On a Periodic Type Boundary Value Problem for a Second Order Linear Hyperbolic System

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In the rectangle $\Omega = [0,\omega_1] \times [0,\omega_2]$ consider the problem

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y + q(x,y), \tag{1}$$

$$u(0,y) = Au(\omega_1,y) + \varphi(y), \quad u(x,0) = Bu(x,\omega_2) + \psi(x), \tag{2}$$

where $P_j \in C(\Omega;\mathbb{R}^{n\times n})$ $(j = 0,1,2)$, $q \in C(\Omega;\mathbb{R}^n)$, $A,B \in \mathbb{R}^{n\times n}$, $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$ and $\psi \in C^1([0,\omega_1];\mathbb{R}^n)$.

Problem (1), (2) is not well-posed, since for its solvability the vector functions φ and ψ should satisfy some compatibility condition. For example, if

$$AB = BA, \tag{3}$$

then for solvability of problem (1), (2) it is necessary that

$$\varphi(0) - B\varphi(\omega_2) = \psi(0) - A\psi(\omega_1). \tag{4}$$

Indeed, for an arbitrary $u \in C(\Omega;\mathbb{R}^n)$, in view of equality (3), we have

$$h \circ \ell(u) = \ell \circ h(u), \tag{5}$$

where

$$\ell(z) = z(0) - Az(\omega_1), \quad h(z) = z(0) - Bz(\omega_2).$$

Consequently, if $u(x,y)$ satisfies condition (2), then equality (5) implies

$$\psi(0) - A\psi(\omega_1) = \ell \circ h(u) = h \circ \ell(u) = \varphi(0) - B\varphi(\omega_2).$$

Notice that, if $u \in C^{1,1}(\Omega;\mathbb{R}^n)$ satisfies condition (2), then

$$h(u_x(x,\,\cdot\,)) = \psi'(x).$$

Therefore,

$$u(0,y) = Au(\omega_1,y) + \varphi(y), \quad u_x(x,0) = Bu_x(x,\omega_2) + \psi'(x). \tag{6}$$

Along with system (1) and conditions (2) and (6) consider their corresponding homogeneous system and conditions

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y, \tag{1_0}$$

$$u(0,y) = Au(\omega_1,y), \quad u(x,0) = Bu(x,\omega_2) \tag{2_0}$$

and

$$u(0,y) = Au(\omega_1,y), \quad u_x(x,0) = Bu_x(x,\omega_2). \tag{6_0}$$

Let $Y(y;x)$ be the fundamental matrix of the differential system

$$\frac{dz}{dy} = P_1(x,y)z,$$

satisfying the initial condition

$$Y(0;x) = I,$$

where I is $n \times n$ identity matrix. By $X(x;y)$ denote the fundamental matrix of the differential system

$$\frac{dz}{dx} = P_2(x,y)z,$$

satisfying the initial condition

$$X(0;y) = I.$$

If problem

$$\frac{dz}{dx} = P_2(x,y)z, \quad z(0) - Az(\omega_1) = 0,$$

has only the trivial solution, then by $G_1(x,s;y)$ denote its Green's matrix, and if problem

$$\frac{dz}{dy} = P_1(x,y)z, \quad z(0) - Bz(\omega_2) = 0$$

has only the trivial solution, then by $G_2(y,t;x)$ denote its Green's matrix.

Theorem 1. *Let the problem*

$$z' = 0, \quad z(0) = A\,z(\omega_1) \tag{7}$$

have only the trivial solution, and let the following inequalities hold:

$$\det\big(I - Y(\omega_2;x)\,B\big) \neq 0 \ \ \text{for} \ \ x \in [0,\omega_1], \tag{8}$$

$$\det\big(I - X(\omega_1;y)\,A\big) \neq 0 \ \ \text{for} \ \ y \in [0,\omega_2]. \tag{9}$$

Then problem $(1),(6)$ *has the Fredholm property. Furthermore, if problem* $(1_0),(6_0)$ *has only the trivial solution, then problem* $(1),(6)$ *has a unique solution* u u *admitting the estimate*

$$\|u\|_{C^{1,1}(\Omega)} \leq M\Big(\|q\|_{C(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C^1([0,\omega_1])}\Big), \tag{10}$$

where M *is a positive number independent of* φ, ψ *and* q.

