On a Dirichlet Type Boundary Value Problem in an Orthogonally Convex Piecewise Smooth Cylinder for a Class of Quasilinear Partial Differential Equations

Tariel Kiguradze, Reemah Alhuzally

Florida Institute of Technology, Melbourne, USA

E-mails: tkigurad@fit.edu; ralhuzally2015@my.fit.edu

In the orthogonally convex cylinder $E=\{(x_1,x_2,x_3)\in\Omega:\ (x_1,x_2)\in D,\ x_3\in(0,\omega_3)\}$, where

$$D=\Big\{(x_1,x_2)\in\Omega:\ \ x_1\in(0,\omega_1),\ \ x_2\in(\gamma_1(x_1),\gamma_2(x_1))\Big\}$$
$$=\Big\{(x_1,x_2)\in\Omega:\ \ x_2\in(0,\omega_2),\ \ x_1\in(\eta_1(x_2),\eta_2(x_2))\Big\},$$

consider the boundary value problem

$$u^{(\mathbf{2})}=f\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{2}}[u]\big), \tag{1}$$

$$u\,\nu_1\Big|_{\partial E}=\nu_1(\mathbf{x})\psi_1(\mathbf{x}),\quad u^{(2,0,0)}\,\nu_2\Big|_{\partial E}=\nu_2(\mathbf{x})\psi_2(\mathbf{x}),\quad u^{(2,2,0)}\,\nu_3\Big|_{\partial E}=\nu_3(\mathbf{x})\psi_3(\mathbf{x}). \tag{2}$$

Here $\mathbf{x}=(x_1,x_2,x_3)$, ∂E is the boundary of E, and $\boldsymbol{\nu}(\mathbf{x})=(\nu_1(\mathbf{x}),\nu_2(\mathbf{x}),\nu_3(\mathbf{x}))$ is the outward unit normal vector at point $\mathbf{x}\in\partial E$, $\mathbf{2}=(2,2,2)$, $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index,

$$\mathcal{D}^{\mathbf{2}}[u]=(u^{(\boldsymbol{\alpha})})_{\boldsymbol{\alpha}\le\mathbf{2}},\quad \widetilde{\mathcal{D}}^{\mathbf{2}}[u]=(u^{(\boldsymbol{\alpha})})_{\boldsymbol{\alpha}<\mathbf{2}},\quad u^{(\boldsymbol{\alpha})}(\mathbf{x})=\frac{\partial^{\alpha_1+\alpha_2+\alpha_3}u(\mathbf{x})}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\,,$$

$f(\mathbf{x},\mathbf{z})$ is a continuous function on $\overline{\mathbf{E}}\times\mathbb{R}^{23}$, $\mathbf{z}=(z_{000},z_{100},z_{010},z_{001},\dots,z_{221},z_{212},z_{122})$, $\psi_i\in C(\overline{E})$ $(i=1,2,3)$ and $\overline{E}$ is the closure of E.

By a solution of problem (1),(2) we understand a *classical* solution, i.e., a function $u\in C^{2,2,2}(E)$ having continuous on $\overline{E}$ partial derivatives $u^{(2,0,0)}$ and $u^{(2,2,0)}$, and satisfying equation (1) and the boundary conditions (2) everywhere in E and ∂E, respectively.

Throughout the paper the following notations will be used:

$\mathbf{0}=(0,0,0)$, $\mathbf{1}=(1,1,1)$, $\boldsymbol{\alpha}_i=(0,\dots,\alpha_i,\dots,0)$, $\boldsymbol{\alpha}_{ij}=\boldsymbol{\alpha}_i+\boldsymbol{\alpha}_j$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)<\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\iff\alpha_i\le\beta_i$ $(i=1,2,3)$ and $\boldsymbol{\alpha}\ne\boldsymbol{\beta}$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)\le\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\iff\boldsymbol{\alpha}<\boldsymbol{\beta}$, or $\boldsymbol{\alpha}=\boldsymbol{\beta}$.

$\|\boldsymbol{\alpha}\|=|\alpha_1|+|\alpha_2|+|\alpha_3|$.

