The Critical Case of the Matrix Differential Equations' Systems

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This paper considers a system of M linear matrix differential equations with coefficients, depicted in the form of absolutely and uniformly convergent Fourier series with slowly variable in a certain sense coefficients and with the frequency (class F). This system is close to the block-diagonal system with slowly changing coefficients. We are looking for a transformation with coefficients of a similar type which brings this system to purely block-diagonal form. Regarding the coefficients of this transformation, chews a quasi-linear system of matrix differential equations, which decays on M independent subsystems, each of which has the form of some auxiliary nonlinear systems. We obtained conditions of existence of the desired transformation for this auxiliary system in a critical case.

1 Basic notation and definitions

Let

$$G(\varepsilon_0) = \Big\{(t;\varepsilon): \quad t \in \mathbb{R}, \quad \varepsilon \in [0;\varepsilon_0), \quad \varepsilon_0 \in \mathbb{R}^*\Big\}.$$

Definition 1.1. Let's say that the function $p(t;\varepsilon)$ belongs to the class $S(m;\varepsilon_0)$ if the following conditions are true

(1) $p\colon G(\varepsilon_0) \to \mathbb{C}$;

(2) $p(t;\varepsilon) \in C^m(G(\varepsilon_0))$ for t;

(3)

$$\frac{d^k p(t;\varepsilon)}{dt^k} = \varepsilon^k p_k(t;\varepsilon) \ \ (0 \leqslant k \leqslant m),$$

where

$$\|p\|_{S(m;\varepsilon_0)} \stackrel{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |p_k(t;\varepsilon)| < +\infty.$$

Definition 1.2. Let's say that the function $f(t;\varepsilon;\theta(t;\varepsilon))$ belongs to the class $F(m;\varepsilon_0;\theta)$ ($m \in \mathbb{N} \cup \{0\}$), if this function can be represented in the following form:

$$f(t;\varepsilon;\theta(t;\varepsilon) = \sum_{n=-\infty}^{+\infty} f_n(t;\varepsilon)\exp(in\theta(t;\varepsilon)),$$

where

(1) $f_n(t;\varepsilon) \in S(m;\varepsilon_0)$ ($n \in \mathbb{Z}$);

(2)

$$\|f\|_{F(m;\varepsilon_0;\theta)} \stackrel{def}{=} \sum_{n=-\infty}^{+\infty} \|f_n\|_{S(m;\varepsilon_0)} < +\infty;$$

(3)
$$\theta(t;\varepsilon)=\int\limits_0^t\varphi(\tau;\varepsilon)\,d\tau,\ \ \varphi\in\mathbb{R}^*,\ \ \varphi\in S(m;\varepsilon_0),\ \ \inf_{G(\varepsilon_0)}\varphi(t;\varepsilon)=\varphi_0>0.$$

Definition 1.3. Let's say that the matrix $A(t;\varepsilon)=(a_{jk}(t;\varepsilon))_{j,k=\overline{1,N}}$ belongs to the class $S_2(m;\varepsilon_0)$ $(m\in\mathbb{N}\cup\{0\})$, in case $a_{jk}\in S(m;\varepsilon_0)$ $(j,k=\overline{1,N})$.

Let's define the norm
$$\|A(t;\varepsilon)\|_{S_2(m;\varepsilon_0)}\overset{def}{=}\max_{1\leqslant j\leqslant N}\sum_{k=1}^N\|a_{jk}(t;\varepsilon)\|_{S(m;\varepsilon_0)}.$$

Definition 1.4. Let's say that the matrix $B(t;\varepsilon;\theta)=(b_{jk}(t;\varepsilon;\theta))_{j,k=\overline{1,N}}$ belongs to the class $F_2(m;\varepsilon_0;\theta)$ $(m\in\mathbb{N}\cup\{0\})$, in case $b_{jk}(t;\varepsilon;\theta)\in F(m;\varepsilon_0;\theta)$ $(j,k=\overline{1,N})$.