Definition. Problem $(1),(6)$ is called well-posed if for every $\varphi \in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi \in C^1([0,\omega_1];\mathbb{R}^n)$ and $q \in C(\Omega;\mathbb{R}^n)$ it has a unique solution u admitting estimate (10), where M is a positive number independent of φ, ψ and q.

Theorem 2. *If problem* $(1),(6)$ *is well-posed, then problem* $(7),(8)$ *has only the trivial solution and inequalities* (9) *and* (10) *hold.*

Theorem 3. *Let inequalities* (9) *and* (10) *hold, and let the matrices* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{n\times n}$ *satisfy condition* (3). *Then:*

(i) *the space of solutions of problem* $(1_0),(2_0)$ *is finite dimensional;*

(ii) *if the homogeneous problem* $(1_0),(2_0)$ *has only the trivial solution, then problem* $(1),(2)$ *is uniquely solvable if and only if the compatibility condition* (4) *holds.*

Corollary 1. *Let* $P_1(x,y) \equiv P_1(x)$, $P_2(x,y) \equiv P_2(y)$, *let the problem* (7) *have only the trivial solution, and let*

$$\det\big(I - \exp(\omega_2 P_1(x))\, B\big) \neq 0 \ \ \text{for} \ \ x \in [0,\omega_1], \tag{11}$$
$$\det\big(I - \exp(\omega_1 P_2(y))\, A\big) \neq 0 \ \ \text{for} \ \ y \in [0,\omega_2]. \tag{12}$$

Then problem (1), (6) *has the Fredholm property.*

Corollary 2. *Let problem* (7) *have only the trivial solution, and let there exist* $\sigma_i \in \{-1,1\}$ $(i=1,2)$ *such that*

$$\begin{aligned} \sigma_1(A^T A - I) \ \ &\text{is positive semi-definite,} \\ \sigma_1 P_1(x,y) \ \ &\text{is positive definite for} \ \ (x,y) \in \Omega \end{aligned}$$

and

$$\begin{aligned} \sigma_2\big(B^T B - I\big) \ \ &\text{is positive semi-definite,} \\ \sigma_2 P_2(x,y) \ \ &\text{is positive definite for} \ \ (x,y) \in \Omega. \end{aligned}$$

Then problem (1) (6) *has the Fredholm property.*

Theorem 4. *Let conditions* (8) *and* (9) *hold, let problem* (7) *have only the trivial solution, let* $\Gamma \in \mathbb{R}_+^{n\times n}$ *be a nonnegative matrix with the* ***spectral radius less than 1***, *and let either*

$$P_1 \in C^{1,0}(\Omega;\mathbb{R}^{n\times n}), \ \ P_1(0,y) = P_1(\omega_1,y), \ \ P_1(\omega_1,y)\, A = A\, P_1(\omega_1,y), \tag{13}$$

and

$$\int_0^{\omega_2}\int_0^{\omega_1} \Big| G_2(y,t;x) G_1(x,s;t) \Big(P_0(s,t) + P_2(s,t) P_1(s,t) - \frac{\partial}{\partial s} P_1(s,t) \Big) \Big| \, ds\, dt \le \Gamma, \tag{14}$$

or

$$P_2 \in C^{0,1}(\Omega;\mathbb{R}^{n\times n}), \ \ P_2(x,0) = P_2(x,\omega_2), \ \ P_2(x,\omega_2)\, B = B\, P_2(x,\omega_2), \tag{15}$$

and

$$\int_0^{\omega_1}\int_0^{\omega_2} \Big| G_1(x,s;y) G_2(y,t;s) \Big(P_0(s,t) + P_1(s,t) P_2(s,t) - \frac{\partial}{\partial t} P_2(s,t) \Big) \Big| \, dt\, ds \le \Gamma. \tag{16}$$