$\boldsymbol{\Xi}=\big\{\boldsymbol{\sigma}\big|\ \mathbf{0}<\boldsymbol{\sigma}<\mathbf{1}\big\}$.

$\boldsymbol{\Upsilon}_{\mathbf{2}}=\big\{\boldsymbol{\alpha}<\mathbf{2}:\ \ \alpha_i=2\ \text{ for some }\ i\in\{1,2,3\}\big\}$.

The variables $z_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha}\in\boldsymbol{\Upsilon}_{\mathbf{2}})$ are called ***principal phase variables*** of the function $f(\mathbf{x},\mathbf{z})$.

$\mathbf{z}=\big(z_{\boldsymbol{\alpha}}\big)_{\boldsymbol{\alpha}<\mathbf{2}}$;$f_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{z})=\frac{\partial f(\mathbf{x},\mathbf{z})}{\partial z_{\boldsymbol{\alpha}}}$.

$\operatorname{supp}\boldsymbol{\alpha}=\{i:\ \alpha_i>0\}$.

$\mathbf{x}_{\boldsymbol{\alpha}}=(\chi(\alpha_1)\,x_1,\chi(\alpha_2)\,x_2,\chi(\alpha_3)x_3)$, where $\chi(\alpha)=0$ if $\alpha=0$, and $\chi(\alpha)=1$ if $\alpha>0$.

$\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}=\mathbf{x}-\mathbf{x}_{\boldsymbol{\alpha}}$.

$\mathbf{x}_{\boldsymbol{\alpha}}$ will be identified with $(x_{i_1},\dots,x_{i_l})$, where $\{i_1,\dots,i_l\}=\operatorname{supp}\boldsymbol{\alpha}$. Furthermore, $\mathbf{x}_{\boldsymbol{\alpha}}$ will be identified with $(\mathbf{x}_{\boldsymbol{\alpha}},\widehat{\mathbf{0}}_{\boldsymbol{\alpha}})$, and $\mathbf{x}$ will be identified with $(\mathbf{x}_{\boldsymbol{\alpha}},\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})$, or with $(\mathbf{x}_{\boldsymbol{\alpha}},\mathbf{x}_{\widehat{\boldsymbol{\alpha}}})$.

$\Omega_{\boldsymbol{\sigma}}=[0,\omega_{i_1}]\times\dots\times[0,\omega_{i_l}]$, where $\{i_1,\dots,i_l\}=\operatorname{supp}\boldsymbol{\sigma}$.

$\Omega_{ij}=(0,\omega_i)\times(0,\omega_j)$ $(1\le i<j\le 3)$.

$C^{\mathbf{m}}(\overline{E})$ is the Banach space of functions $u:\overline{E}\to\mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha}\le\mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}(\overline{E})}=\sum_{\boldsymbol{\alpha}\le\mathbf{m}}\|u^{(\boldsymbol{\alpha})}\|_{C(\overline{E})}.$$

$\widetilde{C}^{\mathbf{m}}(\overline{E})$ is the Banach space of functions $u:\overline{E}\to\mathbb{R}$, having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha}<\mathbf{m})$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}(\overline{E})}=\sum_{\boldsymbol{\alpha}<\mathbf{m}}\|u^{(\boldsymbol{\alpha})}\|_{C(\overline{E})}.$$

If $u_0\in C^{\mathbf{m}}(\overline{E})$ and $r>0$, then $\mathbf{B}^{\mathbf{m}}(u_0;r)=\big\{u\in C^{\mathbf{m}}(\overline{E}):\ \|u-u_0\|_{C^{\mathbf{m}}}\le r\big\}$.

If $u_0\in \widetilde{C}^{\mathbf{m}}(\overline{E})$ and $r>0$, then $\widetilde{\mathbf{B}}^{\mathbf{m}}(u_0;r)=\big\{u\in \widetilde{C}^{\mathbf{m}}(\overline{E}):\ \|u-u_0\|_{\widetilde{C}^{\mathbf{m}}}\le r\big\}$.