Let's define the norm
$$\|B(t;\varepsilon;\theta)\|_{F_2(m;\varepsilon_0;\theta)}\overset{def}{=}\max_{1\leqslant j\leqslant N}\sum_{k=1}^N\|b_{jk}(t;\varepsilon;\theta)\|_{F(m;\varepsilon_0;\theta)}.$$

Note that in case $B_1\in F_2(m;\varepsilon_0;\theta)$, $B_2\in F_2(m;\varepsilon_0;\theta)$, the following conditions are true:

(1) $B_1+B_2, B_1B_2\in F_2(m;\varepsilon_0;\theta)$,

(2) $\|B_1+B_2\|_{F_2(m;\varepsilon_0;\theta)}\le\|B_1\|_{F_2(m;\varepsilon_0;\theta)}+\|B_2\|_{F_2(m;\varepsilon_0;\theta)}$,

(3) $\|B_1B_2\|_{F_2(m;\varepsilon_0;\theta)}\le 2^m\|B_1\|_{F_2(m;\varepsilon_0;\theta)}\cdot\|B_2\|_{F_2(m;\varepsilon_0;\theta)}$.

2 Statement of the problem

The following system of linear matrix equations is considered
$$\frac{dX_j}{dt}=A_j(t,\varepsilon)X_j+\mu\sum_{k=1}^M B_{jk}(t,\varepsilon,\theta)X_k,\ \ j=\overline{1,M},\tag{2.1}$$

where X_j are unknown square matrices of the order N, belonging to some closed bounded region $D\subset\mathbb{C}^{N\times N}$, $\mathbb{C}^{N\times N}$ is the space of complex-valued matrices of dimension $(N\times N)$. Also, let $A_j(t,\varepsilon)\in S_2(m;\varepsilon_0)$, $B_{kj}(t,\varepsilon,\theta)\in F_2(m;\varepsilon_0;\theta)$, $\mu\in(0,1)$ be real parameter.

We are looking for the transformation
$$X_j=Y_j+\sum_{\substack{k=1\\k\neq j}}^M Q_{jk}(t,\varepsilon,\theta(t,\varepsilon),\mu)Y_k,\ \ j=\overline{1,M},\tag{2.2}$$

in which $Q_{jk}(t,\varepsilon,\theta(t,\varepsilon),\mu)$ $(j,k=\overline{1,M})$ are unknown square matrices of dimension $N\times N$ that belong to the class $F_2(m_1;\varepsilon_1;\theta)$ $(m_1\le m_0;\ \varepsilon_1\le\varepsilon_0)$ which brings system (2.1) to the form
$$\frac{dY_j}{dt}=V_j(t,\varepsilon,\theta,\mu)Y_j,\tag{2.3}$$

where $V_j(t,\varepsilon,\theta,\mu)\in F_2(m_1;\varepsilon_0;\theta)$.

Using transformation (2.2) with respect to unknown functions $Q_{jk}(t,\varepsilon,\theta,\mu)$ $(j=\overline{1,M})$ we will get the system

$$\frac{dQ_{jk}}{dt}=A_j(t,\varepsilon)Q_{jk}-Q_{jk}A_k(t,\varepsilon)+\mu(B_{jj}(t,\varepsilon,\theta)Q_{jk}-Q_{jk}B_{kk}(t,\varepsilon,\theta))$$
$$+\mu B_{jk}(t,\varepsilon,\theta)+\mu\sum_{\substack{s=1\\ s\neq j,\, s\neq k}}^{M}B_{js}(t,\varepsilon,\theta)Q_{sk}-\mu Q_{jk}\sum_{\substack{s=1\\ s\neq k}}^{M}B_{ks}(t,\varepsilon,\theta)Q_{sk},\quad j,k=\overline{1,M},\quad j\neq k. \tag{2.4}$$

So, system (2.1) turns into

$$\frac{dY_j}{dt}=V_j(t,\varepsilon,\theta,\mu)Y_j=\Big(\mu B_{jj}(t,\varepsilon,\theta)+\Lambda(t,\varepsilon)+\sum_{\substack{s=1\\ s\neq j}}^{M}B_{js}(t,\varepsilon,\theta)Q_{sj}\Big)Y_j,\quad j=\overline{1,M}. \tag{2.5}$$

The following lemma takes place.

