The Boundary Value Problem for One Class of Nonlinear Systems of Partial Differential Equations

Sergo Kharibegashvili

Andrea Razmadze Mathematical Institute of Ivane Javakhishvili Tbilisi State University
Tbilisi, Georgia
E-mail: kharibegashvili@yahoo.com

In Euclidean space $\mathbb{R}^{n+1}$ of variables $x=(x_1,\dots,x_n)$ and t consider a nonlinear system of partial differential equations of the form

$$L_f u := \frac{\partial^{4k} u}{\partial t^{4k}} - \sum_{i,j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + f(u) = F, \tag{1}$$

where $f=(f_1,\dots,f_N)$, $F=(F_1,\dots,F_N)$ are the given and $u=(u_1,\dots,u_N)$ is an unknown vector functions, $N \geq 2$; A_{ij} are the given constant quadratic matrices of order N, besides $A_{ij}=A_{ji}$, $i,j=1,\dots,n$, $n\geq 2$, k is a natural number.

For system (1) consider the following boundary value problem: in cylindrical domain $D_T := \Omega\times(0,T)$, where Ω is an open Lipschitz domain in $\mathbb{R}^n$, find a solution $u=u(x,t)$ to system (1.1) according to the following boundary conditions

$$u\big|_{\Gamma} = 0, \tag{2}$$

$$\frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0\cup\Omega_T} = 0, \quad i=0,\dots,2k-1, \tag{3}$$

where $\Gamma := \partial\Omega\times(0,T)$ is a lateral face of the cylinder D_T, $\Omega_0 :\ x\in\Omega,\ t=0$ and $\Omega_T :\ x\in\Omega,\ t=T$ are upper and lower bases of this cylinder, respectively.

Denote by $C^{2,4k}(\overline{D}_T)$ the space of continuous in $\overline{D}_T$ vector functions $u=(u_1,\dots,u_N)$, having continuous in $\overline{D}_T$ partial derivatives $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i \partial x_j}$, $\frac{\partial^l u}{\partial t^l}$, $i,j=1,\dots,n$; $l=1,\dots,4k$. Let

$$C_0^{2,4k}(\overline{D}_T,\partial D_T) := \Big\{ u\in C^{2,4k}(\overline{D}_T) :\ \ u\big|_{\Gamma}=0,\ \ \frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0\cup\Omega_T} = 0, \ \ i=0,\dots,2k-1 \Big\}.$$

Consider the Hilbert space $W_0^{1,2k}(D_T)$, which is obtained by completion with respect to the norm

$$\|u\|^2_{W_0^{1,2k}(D_T)} = \int\limits_{D_T} \bigg[|u|^2 + \sum_{i=1}^{2k} \Big| \frac{\partial^i u}{\partial t^i} \Big|^2 + \sum_{i=1}^{n} \Big| \frac{\partial u}{\partial x_i}\Big|^2 \bigg]\, dx\, dt \tag{4}$$

of the classical space $C_0^{2,4k}(\overline{D}_T,\partial D_T)$, where $|\cdot|$ is the norm in the space $\mathbb{R}^N$.

Remark 1. From (4) it follows that if $u\in W_0^{1,2k}(D_T)$, then $u\in \overset{\circ}{W}{}_2^1(D_T)$ and $\frac{\partial^i u}{\partial t^i}\in L_2(D_T)$, $i=1,\dots,2k$. Here $W_2^1(D_T)$ is a well-known Sobolev space consisting of elements from $L_2(D_T)$ and having generalized partial derivatives of the first order from $L_2(D_T)$, and

$$\overset{\circ}{W}{}_2^1(D_T) = \big\{ u\in W_2^1(D_T) :\ \ u|_{\partial D_T} = 0 \big\},$$

where the equality $u|_{\partial D_T}=0$ must be understood in the sense of the trace theory.

Below, we impose on nonlinear vector function $f=(f_1,\dots,f_N)$ from (1) the following requirements

$$f\in C(\mathbb{R}^N),\ \ |f(u)|\le M_1+M_2|u|^\alpha,\ \ u\in\mathbb{R}^N, \tag{5}$$

where $M_i=const\ge 0$, $i=1,2$, and

$$0\le\alpha=const<\frac{n+1}{n-1}\,. \tag{6}$$

Remark 2. The embedding operator $I:W_2^1(D_T)\to L_q(D_T)$ represents a linear continuous compact operator for $1<q<\frac{2(n+1)}{n-1}$, $n>1$. At the same time the Nemitsky operator $K:L_q(D_T)\to L_2(D_T)$, acting by the formula $Ku=f(u)$, where $u=(u_1,\dots,u_N)\in L_q(D_T)$ and vector function $f=(f_1,\dots,f_N)$ satisfies condition (5), is continuous and bounded if $q\ge 2\alpha$. Therefore, if $\alpha<\frac{n+1}{n-1}$, then there exists such number q that $1<q<\frac{2(n+1)}{n-1}$ and $q\ge 2\alpha$. Therefore, in this case the operator

$$K_0=KI:W_2^1(D_T)\to L_2(D_T)$$

is continuous and compact. Then from $u\in W_2^1(D_T)$ it follows that $f(u)\in L_2(D_T)$ and, if $u^m\to u$ in the space $W_2^1(D_T)$, then $f(u^m)\to f(u)$ in $L_2(D_T)$.