Then problem (1) (6) *is uniquely solvable.*

Consider the system

$$u_{xy} = P_0(x,y) u + u_x + u_y + q(x,y). \tag{17}$$

Theorem 5. *Let problem* (7) *have only the trivial solution,*

$$\begin{gathered} P_0(x,y) = P_0^T(x,y) \ \ \text{for} \ \ x,y) \in \Omega, \\ A^T A - I \ \ \text{be positive semi-definite,} \\ B^T B - I \ \ \text{be positive semi-definite,} \\ I - A^T A - B^T B + B^T A^T A B \ \ \text{be positive semi-definite,} \end{gathered}$$

and let ***one of the following three conditions*** *hold:*

(i) $P_0 \in C^{1,0}(\Omega;\mathbb{R}^{n\times n})$ *and*

$$\begin{aligned}
P_0(\omega_1,y)-A^TP_0(0,y)A \ \ &\textit{is positive semi-definite for } y\in[0,\omega_2],\\
P_0(x,y)+\frac{1}{2}\,\frac{\partial P_0(x,y)}{\partial x} \ \ &\textit{is negative semi-definite for } (x,y)\in\Omega,\\
\int\limits_0^{\omega_1} P_0(s,y)\,ds \ \ &\textit{is negative definite for } y\in[0,\omega_2];
\end{aligned}$$

(ii) $P_0 \in C^{0,1}(\Omega;\mathbb{R}^{n\times n})$ *and*

$$\begin{aligned}
P_0(x,\omega_2)-B^TP_0(x,\omega_2)B \ \ &\textit{is positive semi-definite for } x\in[0,\omega_1],\\
P_0(x,y)+\frac{1}{2}\,\frac{\partial P_0(x,y)}{\partial y} \ \ &\textit{is negative semi-definite for } (x,y)\in\Omega,\\
\int\limits_0^{\omega_2} P_0(x,t)\,dt \ \ &\textit{is negative definite for } x\in[0,\omega_1];
\end{aligned}$$

(iii) $P_0 \in C^{1}(\Omega;\mathbb{R}^{n\times n})$ *and*

$$\begin{aligned}
P_0(\omega_1,y)-A^TP_0(0,y)A \ \ &\textit{is positive semi-definite for } y\in[0,\omega_2],\\
P_0(x,\omega_2)-B^TP_0(x,\omega_2)B \ \ &\textit{is positive semi-definite for } x\in[0,\omega_1],\\
P_0(x,y)+\frac{1}{4}\Big(\frac{\partial P_0(x,y)}{\partial x}+\frac{\partial P_0(x,y)}{\partial y}\Big) \ \ &\textit{is negative semi-definite for } (x,y)\in\Omega,\\
\int\limits_0^{\omega_1}\int\limits_0^{\omega_2} P_0(s,t)\,dt\,ds \ \ &\textit{is negative definite.}
\end{aligned}$$

Then problem (17), (6) *is uniquely solvable.*

Consider the case, where $P_i(x,y)\equiv P_i$ $(i=0,1,2)$ and $A=I$, i.e. consider the problem

$$u_{xy}=P_0u+P_1u_x+P_2u_y+q(x,y), \tag{18}$$
$$u(0,y)=u(\omega_1,y)+\varphi(y), \quad u(x,0)=Bu(x,\omega_2)+\psi(x). \tag{19}$$

Theorem 6. *Let*

$$\det\big(I-\exp(\omega_2P_1)B\big)\neq 0,$$
$$\det\big(I-\exp(\omega_1P_2)\big)\neq 0,$$

and let the compatibility condition

$$\varphi(0)-B\varphi(\omega_2)=\psi(0)-\psi(\omega_1)$$

hold. Then problem (18), (19) *is uniquely solvable if and only if*

$$\det\big(I-\exp(\omega_1\Lambda_k)\,B\big)\neq 0 \ \ \textit{for } k\in\mathbb{Z},$$

where

$$\Lambda_k=\Big(i\,\frac{2\pi}{\omega_1}\,kI-P_2\Big)\Big(P_0+i\,\frac{2\pi}{\omega_1}\,kP_1\Big).$$

Consider the case $n=1$. For the equation

$$u_{xy} = p_0(y)u + p_1(y)u_x + p_2(y)u_y + q(x,y) \tag{20}$$

consider the boundary conditions

$$u(0,y) = u(\omega_1, y), \quad u(x,0) = bu(x,\omega_2). \tag{21}$$

Theorem 7. *Let the following inequalities hold:*

$$p_0(y)\,p_1(y)\,p_2(y) < 0 \ \ \textit{for} \ \ y \in [0,\omega] \tag{22}$$

and

$$(1-b)\,p_1(y) \geq 0 \ \ \textit{for} \ \ y \in [0,\omega].$$

Then problem (20), (21) *is uniquely solvable. In particular, if* $b=1$*, then the doubly periodic problem* (20), (21) *is uniquely solvable if inequality* (22) *holds.*

References

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