The boundary conditions (2) can be written int the following way

$$\begin{gathered}u(\eta_k(x_2),x_2,x_3)=\varphi_{1k}(x_2,x_3),\quad u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=\varphi_{2k}(x_1,x_3),\\ u^{(2,2,0)}(x_1,x_2,(k-1)\,\omega_3)=\varphi_{3k}(x_1,x_2)\ \ (k=1,2),\end{gathered}\tag{3}$$

where

$$\begin{gathered}\varphi_{1k}(x_2,x_3)=\psi_1(\eta_k(x_2),x_2,x_3),\quad \varphi_{2k}(x_1,x_3)=\psi_2(x_1,\gamma_k(x_1),x_3),\\ \varphi_{3k}(x_1,x_2)=\psi_3(x_1,x_2,(k-1)\,\omega_3)\ \ (k=1,2).\end{gathered}\tag{4}$$

Along with problem (1), (3) consider the linear homogeneous problem

$$u^{(\mathbf{2})}=\sum_{\alpha<\mathbf{2}}p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})},\tag{1_0}$$

$$u(\eta_k(x_2),x_2,x_3)=0,\ \ u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=0,\ \ u^{(2,2,0)}(x_1,x_2,(k-1)\,\omega_3)=0\ \ (k=1,2).\tag{3_0}$$

For each $\boldsymbol{\sigma}\in\boldsymbol{\Xi}$ in the domain $\Omega_{\boldsymbol{\sigma}}$ consider the homogeneous boundary value problem depending on the parameter $\mathbf{x}_{\widehat{\boldsymbol{\sigma}}}\in\Omega_{\widehat{\boldsymbol{\sigma}}}$:

$$v^{(2,0,0)}=p_{022}(\mathbf{x}_1,\widehat{\mathbf{x}}_1)v+p_{122}(\mathbf{x}_1,\widehat{\mathbf{x}}_1)v^{(1,0,0)},\tag{1_{100}}$$

$$v(\eta_1(\mathbf{x}_2),\widehat{\mathbf{x}}_1)=0,\quad v(\eta_2(\mathbf{x}_2),\widehat{\mathbf{x}}_1)=0;\tag{2_{100}}$$

$$v^{(0,2,0)}=p_{202}(\mathbf{x}_2,\widehat{\mathbf{x}}_2)v+p_{212}(\mathbf{x}_2,\widehat{\mathbf{x}}_2)v^{(0,1,0)},\tag{1_{010}}$$

$$v(\gamma_1(\mathbf{x}_1),\widehat{\mathbf{x}}_2)=0,\quad v(\gamma_2(\mathbf{x}_1),\widehat{\mathbf{x}}_2)=0;\tag{3_{010}}$$

$$v^{(0,0,2)}=p_{220}(\mathbf{x}_3,\widehat{\mathbf{x}}_3)v+p_{221}(\mathbf{x}_3,\widehat{\mathbf{x}}_3)v^{(0,0,1)},\tag{1_{001}}$$

$$v(0,\widehat{\mathbf{x}}_3)=0,\quad v(\omega_3,\widehat{\mathbf{x}}_3)=0;\tag{3_{001}}$$

$$v^{(\mathbf{2}_{12})}=\sum_{\boldsymbol{\alpha}<\mathbf{2}_{12}}p_{\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{12}}(\mathbf{x}_{12},\widehat{\mathbf{x}}_{12})v^{(\boldsymbol{\alpha})},\tag{1_{110}}$$

$$v(\eta_k(\mathbf{x}_2),\widehat{\mathbf{x}}_{12})=0,\quad v^{(2,0,0)}(\gamma_k(\mathbf{x}_1),\widehat{\mathbf{x}}_{12})=0\ \ (k=1,2);\tag{3_{110}}$$

$$v^{(\mathbf{2}_{13})}=\sum_{\boldsymbol{\alpha}<\mathbf{2}_{13}} p_{\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{13}}(\mathbf{x}_{13},\widehat{\mathbf{x}}_{13})v^{(\boldsymbol{\alpha})}, \tag{1_{101}}$$

