Optimal Regularity Results for the One-Dimensional Prescribed Curvature Equation Via the Strong Maximum Principle

Julián López-Gómez

Instituto Interdisciplinar de Matemáticas, Universidad Complutense de Madrid
Madrid, Spain
E-mail: jlopezgo@ucm.es

Pierpaolo Omari

Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste
Trieste, Italy
E-mail: omari@units.it

1 Introduction

This contribution is based on our recent paper [1], where we establish a novel, extended, version of the strong maximum principle for a general class of second order ordinary differential equations

$$v'' = g(t, v, v'),$$

in the absence of any assumption of continuity or monotonicity on the function g, and where, exploiting this tool, we provide some optimal regularity results for the bounded variation solutions, positive and nodal, of the non-autonomous curvature equation

$$-\Big(\frac{u'}{\sqrt{1+(u')^2}}\Big)' = f(x, u), \tag{1.1}$$

f being an arbitrary function prescribing the curvature of the graph of u.

The analysis carried out in [1] allows us, through a completely different technical device, to extend most of the results we previoulsly obtained in [2–5], for the positive bounded variation solutions of (1.1) under homogeneous Neumann boundary condition and the structural assumption $f(x,s) = h(x)k(s)$, to more general classes of equations and to, possibly non-homogeneous, Dirichlet, Neumann, Robin, or even periodic boundary value problems. Furthermore, we are able to produce a new interpretation of the assumptions used in our previous works, clarifying their meaning and displaying some deep, though previously hidden, connections with the strong maximum principle.

2 A variant of the strong maximum principle

The main result of this section is the following version of the strong maximum principle for second order ordinary differential equations with possibly discontinuous and non-monotone right-hand sides. In this respect, the Keller–Osserman assumption (G) stated below is independent of the conditions required by the classical Vázquez strong maximum principle in [6] and by its extensions given by Pucci and Serrin in [7], where G' is always supposed to be continuous and increasing. Accordingly, this result yields, in the one-dimensional setting, a completion and a sharpening of its counterparts in [6] or [7]; its proof, delivered in [1], being also more delicate than in the classical situations.

Theorem 2.1. *Let $g : (\alpha, \omega) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given function and let $v \in W^{2,1}_{\mathrm{loc}}(\alpha, \omega) \cap W^{1,1}(\alpha, \omega)$ be a non-trivial non-negative solution of the differential equation*

$$v''(t) = g(t, v(t), v'(t)) \text{ for almost all } t \in (\alpha, \omega).$$

Assume that:

(G) *there exist a constant $\varepsilon > 0$ and an absolutely continuous function $G : [0, \varepsilon] \to \mathbb{R}$ such that*

$$0 \le g(t, v(t), v'(t)) \le G'(v(t)) \text{ for almost all } t \in (\alpha, \omega) \\ \text{for which } 0 < v(t) \le \varepsilon \text{ and } |v'(t)| \le \varepsilon,$$

and either

$$G(s) = 0 \text{ for all } s \in (0, \varepsilon],$$

or

$$G(s) > 0 \text{ for all } s \in (0, \varepsilon] \text{ and } \int_0^\varepsilon \frac{1}{\sqrt{G(s)}}\, ds = +\infty. \tag{2.1}$$

Then, v is strongly positive in the sense that the following properties hold true:

- *$v(t) > 0$ for all $t \in (\alpha, \omega)$;*
- *$v'(\alpha^+) > 0$ if $v(\alpha) = 0$ and $v'(\alpha^+)$ exists;*
- *$v'(\omega^-) < 0$ if $v(\omega) = 0$ and $v'(\omega^-)$ exists.*

3 Optimal regularity results for the prescribed curvature equation

In this section we discuss the regularity properties of the bounded variation solutions of the one-dimensional non-autonomous prescribed curvature equation

$$-\left(\frac{u'}{\sqrt{1 + (u')^2}}\right)' = f(x, u), \quad a < x < b, \tag{3.1}$$

where $f : (a, b) \times \mathbb{R} \to \mathbb{R}$ is any given function. We begin by recalling the notion of bounded variation solution of equation (3.1). To this end, for any $v \in BV(a, b)$, we denote by $Dv = D^a v\, dx + D^s v$ the Lebesgue–Nikodym decomposition, with respect to the Lebesgue measure dx in $\mathbb{R}$, of the Radon measure Dv in its absolutely continuous part $D^a v\, dx$, with density function $D^a v$, and its singular part $D^s v$. Further, $\frac{D^s v}{|D^s v|}$ stands for the density function of $D^s v$ with respect to its absolute variation $|D^s v|$. Finally, for every $x_0 \in [a, b)$, $v(x_0^+)$ denotes the right trace of v at x_0 and, for every $x_0 \in (a, b]$, $v(x_0^-)$ denotes the left trace of v at x_0.

