in \mathbb{R}^{k+1} : \;\; 0 \le t_1 < \cdots \le t_{k+1}\}$, we define the set $\mathcal{M}[P_A]$ and the function

$$\Psi_A(\tau) = \psi[P_A](\tau) = \overline{\lim_{t\to+\infty}} \frac{1}{t}\ln P_A(t\tau)$$

according to Definitions 1 and 2. By [10] (see also [12]) we have the following statements.

Proposition 1. *The equality*

$$\mathcal{M}[P_A] = \mathcal{M}[\exp\Psi_A]$$

holds.

Proposition 2. *Let $\lambda \in \mathcal{M}[\Psi_A]$. If for some sequence of vectors $\tau_j \in K \subset \mathbb{R}^{k+1}$, such that $\|\tau_j\| \to \infty$ and $\tau_j\|\tau_j\|^{-1} \to \xi \in \mathbb{R}^{k+1}$ as $j \to \infty$, we have*

$$\lim_{j\to\infty} \|\tau_j\|^{-1}(\lambda\tau_j + \ln P_A(\tau_j)) = 0,$$

then $\lambda\xi + \Psi_A(\xi) = 0$ and $\lambda\xi + \Psi_A(\xi) \ge 0$ for all $\xi \in K$.

We cannot use these results to calculate $\nabla_\sigma(A)$, since in (6) the length of τ can increase indefinitely as m increases. However, we can apply Propositions 1 and 2 to obtain some information on finite-point approximations of $\nabla_\sigma(A)$.

Let $\mathcal{D}_0^k(m)$ be a subset of $\mathcal{D}_0(m)$ containing sequences with at most k elements.

Definition 4 ([13])**.** The number

$$\nabla^k_\sigma(A) = \overline{\lim_{m\to\infty}}\, m^{-1} \max_{\tau\in\mathcal{D}_0^k(m)} \big(\Xi_A(\tau) - \sigma\|\tau\|_{\rm i}\big)$$

is said to be the k-point approximation for $\nabla_\sigma(A)$.

Proposition 3. *If $(\sigma,\mu) \in \mathbb{R}^2$ is an extreme point for the epigraph of $\nabla^k_\sigma(A)$, then the vector $(-\sigma,\ldots,-\sigma,-\mu) \in (\mathbb{R}^{k+1})^*$ is a characteristic vector for P_A.*

Corollary. *If $(\sigma,\mu) \in \mathbb{R}^2$ is an extreme point for the epigraph of $\nabla^k_\sigma(A)$, then*

$$\sigma\sum_{i=1}^k \xi_i + \mu\xi_{k+1} \le \Psi_A(\xi)$$

for all $\xi \in K$ and there exists some $\xi^0 \in K$ such that

$$\sigma\sum_{i=1}^k \xi^0_i + \mu\xi^0_{k+1} = \Psi_A(\xi^0).$$

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