Remark 3. Let $u\in C_0^{2,4k}(\overline{D}_T,\partial D_T)$ be a classical solution of problem (1)–(3). Multiplying scalarly both parts of system (1) by an arbitrary vector function $\varphi\in C_0^{2,4k}(\overline{D}_T,\partial D_T)$ and integrating by parts the obtained equality on the domain D_T, we have

$$\int\limits_{D_T}\Big[\frac{\partial^{2k}u}{\partial t^{2k}}\,\frac{\partial^{2k}\varphi}{\partial t^{2k}}+\sum_{i,j=1}^n A_{ij}\,\frac{\partial u}{\partial x_i}\,\frac{\partial\varphi}{\partial x_j}\Big]\,dx\,dt+\int\limits_{D_T}f(u)\varphi\,dx\,dt=\int\limits_{D_T}F\varphi\,dx\,dt \tag{7}$$
$$\forall\,\varphi\in C_0^{2,4k}(\overline{D}_T,\partial D_T).$$

We consider equality (7) as a basis for defining a weak generalized solution of problem (1)–(3).

Definition 1. Let the vector function f satisfy conditions (5), (6) and $F\in L_2(D_T)$. A vector function $u\in W_0^{1,2k}(D_T)$ is called a weak generalized solution of problem (1)–(3) if the integral equality (7) is valid for any vector function $\varphi\in W_0^{1,2k}(D_T)$, i.e.,

$$\int\limits_{D_T}\Big[\frac{\partial^{2k}u}{\partial t^{2k}}\cdot\frac{\partial^{2k}\varphi}{\partial t^{2k}}+\sum_{i,j=1}^n A_{ij}\,\frac{\partial u}{\partial x_i}\,\frac{\partial\varphi}{\partial x_j}\Big]\,dx\,dt+\int\limits_{D_T}f(u)\varphi\,dx\,dt=\int\limits_{D_T}F\varphi\,dx\,dt \tag{8}$$
$$\forall\,\varphi\in W_0^{1,2k}(D_T).$$

Note that due to Remark 2, the integral $\int\limits_{D_T}f(u)\varphi\,dx\,dt$ in equality (8) is defined correctly, since from $u\in W_0^{1,2k}(D_T)$ it follows $f(u)\in L_2(D_T)$ and, therefore, $f(u)\varphi\in L_1(D_T)$.

It is easy to verify that if the solution u of problem (1)–(3) belongs to the class $C_0^{2,4k}(D_T,\partial D_T)$ in the sense of Definition 1, then it will also be a classical solution of this problem.

Below we assume that the operator

$$\sum_{i,j=1}^n A_{ij}\,\frac{\partial^2}{\partial x_i\partial x_j} \tag{9}$$

is strictly elliptic, i.e., the matrix $Q(\xi)=\sum\limits_{i,j=1}^n A_{ij}\xi_i\xi_j$ is positively defined for each $\xi=(\xi_1,\dots,\xi_n)\in\mathbb{R}^n\setminus\{(0,\dots,0)\}$:

$$(Q(\xi)\eta,\eta)_{\mathbb{R}^N}>0\quad\forall\,\eta\in\mathbb{R}^N\setminus\{(0,\dots,0)\}, \tag{10}$$

where $(\,\cdot\,,\,\cdot\,)_{\mathbb{R}^N}$ is a standard scalar product in the euclidian space $\mathbb{R}^N$. Note that in the scalar case the operator from (9) represents an elliptic operator and in this case the linear part of the operator L_f from (1), i.e. L_0 is semielliptic.

At fulfilment of condition (10) in the space $C_0^{2,4k}(\overline{D}_T, \partial D_T)$, together with the scalar product

$$(u,v)_o = \int\limits_{D_T} \bigg[uv + \sum_{i=1}^{2k} \frac{\partial^i u}{\partial t^i}\,\frac{\partial^i v}{\partial t^i} + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_i}\bigg]\,dx\,dt \tag{11}$$

with norm $||\cdot||_0 = ||\cdot||_{W_0^{1,2k}(D_T)}$, defined by the right-hand side of equality (4), let us introduce the following scalar product

$$(u,v)_1 = \int\limits_{D_T} \bigg[\frac{\partial^{2k} u}{\partial t^{2k}}\,\frac{\partial^{2k} v}{\partial t^{2k}} + \sum_{i,j=1}^{n} A_{ij}\,\frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_j}\bigg]\,dx\,dt \tag{12}$$

with norm

$$\|u\|_1^2 = \int\limits_{D_T} \bigg[\Big|\frac{\partial^{2k} u}{\partial t^{2k}}\Big|^2 + \sum_{i,j=1}^{n} A_{ij}\,\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\bigg]\,dx\,dt, \tag{13}$$

where $u,v \in C_0^{2,4k}(\overline{D}_T, \partial D_T)$.