Definition 3.1. A function $u \in BV(a, b)$ is a bounded variation solution of (3.1) if $f(\,\cdot\,, u(\,\cdot\,)) \in L^1(a, b)$ and

$$\int_a^b \frac{D^a u(x) D^a \phi(x)}{\sqrt{1 + (D^a u(x))^2}}\, dx + \int_a^b \frac{D^s u}{|D^s u|}(x)\, D^s \phi = \int_a^b f(x, u(x)) \phi(x)\, dx$$

for all $\phi \in BV(a, b)$ such that $|D^s \phi|$ is absolutely continuous with respect to $|D^s u|$ and $\phi(a^+) = \phi(b^-) = 0$.

We begin with a partial regularity result: it establishes that a bounded variation solution u of (3.1) can lose its regularity at the endpoints, but never at the interior points, of the intervals where the function $f(\,\cdot\,,u(\,\cdot\,))$ has a definite sign; whereas, u can be singular at an interior point of its domain if such a point separates two adjacent intervals where $f(\,\cdot\,,u(\,\cdot\,))$ changes sign. In both cases, the derivative u' blows up, but, in the latter one, u can further exhibit a jump discontinuity.

Theorem 3.1. *Let u be a bounded variation solution of equation* (3.1). *Then, the following statements hold.*

(i) *If $f(x,u(x))\geq 0$ for almost all $x\in(a,b)$, then u is concave and either $u\in W^{2,1}(a,b)$, or $u\in W^{2,1}_{\rm loc}[a,b)\cap W^{1,1}(a,b)$ and $u'(b^-)=-\infty$, or $u\in W^{2,1}_{\rm loc}(a,b]\cap W^{1,1}(a,b)$ and $u'(a^+)=+\infty$, or $u\in W^{2,1}_{\rm loc}(a,b)\cap W^{1,1}(a,b)$, $u'(a^+)=+\infty$, and $u'(b^-)=-\infty$. In all cases, u satisfies equation* (3.1) *for almost all $x\in(a,b)$.*

(ii) *If $f(x,u(x))\leq 0$ for almost all $x\in(a,b)$, then u is convex and either $u\in W^{2,1}(a,b)$, or $u\in W^{2,1}_{\rm loc}[a,b)\cap W^{1,1}(a,b)$ and $u'(b^-)=+\infty$, or $u\in W^{2,1}_{\rm loc}(a,b]\cap W^{1,1}(a,b)$ and $u'(a^+)=-\infty$, or $u\in W^{2,1}_{\rm loc}(a,b)\cap W^{1,1}(a,b)$, $u'(a^+)=-\infty$, and $u'(b^-)=+\infty$. In all cases, u satisfies equation* (3.1) *for almost all $x\in(a,b)$.*

(iii) *If there is $c\in(a,b)$ such that $f(x,u(x))\geq 0$ for almost all $x\in(a,c)$ and $f(x,u(x))\leq 0$ for almost all $x\in(c,b)$, then $u_{|(a,c)}$ is concave, $u_{|(c,b)}$ is convex, and either $u\in W^{2,1}_{\rm loc}(a,b)\cap W^{1,1}(a,b)$, or $u_{|(a,c)}\in W^{2,1}_{\rm loc}(a,c)\cap W^{1,1}(a,c)$, $u_{|(c,b)}\in W^{2,1}_{\rm loc}(c,b)\cap W^{1,1}(c,b)$, $u(c^-)\geq u(c^+)$, and $u'(c^-)=-\infty=u'(c^+)$. Moreover, in case $u(c^-)>u(c^+)$, we have that*

$$D^s u=(u(c^+)-u(c^-))\,\delta_c,$$

where δ_c stands for the Dirac measure concentrated at c. In any circumstances, u satisfies equation (3.1) *for almost all $x\in(a,b)$.*