$$v(\eta_k(\mathbf{x}_2),\widehat{\mathbf{x}}_{13})=0,\quad v^{(2,0,0)}((k-1)\,\omega_3,\widehat{\mathbf{x}}_{13})=0\ \ (k=1,2); \tag{3_{101}}$$

$$v^{(\mathbf{2}_{23})}=\sum_{\boldsymbol{\alpha}<\mathbf{2}_{23}} p_{\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{23}}(\mathbf{x}_{23},\widehat{\mathbf{x}}_{23})v^{(\boldsymbol{\alpha})}, \tag{1_{011}}$$

$$v(\gamma_k(\mathbf{x}_1),\widehat{\mathbf{x}}_{23})=0,\quad v^{(2,0,0)}((k-1)\,\omega_3,\widehat{\mathbf{x}}_{23})=0\ \ (k=1,2). \tag{3_{011}}$$

Definition 1. Problem $(1_{\boldsymbol{\sigma}}),(3_{\boldsymbol{\sigma}})$ $(\boldsymbol{\sigma}\in\boldsymbol{\Xi})$ is called $\boldsymbol{\sigma}$-associated problem of problem $(1_0),(3_0)$.

Along with problem $(1),(2)$ consider the perturbed problem

$$u^{(\mathbf{2})}=f\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{2}}[u]\big)+\widetilde{f}\big(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{2}}[u]\big), \tag{5}$$

$$\begin{gathered}u(\eta_k(x_2),x_2,x_3)=\varphi_{1k}(x_2,x_3)+\widetilde{\varphi}_{1k}(x_2,x_3),\\ u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=\varphi_{2k}(x_1,x_3)+\widetilde{\varphi}_{2k}(x_1,x_3),\\ u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3)=\varphi_{3k}(x_1,x_2)+\widetilde{\varphi}_{3k}(x_1,x_2)\ \ (k=1,2),\end{gathered} \tag{6}$$

where

$$\widetilde{\varphi}_{1k}(x_2,x_3)=\widetilde{\psi}_1(\eta_k(x_2),x_2,x_3),\quad \widetilde{\varphi}_{2k}(x_1,x_3)=\widetilde{\psi}_2(x_1,\gamma_k(x_1),x_3),$$
$$\widetilde{\varphi}_{3k}(x_1,x_2)=\widetilde{\psi}_3(x_1,x_2,(k-1)\,\omega_3)\ \ (k=1,2). \tag{7}$$

A vector function $(\widetilde{f};\widetilde{\psi}_1,\widetilde{\psi}_2,\widetilde{\psi}_3)$ is said to be an *admissible perturbation* if $\widetilde{f}\in C(\Omega\times\mathbb{R}^{23})$ is locally Lipschitz continuous with respect to the *principal* phase variables, $\widetilde{\psi}_1\in C^{2,2,2}(\overline{E})$, $\widetilde{\psi}_2\in C^{0,2,2}(\overline{E})$ and $\widetilde{\psi}_2\in C^{0,0,2}(\overline{E})$.

Definition 2. Let u_0 be a solution of problem $(1),(2)$, and $r>0$. We say that problem $(1),(2)$ is (u_0,r)-*well-posed*, if:

(I) $u_0(\mathbf{x})$ is the unique solution of problem $(1),(2)$ in the ball $\widetilde{\mathbf{B}}^{\mathbf{2}}(u_0;r)$;