It is proved the validity of the following inequalities

$$c_1\|u\|_0 \le ||u||_1 \le c_2\|u\|_0 \quad \forall\, u \in C_0^{2,4k}(\overline{D}_T, \partial D_T)$$

with positive constants c_1 and c_2, not dependent on u. Hence it follows that if we complete the space $C_0^{2,4k}(\overline{D}_T, \partial D_T)$ under norm (13), then in view of (11) we obtain the same Hilbert space $W_0^{1,2k}(D_T)$ with equivalent scalar products (11) and (12). Further, it can be proved the unique solvability of the linear problem correspondent to (1)–(3), i.e., when $f = 0$: for any $F \in L_2(D_T)$ there exists the unique solution $u = L_0^{-1}F \in W_0^{1,2k}(D_T)$ of this problem, where the linear operator

$$L_0^{-1} : L_2(D_T) \to W_0^{1,2k}(D_T)$$

is continuous. Thus, the nonlinear problem (1)–(3) is reduced to the following functional equation

$$u = L_0^{-1}\big[-f(u) + F\big] \tag{14}$$

in the Hilbert space $W_0^{1,2k}(D_T)$.

At fulfillment of the condition

$$\lim_{|u|\to\infty} \inf \frac{uf(u)}{|u|^2} \ge 0 \tag{15}$$

it can be proved the a priori estimate of the solution $u \in W_0^{1,2k}(D_T)$ of equation (14), whence due to Remark 2 we have the solvability of this equation, and, therefore, of problem (1)–(3) in the space $W_0^{1,2k}(D_T)$. Therefore the following theorem is valid.

Theorem 1. *Let conditions* (5), (6), (10) *and* (15) *be fulfilled. Then for any $F \in L_2(D_T)$ problem* (1)–(3) *has at least one generalized solution u in the space $W_0^{1,2k}(D_T)$.*

Remark 4. If conditions (5), (6) and (10) are fulfilled and the mapping $f(u) : \mathbb{R}^N \to \mathbb{R}^N$ satisfies the condition

$$(f(u) - f(v))(u - v) \ge 0 \quad \forall\, u,v \in \mathbb{R}^N, \tag{16}$$

then the solution of this problem is unique.

Thus, the following theorem is valid.

Theorem 2. *Let conditions* (5), (6), (10) *and* (15), (16) *be fulfilled. Then for any* $F \in L_2(D_T)$ *problem* (1)–(3) *has a unique weak generalized solution* u *in the space* $\overset{\circ}{W}{}^{1,2k}_0(D_T)$*.*

As the examples show, if the conditions imposed on the nonlinear vector function f are violated, then problem (1)–(3) may not have a solution. Indeed, consider the particular case of system (1), when it is split in the main part, i.e., $A_{ij} = a_{ij} I_N$, where I_N is a unit matrix of order N, and a_{ij} are numbers such that the operator $\sum\limits_{i,j=1}^{n} a_{ij}\partial^2/\partial x_i \partial x_j$ is a scalar elliptic operator.

Consider the following requirement imposed on the vector function f: there exist numbers $l_1, \dots, l_N$, $\sum\limits_{i=1}^{N} |l_i| \neq 0$ such that

$$\sum_{i=1}^{N} l_i f_i(u) \leq -d_0 \Big| \sum_{i=1}^{N} l_i u_i \Big|^{\beta} \quad \forall\, u \in \mathbb{R}^N, \;\; 1 < \beta = const < \frac{n+1}{n-1}\,, \tag{17}$$

where $d_0 = const > 0$. Let the domain Ω be given by the equation $\partial\Omega : \omega(x) = 0$, where $\nabla_x \omega|_{\partial\Omega} \neq 0$, $\omega|_{\Omega} > 0$, $\nabla_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ and $\omega \in C^2(\mathbb{R}^n)$.

Theorem 3. *Let the vector function* f *satisfy conditions* (5), (6), (10) *and* (17). *Let* $F^0 = (F_1^0, \dots, F_N^0) \in L_2(D_T)$, $G = \sum\limits_{i=1}^{N} l_i F_i^0 \geq 0$ *and* $||G||_{L_2(D_T)} \neq 0$. *Then there exists a number* $\mu_0 = \mu_0(G, \beta) > 0$ *such that for* $\mu > \mu_0$ *problem* (1)–(3) *cannot have a weak generalized solution in the space* $\overset{\circ}{W}{}^{1,2k}_0(D_T)$ *for* $F = \mu F_0$*.*

Remark 5. Consider one class of vector functions f:

$$f_i(u_1, \dots, u_N) = \sum_{j=1}^{N} a_{ij} |u_j|^{\beta_{ij}} + b_i, \;\; i = 1, \dots, N,$$

where constants a_{ij}, β_{ij} and b_i satisfy the inequalities

$$a_{ij} > 0, \;\; 1 < \beta_{ij} < \frac{n+1}{n-1}\,, \;\; \sum_{k=1}^{N} b_k > 0, \;\; , i, j = 1, \dots, N.$$

It is easy to verify that this class satisfies condition (17).

Acknowledgements

The work was supported by the Shota Rustavely National Science Foundation, Grant # FR-21-7307.