(iiii) *If there is $c\in(a,b)$ such that $f(x,u(x))\leq 0$ for almost all $x\in(a,c)$ and $f(x,u(x))\geq 0$ for almost all $x\in(c,b)$, then $u_{|(a,c)}$ is convex, $u_{|(c,b)}$ is concave, and either $u\in W^{2,1}_{\rm loc}(a,b)\cap W^{1,1}(a,b)$, or $u_{|(a,c)}\in W^{2,1}_{\rm loc}(a,c)\cap W^{1,1}(a,c)$, $u_{|(c,b)}\in W^{2,1}_{\rm loc}(c,b)\cap W^{1,1}(c,b)$, $u(c^-)\leq u(c^+)$, and $u'(c^-)=+\infty=u'(c^+)$. Moreover, in case $u(c^-)<u(c^+)$,* (3.1) *holds. In any circumstances, u satisfies equation* (3.1) *for almost all $x\in(a,b)$.*

Our next two results, Theorems 3.2 and 3.3, establish the complete regularity of the bounded variation solutions u of (3.1). Precisely, Theorem 3.2 guarantees the regularity at the endpoints of any interval where the sign of $f(\,\cdot\,,u(\,\cdot\,))$ is constant, by imposing at these points a suitable control, expressed by any of the conditions (j)–(jjjj), on the decay rate to zero of $f(\,\cdot\,,u(\,\cdot\,))$ Theorem 3.3, instead, guarantees the regularity of u at any interior point, z, separating two adjacent interval where $f(\,\cdot\,,u(\,\cdot\,))$ changes sign, by imposing a similar decay property to $f(\,\cdot\,,u(\,\cdot\,))$ either on the left, or on the right, of z, as expressed by the conditions (h) or (hh). From [3–5] we also know that these assumptions on the decay rate of $f(\,\cdot\,,u(\,\cdot\,))$ are optimal, in the sense that, if they fail at some point, the derivative u' might blow-up there, and the solution u might even develop a jump discontinuity.

The proof of Theorems 3.2 and 3.3 presented in [1] is completely new and it relies on the use of the strong maximum principle as expressed by Theorem 2.1. Our approach, besides being far more general and versatile, displays the following striking fact: it turns out that the assumption yielding the regularity of a solution u of (3.1), through a control on the decay rate to zero of $f(\,\cdot\,,u(\,\cdot\,))$ at some point z, is precisely the Keller–Osserman condition (2.1) required by Theorem 2.1 so that the strong maximum principle holds for the differential equation

$$\left(\frac{v'}{\sqrt{1+(v')^2}}\right)' = f(z+v,t) \iff v'' = f(z+v,t)(1+(v')^2)^{\frac{3}{2}}, \tag{3.2}$$

satisfied by the shift $v = w - z$ of a local inverse w of u. Note that, as f is not assumed to satisfy any regularity condition, the right-hand side of (3.2), that is, the function

$$g(t,s,\xi) := f(z+s,t)(1+\xi^2)^{\frac{3}{2}},$$

may be discontinuous, besides in t, in the state variable s as well. Note that this could happen even if f were a Carathéodory function and thus g would be continuous in t and ξ, but just Lebesgue measurable with respect to s. Essentially, we establish that the validity of the strong maximum principle for equation (3.2) yields the regularity for the solutions of (3.1). As a consequence, the bounded variation solutions of (3.1) can develop singularities only when the conclusions of the strong maximum principle fail for (3.2). This appears to be a quite remarkable achievement that illuminates and clarify the otherwise apparently exotic conditions we introduced in [3].

Theorem 3.2. *Let u be a bounded variation solution of* (3.1)*. Then the following assertions hold.*

(j) *If $f(x,u(x)) \geq 0$ for almost all $x \in (a,b)$ and there exist $\delta > 0$ and $\mu \in L^1(a,a+\delta)$ such that*

- *$f(x,u(x)) \leq \mu(x)$ for almost all $x \in (a,a+\delta)$,*
- *$M(x) := \int_a^x \mu(t)\,dt > 0$ for all $x \in (a,a+\delta]$, and $\int_a^{a+\delta} \frac{1}{\sqrt{M(x)}}\,dx = +\infty$,*

then $u \in W^{2,1}_{\mathrm{loc}}[a,b) \cap W^{1,1}(a,b)$.

(jj) *If $f(x,u(x)) \geq 0$ for almost all $x \in (a,b)$ and there exist $\delta > 0$ and $\mu \in L^1(b-\delta,b)$ such that*

- *$f(x,u(x)) \leq \mu(x)$ for almost all $x \in (b-\delta,b)$,*
- *$M(x) := \int_x^b \mu(t)\,dt > 0$ for all $x \in [b-\delta,b)$, and $\int_{b-\delta}^{b} \frac{1}{\sqrt{M(x)}}\,dx = +\infty$,*

then $u \in W^{2,1}_{\mathrm{loc}}(a,b] \cap W^{1,1}(a,b)$.