Lemma 2.1. *Let the matrices $A_j(t,\varepsilon)$ $(j=\overline{1,M})$ in system (2.4) be such that there are matrices $L_j(t,\varepsilon)$ $(j=\overline{1,M})$, for which the following conditions are true:*

(1) $L_j(t,\varepsilon)\in S_2(m;\varepsilon)$ $(j=\overline{1,M})$;

(2) $|\det(L_j(t,\varepsilon))|\geq a_0>0$ $(j=\overline{1,M})$;

(3)
$$L_j^{-1}(t,\varepsilon)A_j(t,\varepsilon)L_j(t,\varepsilon)=\triangle_j(t,\varepsilon)\ \ (j=\overline{1,M}),$$
in which $\triangle_j(t,\varepsilon)$ $(j=\overline{1,M})$ – lower triangular matrices of the Nth order of the class $S_2(m;\varepsilon_0)$.

Then using the transformation

$$Q_{jk}=L_j(t,\varepsilon)Y_{jk}L_k^{-1}(t,\varepsilon)\ \ (j,k=\overline{1,M},\quad j\neq k), \tag{2.6}$$

system (2.4) is reduced to the next system

$$\frac{dY_{jk}}{dt}=\triangle_j(t,\varepsilon)Y_{jk}-Y_{jk}\triangle_k(t,\varepsilon)-L^{-1}\frac{dL_j}{dt}Y_{jk}-Y_{jk}L_k^{-1}(t,\varepsilon)\frac{dL_k}{dt}$$
$$+\mu(L_j^{-1}(t,\varepsilon)B_{jj}(t,\varepsilon,\theta)L_j(t,\varepsilon)Y_{jk}-Y_{jk}L_k^{-1}(t,\varepsilon)B_{kk}(t,\varepsilon,\theta)L_k(t,\varepsilon))$$
$$+\mu L_j^{-1}(t,\varepsilon)B_{jk}(t,\varepsilon,\theta)L_k(t,\varepsilon)+\mu\sum_{\substack{s=1\\ s\neq j,\, s\neq k^c}}^{M}L_j^{-1}(t,\varepsilon)B_{js}(t,\varepsilon,\theta)L_s(t,\varepsilon)Y_{sk}$$
$$-\mu Y_{jk}\sum_{\substack{s=1\\ s\neq k^c}}^{M}L_k^{-1}B_{ks}(t,\varepsilon,\theta)L_s(t,\varepsilon)Y_{sk},\quad j,k=\overline{1,M}\ \ (j\neq k). \tag{2.7}$$

3 Main results

Lemma 3.1. *Let the following system of matrix differential equations be given:*

$$\frac{dY_j}{dt}=D_{j1}(t,\varepsilon)Q_{jk}-Q_{jk}D_{j2}(t,\varepsilon)+\mu F_j(t,\varepsilon,\theta)+\mu\sum_{s=1}^{M}P_{js1}(t,\varepsilon,\theta)Y_sP_{js2}(t,\varepsilon,\theta)$$
$$-\mu Y_j\sum_{s=1}^{M}R_{js1}(t,\varepsilon,\theta)Y_sR_{js2}(t,\varepsilon,\theta)-\varepsilon H_{j1}(t,\varepsilon)Y_j-\varepsilon Y_jH_{j2}(t,\varepsilon),\quad j=\overline{1,M}, \tag{3.1}$$

where $D_{j1}(t,\varepsilon)=(d^{j1}_{\alpha\beta}(t,\varepsilon))_{\alpha,\beta=\overline{1,N}}$, $D_{j2}(t,\varepsilon)=(d^{j2}_{\alpha\beta}(t,\varepsilon))_{\alpha,\beta=\overline{1,N}}$ *– lower triangular matrices of the class* $S_2(m;\varepsilon_0)$, $F_j(t,\varepsilon,\theta)$, $P_{js1}(t,\varepsilon,\theta)$, $P_{js2}(t,\varepsilon,\theta)$, $R_{js1}(t,\varepsilon,\theta)$, $R_{js2}(t,\varepsilon,\theta)$ *is in the class* $F_2(m;\varepsilon_0;\theta)$, $H_{j1}(t,\varepsilon)$, $H_{j2}(t,\varepsilon)$ *are in the class* $S_2(m-1;\varepsilon_0)$,$\mu\in(0,1)$ *is a real parameter. And let the conditions be fulfilled:*