(II) there exist positive constant δ_0 and an increasing continuous $\varepsilon:[0,\delta_0]\to[0,+\infty)$ such that $\varepsilon(0)=0$ and for any $\delta\in(0,\delta_0]$ and an arbitrary admissible perturbation $(\widetilde{f};\widetilde{\psi}_1,\widetilde{\psi}_2,\widetilde{\psi}_3)$ satisfying the conditions

$$|\widetilde{f}_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{z})|\le\delta_0\ \text{ for }\ (\mathbf{x},\mathbf{z})\in\Omega\times\mathbb{R}^{23}\ \ (\boldsymbol{\alpha}\in\boldsymbol{\Upsilon}_{\mathbf{m}}), \tag{8}$$

$$|\widetilde{f}(\mathbf{x},\mathbf{z})|\le\delta\ \text{ for }\ (\mathbf{x},\mathbf{z})\in\Omega\times\mathbb{R}^{23}, \tag{9}$$

$$\|\widetilde{\psi}_1\|_{C^{2,2,2}(\overline{E})}+\|\widetilde{\psi}_2\|_{C^{0,2,2}(\overline{E})}+\|\widetilde{\psi}_3\|_{C^{0,0,2}(\overline{E})}\le\delta, \tag{10}$$

problem $(4),(5)$ has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{2}}(u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{2}}(u_0;\varepsilon(\delta))$.

Definition 3. Let u_0 be a solution of problem $(1),(2)$, and $r>0$. We say that problem $(1),(2)$ is *strongly* (u_0,r)-*well-posed*, if:

(I) $u_0(\mathbf{x})$ is the unique solution of problem $(1),(2)$ in the ball $\widetilde{\mathbf{B}}^{\mathbf{2}}(u_0,r)$;

(II) there exist a positive constants δ_0 and M such that for any $\delta\in(0,\delta_0]$ and an arbitrary admissible perturbation $(\widetilde{f};\widetilde{\psi}_1,\widetilde{\psi}_2,\widetilde{\psi}_3)$ satisfying conditions (7)–(9), problem $(4),(5)$ has at least one solution in the ball $\widetilde{\mathbf{B}}^{\mathbf{2}}(u_0;r)$, and each such solution belongs to the ball $\widetilde{\mathbf{B}}^{\mathbf{2}}(u_0;M\delta)$.

Definition 4. Problem $(1),(2)$ is called well-posed (strongly well-posed), if it is (u_0,r)-well-posed (strongly (u_0,r)-well-posed) for every $r>0$.

Definition 5. A solution u_0 of problem $(1),(2)$ is called *strongly isolated*, if problem $(1),(2)$ is strongly (u_0,r)-well-posed for some $r>0$.

Theorem 1. *Let*

$$\eta_k\in C^2([0,\omega_2])\ \ (k=1,2), \tag{11}$$

let the function $f(\mathbf{x},\mathbf{z})$ be continuously differentiable with respect to the phase variables, and let there exist functions $P_{i\boldsymbol{\alpha}}(\mathbf{x})\in C(\overline{E})$ $(\boldsymbol{\alpha}<\mathbf{2};\ i=1,2)$ such that:

(E_1)

$$P_{1\boldsymbol{\alpha}}(\mathbf{x})\le f_{\boldsymbol{\alpha}}(\mathbf{x},\mathbf{z})\le P_{2\boldsymbol{\alpha}}(\mathbf{x})\ \ \text{for}\ \ (\mathbf{x},\mathbf{Z})\in\overline{E}\times\mathbb{R}^{23}\quad(\boldsymbol{\alpha}<\mathbf{2}); \tag{12}$$

(E_2) *For every $\boldsymbol{\sigma}\in\Xi\cup\{\mathbf{1}\}$,*[1] *$\widehat{\mathbf{x}}_{\boldsymbol{\sigma}}\in\overline{E}_{\widehat{\boldsymbol{\sigma}}}$ and arbitrary measurable functions $p_{\boldsymbol{\alpha}}\in L^\infty(E_{\boldsymbol{\sigma}})$ $(\boldsymbol{\alpha}<\mathbf{2}_{\boldsymbol{\sigma}})$ satisfying the inequalities*

$$P_{1\,\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{\boldsymbol{\sigma}}}(\mathbf{y},\widehat{\mathbf{x}}_{\boldsymbol{\sigma}})\le p_{\boldsymbol{\alpha}}(\mathbf{y})\le P_{2\,\boldsymbol{\alpha}+\widehat{\mathbf{2}}_{\boldsymbol{\sigma}}}(\mathbf{y},\widehat{\mathbf{x}}_{\boldsymbol{\sigma}})\ \ \text{for}\ \ \mathbf{y}\in E_{\boldsymbol{\sigma}}\ \ (\boldsymbol{\alpha}<\mathbf{2}_{\boldsymbol{\sigma}}), \tag{13}$$

the $\boldsymbol{\sigma}$-associated problem $(1_{\boldsymbol{\sigma}})$, $(3_{\boldsymbol{\sigma}})$ has only the trivial solution in $AC^{\mathbf{1}}(\overline{E}_{\boldsymbol{\sigma}})$;