(jjj) *If $f(x,u(x)) \leq 0$ for almost all $x \in (a,b)$ and there exist $\delta > 0$ and $\nu \in L^1(a,a+\delta)$ such that*

- *$f(x,u(x)) \geq \nu(x)$ for almost all $x \in (a,a+\delta)$,*
- *$N(x) := \int_a^x \nu(t)\,dt < 0$ for all $x \in (a,a+\delta]$, and $\int_a^{a+\delta} \frac{1}{\sqrt{-N(x)}}\,dx = +\infty$,*

then $u \in W^{2,1}_{\mathrm{loc}}[a,b) \cap W^{1,1}(a,b)$.

(jjjj) *If $f(x,u(x)) \leq 0$ for almost all $x \in (a,b)$ and there exist $\delta > 0$ and $\nu \in L^1(b-\delta,b)$ such that*

- *$f(x,u(x)) \geq \nu(x)$ for almost all $x \in (b-\delta,b)$,*
- *$N(x) := \int_x^b \nu(t)\,dt < 0$ for all $x \in [b-\delta,b)$, and $\int_{b-\delta}^{b} \frac{1}{\sqrt{-N(x)}}\,dx = +\infty$,*

then $u \in W^{2,1}_{\rm loc}(a,b] \cap W^{1,1}(a,b)$.

Theorem 3.3. *Let* u *be a bounded variation solution of equation* (3.1)*. Then the following statements hold.*

(h) *If there is* $c \in (a,b)$ *such that* $f(x,u(x)) \geq 0$ *for almost all* $x \in (a,c)$ *and* $f(x,u(x)) \leq 0$ *for almost all* $x \in (c,b)$ *and either there exist* $\delta > 0$ *and* $\mu \in L^1(c-\delta,c)$ *such that*

- $f(x,u(x)) \leq \mu(x)$ *for almost all* $x \in (c-\delta,c)$,
- $M(x) := \displaystyle\int\limits_x^c \mu(t)\,dt > 0$ *for all* $x \in [c-\delta,c)$, *and* $\displaystyle\int\limits_{c-\delta}^c \frac{1}{\sqrt{M(x)}}\,dx = +\infty$,

or there exist $\delta > 0$ *and* $\nu \in L^1(c,c+\delta)$ *such that*

- $f(x,u(x)) \geq \nu(x)$ *for almost all* $x \in (c,c+\delta)$,
- $N(x) := \displaystyle\int\limits_c^x \nu(t)\,dt < 0$ *for all* $x \in (c,c+\delta]$, *and* $\displaystyle\int\limits_c^{c+\delta} \frac{1}{\sqrt{-N(x)}}\,dx = +\infty$,

then $u \in W^{2,1}_{\rm loc}(a,b) \cap W^{1,1}(a,b)$.

(hh) *If there is* $c \in (a,b)$ *such that* $f(x,u(x)) \leq 0$ *for almost all* $x \in (a,c)$ *and* $f(x,u(x)) \geq 0$ *for almost all* $x \in (c,b)$ *and either there exist* $\delta > 0$ *and* $\nu \in L^1(c-\delta,c)$ *such that*

- $f(x,u(x)) \geq \nu(x)$ *for almost all* $x \in (c-\delta,c)$,
- $N(x) := \displaystyle\int\limits_x^c \nu(t)\,dt < 0$ *for all* $x \in [c-\delta,c)$, *and* $\displaystyle\int\limits_{c-\delta}^c \frac{1}{\sqrt{-N(x)}}\,dx = +\infty$,

or there exist $\delta > 0$ *and* $\mu \in L^1(c,c+\delta)$ *such that*

- $f(x,u(x)) \leq \mu(x)$ *for almost all* $x \in (c,c+\delta)$,
- $M(x) := \displaystyle\int\limits_c^x \nu(t)\,dt > 0$ *for all* $x \in (c,c+\delta]$, *and* $\displaystyle\int\limits_c^{c+\delta} \frac{1}{\sqrt{M(x)}}\,dx = +\infty$,

then $u \in W^{2,1}_{\rm loc}(a,b) \cap W^{1,1}(a,b)$.

References

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