(1^0)

$$\inf_{G(\varepsilon_0)}\big|d^{j1}_{\alpha\beta}(t,\varepsilon)-d^{k1}_{\alpha\beta}(t,\varepsilon)-in\varphi(t,\varepsilon)\big|\geq b_0>0,$$
$$\inf_{G(\varepsilon_0)}\big|d^{j2}_{\alpha\beta}(t,\varepsilon)-d^{k2}_{\alpha\beta}(t,\varepsilon)-in\varphi(t,\varepsilon)\big|\geq b_0>0\ \ \forall\,n\in\mathbb{Z},\ \ j,k=\overline{1,N},\ \ j\neq k.$$

(2^0)

$$d^{j1}_{\alpha\beta}(t,\varepsilon)-d^{k2}_{\alpha\beta}(t,\varepsilon)=i\omega_{jk}(t,\varepsilon),\ \ \omega_{jk}(t,\varepsilon)\in\mathbb{R},$$
$$\inf_{G(\varepsilon_0)}|\omega_{jk}(t,\varepsilon)-n\varphi(t,\varepsilon)|\geq b_0>0\ \ \forall\,n\in\mathbb{Z},\ \ j,k=\overline{1,N}.$$

Then there exist constants $\mu_1\in(0;\mu_0),\varepsilon_2\in(0;\mu_0)$ *such that for all* $\mu\in[0;\mu_2)$ *and for all* $\varepsilon\in(0,\varepsilon_2)$*, system* (3.1) *has a partial solution of the class* $F_2(m-1;\varepsilon_2;\theta)$.

Condition (2^0) shows that in this case we are dealing with critical by chance, as opposed to work [8], in which it is assumed that

$$\big|\mathrm{Re}\,(d^{j1}_{\alpha\beta}(t,\varepsilon)-d^{k2}_{\alpha\beta}(t,\varepsilon))\big|\geq\gamma>0\ \ (j=\overline{1,M},\ \ k=\overline{1,N}).$$

The next theorem takes place.

Theorem 3.1. *Let system* (2.4) *satisfy the conditions of Lemma 3.1, and let system* (2.7)*, obtained by transformation* (2.6)*, for each* $k=\overline{1,M}$ *satisfy all the conditions of Lemma 3.1. Then there exist* $\mu_4\in(0;1)$, $\varepsilon_4(\mu)\in(0;\varepsilon_0)$ *such that for all* $\mu\in(0;\varepsilon_4)$ *and for all* $\varepsilon\in(0;\varepsilon_4(\mu))$ *there exists the transformation of the form* (2.2)*, in which the coefficients* $Q_{jk}(t,\varepsilon,\theta(t,\varepsilon),\mu)$ *belong to the class* $F_2(m-1;\varepsilon_4(\mu);\theta)$*, that brings system* (2.1) *to the form* (2.3)*, in which* $V_j(t,\varepsilon,\theta,\mu)$ *are determined by formulas* (2.5).

For matrix systems of this type, such a result was not obtained before. In previous works [9] a matrix differential equation was considered:

$$\frac{dX}{dt}=A(t,\varepsilon)X-XB(t,\varepsilon)+P(t;\varepsilon_0;\theta)+\mu\Phi(t;\varepsilon_0;\theta;X),\tag{3.2}$$

where X is an unknown square matrix of order N, that belongs to some closed limited area $D\subset\mathbb{C}^{N\times N}$, where $\mathbb{C}^{N\times N}$ is the space of complex-valued matrices of dimention $N\times N$, $A(t;\varepsilon),B(t,\varepsilon)\in S_2(m;\varepsilon_0)$, $P(t;\varepsilon_0;\theta)\in F_2(m;\varepsilon_0;\theta)$. It is also assumed that $\Phi(t;\varepsilon_0;\theta;X)$ is a matrix-function that belongs to the class $F_2(m;\varepsilon_0;\theta)$ with respect to m,ε_0,θ and is continuous over X in D. μ is a real parameter.

For equation (3.2) in the critical case, the issue of the presence of partial class solutions was studied $F(m_1;\varepsilon_1;\theta)$ $(m_1\leq m;\ \varepsilon_1\leq\varepsilon_0)$.

The results of the works [1–7, 10] were used for obtaining our results.

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