(E_3) *the problem*

$$u^{(\mathbf{2})}=\sum_{\boldsymbol{\alpha}<\mathbf{2}}P_{1\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})},$$

$$u(\eta_k(x_2),x_2,x_3)=0,\ \ u^{(2,0,0)}(x_1,\gamma_k(x_1),x_3)=0,\ \ u^{(2,2,0)}(x_1,x_2,(k-1)\omega_3)=0\ \ (k=1,2),$$

is well-posed. Then problem $(1),(3)$ is strongly well-posed, and its solution belongs to $C^{2,2,2}(\overline{E})$.

Theorem 2. *Let condition (11) hold, the function $f(\mathbf{x},\mathbf{z})$ be continuously differentiable with respect to the phase variables, and let u_0 be a solution of problem $(1),(3)$. Then problem $(1),(3)$ is strongly (u_0,r)-well-posed for some $r>0$ if and only if the linear homogeneous problem $(1_0),(3_0)$ is well-posed, where*

$$p_{\boldsymbol{\alpha}}(\mathbf{x})=f_{\boldsymbol{\alpha}}(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{2}}[u_0(\mathbf{x})])\ \ (\boldsymbol{\alpha}<\mathbf{2}).$$

Consider the equations

$$u^{(\mathbf{2})}=f(\mathbf{x},\widetilde{\mathcal{D}}^{\mathbf{2}}[u])+q(\mathbf{x},\mathcal{D}^{\mathbf{1}}[u]), \tag{14}$$

$$u^{(\mathbf{2})}=\big(p_1(\mathbf{x})u^{(1,0,0)}\big)^{(1,0,0)}+\big(p_2(\mathbf{x})u^{(0,1,0)}\big)^{(0,1,0)}+\big(p_3(\mathbf{x})u^{(0,0,1)}\big)^{(0,0,1)}+p_0(\mathbf{x},u), \tag{15}$$

$$u^{(\mathbf{2})}=\sum_{\boldsymbol{\alpha}<\mathbf{2}}\rho_{\boldsymbol{\alpha}}\big(\mathbf{x},\mathcal{D}^{\mathbf{1}}[u]\big)\,u^{(\boldsymbol{\alpha})}+q\big(\mathbf{x},\mathcal{D}^{\mathbf{1}}[u]\big), \tag{16}$$

$$\begin{aligned}u^{(\mathbf{2})}=\big(p_1(\mathbf{x},u)u^{(1,0,0)}\big)^{(1,0,0)}&+\big(p_2(\mathbf{x},u)u^{(0,1,0)}\big)^{(0,1,0)}\\&+\big(p_3(\mathbf{x},u)u^{(0,0,1)}\big)^{(0,0,1)}+p_0(\mathbf{x},u)+q\big(\mathbf{x},\mathcal{D}^{\mathbf{1}}[u]\big).\end{aligned} \tag{17}$$

Theorem 3. *Let the function f satisfy all of the conditions of Theorem 1, and let $q\in C(\Omega\times\mathbb{R}^8)$ be such that*

$$\lim_{\|\mathbf{z}\|\to+\infty}\frac{|q(\mathbf{x},\mathbf{z})|}{\|\mathbf{z}\|}=0\ \ \text{uniformly on}\ \overline{E}. \tag{18}$$

Then problem $(14),(3)$ is solvable and its every solution belongs to $C^{2,2,2}(\overline{E})$.

[1] If $\boldsymbol{\sigma}=\mathbf{1}$, then by $(1_{\boldsymbol{\sigma}}),(3_{\boldsymbol{\sigma}})$ we understand the homogeneous problem (1_0), (3_0).

Corollary. *Let condition* (11) *hold,*

$$(-1)^{k-1}\eta_k''(x_2) \geq 0 \ \ for \ \ x_2 \in (0,\omega_2) \ \ (k=1,2), \tag{19}$$

and let $p_1 \in C^{1,0,0}(\overline{E})$, $p_2 \in C^{0,1,0}(\overline{E})$, $p_3 \in C^{0,0,1}(\overline{E})$, $p_0 \in C(\overline{E}\times\mathbb{R})$ *satisfy the inequalities*

$$p_1(\mathbf{x}) \leq 0, \quad p_2(\mathbf{x}) \leq 0, \quad p_3(\mathbf{x}) \leq 0 \ \ for \ \ \mathbf{x} \in \overline{E}, \tag{20}$$

$$\big(p_0(\mathbf{x},z_1) - p_0(\mathbf{x},z_2)\big)(z_1-z_2) \geq 0 \ \ for \ \ (\mathbf{x}_1,x_2,z) \in \overline{E}\times\mathbb{R}. \tag{21}$$

Then problem (15), (3) *is strongly well-posed and its solution belongs to* $C^{2,2,2}(\overline{E})$.

Theorem 4. *Let conditions* (11) *and* (18) *hold, and let there exist functions* $P_{i\boldsymbol{\alpha}}(\mathbf{x}) \in C(\overline{E})$ $(\boldsymbol{\alpha} < \mathbf{2}$; $i=1,2)$ *satisfying conditions* (E_2) *and* (E_2) *of Theorem 1 such that:*

$$P_{1\alpha}(\mathbf{x}) \leq \rho_\alpha(\mathbf{x},\mathbf{z}) \leq P_{2\alpha}(\mathbf{x}) \ \ for \ \ (\mathbf{x},\mathbf{z}) \in \overline{E}\times\mathbb{R}^8 \ \ (\boldsymbol{\alpha} < \mathbf{2}). \tag{22}$$

Then problem (16), (3) *is solvable and its every solution belongs to* $C^{2,2,2}(\overline{E})$.

Theorem 5. *Let conditions* (11), (18) *and* (19) *hold, and let* $p_k \in C^1(\overline{E}\times\mathbb{R})$ $(k=1,2,3)$ *satisfy the inequalities*

$$p_k(\mathbf{x},z) \leq 0 \ \ for \ \ (\mathbf{x},z) \in \overline{E}\times\mathbb{R} \ \ (k=1,2,3),$$
$$p_0(\mathbf{x},z)z \geq 0 \ \ for \ \ (\mathbf{x},z) \in \overline{\Omega}\times\mathbb{R}.$$

Then problem (17), (3_0) *is solvable and its every solution belongs to* $C^{2,2,2}(\overline{E})$.

References

[1] T. Kiguradze, On the correctness of the Dirichlet problem in a characteristic rectangle for fourth order linear hyperbolic equations. *Georgian Math. J.* **6** (1999), no. 5, 447–470.

[2] T. Kiguradze, On the Dirichlet problem in a characteristic rectangle for fourth order linear singular hyperbolic equations. *Georgian Math. J.* **6** (1999), no. 6, 537–552.

[3] T. Kiguradze and R. Alhuzally, On a Dirichlet type boundary value problem in an orthogonally convex cylinder for a class of linear partial differential equations. *Abstracts of the International Workshop on the Qualitative Theory of Differential Equations – QUALITDE-2021, Tbilisi, Georgia, December 18-20*, pp. 114–119; http://www.rmi.ge/eng/QUALITDE-2019/Kiguradze_T_AlHuzally_workshop_2021.pdf.

[4] T. Kiguradze and V. Lakshmikantham, On the Dirichlet problem for fourth-order linear hyperbolic equations. *Nonlinear Anal.* **49** (2002), no. 2, Ser. A: Theory Methods, 197–219.